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THE INVOLUTION CURVE DETERMINED FROM A SPECIAL PENCIL OF n-ICS*

BY
JOSEPHINE H. CHANLER

Introduction. In *The geometry of the Weddle manifold* W_p [1]† there arises the necessity for investigating involution curves, some of which cover W_p, while others cover the sections of W_p by its F-loci. Since the points of an involution curve V_k are in (1, 1) correspondence with the k-ads from the members of a pencil of n-ics, V_k may be constructed as follows: Assign the parameter t to the points of a norm curve N^k in S_k. The hyperplanes of the curve given by a general member of the pencil intersect in $C_{n,k}$ points, the locus of which is the curve V_k of order $C_{n-1,k-1}$. The involution curves of W_p are particularized by the fact that the pencil of n-ics contains one member with a number of double roots. In the study of these it is important to determine the genus; it is likewise necessary to discover what multiple points are introduced by the special n-ic if we are to ascertain the effect of the Cremona transformations associated with W_p. The article just cited makes no attempt to solve these problems except for special values of n, k. In the present paper (§§1–4) we solve them for general n, k where one n-ic of the pencil possesses j distinct double roots $(1 \leqq j \leqq n/2)$.

We determine the genus of V_k by applying to the correspondence set up between the points of N^k and the points of V_k Zeuthen's formula:

$$\eta - \eta' = 2\alpha(p'-1) - 2\alpha'(p-1).$$

Here α, α' are the indices of the correspondence, p, p' the genera of the curves, and η, η' the numbers of branch points of the curves. η, η' may be calculated in either of two ways. By the first method, the point t_1 of N^k is counted $v_1 - 1$ times as a branch point if v_1 of the points y of V_k corresponding to t_1 coalesce at γ on the same branch of V_k. Hence we must first investigate the multiple points of V_k. This method I use in §1, and used originally in §2. The second much more elegant method, which may be applied at once without a study of the multiple points, was pointed out to me by O. Zariski. It depends on the following criterion for the multiplicity of a coincidence in an (α, α') correspondence between two algebraic curves V and V': Let P_0 be a point on the

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† Numbers in brackets refer to the bibliography at the end of the paper.

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Riemann surface of V; as a variable point P of V, very near P_0, turns around P_0, the corresponding α' points on V' may be permuted among themselves. If this permutation consists of cycles of periods $v_1, v_2, \cdots, v_m$, $(\sum_{i=1}^{m} v_i = \alpha')$, then to the point P_0 there corresponds on V', m coincidences of multiplicities $v_1, v_2, \cdots, v_m$ respectively. P_0 then contributes to η the value $\sum_{i=1}^{m}(v_i - 1)$. By means of this method we readily study the special cases of §6, where some or all of the j double roots of the special n-ics coincide. It is interesting to note that η, η' are not always identical for the two definitions, though their difference is the same for each.

In §3 we determine the multiple points of V_k for $j > 1$ by comparing the value of its genus as obtained in §2 to the value determined by the first method. In §4 we check this result by a projective study of V_k as part of the intersection Γ of $k-1$ hypersurfaces. These hypersurfaces are represented by the vanishing of $k-1$ determinants of highest order selected in a special way from a matrix whose elements are binary n-ics. The order of the manifold represented by the vanishing of all the determinants of highest order in such a matrix has been studied by Brill [2] as the order of a restricted system of equations. One reason for the introduction of this second proof is the light that it throws upon such manifolds; in particular the machinery set up for our work shows that Brill's formulas may be more easily obtained from the study of a dual matrix whose elements are linear forms in several variables. We thus prove that Brill's problem reduces to a special case of those investigated by Salmon, Roberts, Cayley, and Pieri. A more important reason for the projective method is that it introduces interesting manifolds associated with V_k; also it seems adapted to the study of manifolds which are obtained from V_k by generalization.

In §5 we prove an interesting identity involving binomial coefficients which I have not found in the literature.

1. **The involution curve of the general pencil.** The pencil of n-ics $[(a_1 t)^n, (a_2 t)^n]$ contains $2(n-1)$ members with a double root each. We have for N^k, V_k:

$$\alpha = k, \qquad \alpha' = C_{n-1,k-1}, \qquad p = 0, \qquad p' = p_k. \tag{1}$$

Since the involution curve has no multiple points, we easily calculate η, η' by the first method:

$$\eta = 2(n-1)(k-1)C_{n-2,k-1}; \qquad \eta' = 2(n-1)C_{n-2,k-2}. \tag{2}$$

Substituting these values in Zeuthen's formula, we have

$$p_k = (n-k)C_{n-1,k-1} - C_{n,k} + 1. \tag{3}$$

This is unaltered when k and $n-k$ are interchanged, checking the fact that k-ads and complementary $(n-k)$-ads in the pencil of n-ics determine two curves in (1, 1) correspondence.

2. **The genus of the involution curve V_k for which one n-ic possesses several double roots.** Let the pencil contain one n-ic with j distinct double roots, while there are $2(n-1)-j$ n-ics with a double root each. Since the n-ics of the latter type behave as before with respect to η, η', we have for these

$$\eta_0 = (2n-2-j)(k-1)C_{n-2,k-1}, \qquad \eta_0' = (2n-2-j)C_{n-2,k-2}$$

so that

$$\eta_0 - \eta_0' = (2n-2-j)(n-k-1)C_{n-2,k-2}. \tag{1}$$

To determine the special n-ic's contributions η_s, η_s' to η, η', we use Zariski's argument.

Let the special n-ic of the pencil correspond to the value x_0 of the parameter x. As x very near x_0 turns around x_0, the roots of the n-ic corresponding to x will be permuted, the permutation consisting of j cycles of period two and $n-2j$ cycles of period one. For convenience we write this permutation

$$S = (t_{11}t_{12})\cdots(t_{i1}t_{i2})\cdots(t_{j1}t_{j2})(\tau_1)(\tau_2)\cdots(\tau_{n-2j}). \tag{2}$$

A corresponding permutation Σ takes place among the k-ads of t's, τ's chosen from the roots of the n-ic. As x approaches x_0, a letter of the ith cycle approaches the point P_i of N^k corresponding to the double root of the special n-ic, and τ_h approaches the point Q_h of N^k. The k-ads of t's and τ's approach points $\pi(t, \tau)$ of V_k. To find the number of coincidences on V_k corresponding to P_i or Q_h, we let t_i or τ_h turn around P_i or Q_h, and write down the corresponding permutation on the k-ads involving t_i or τ_h. From this it is seen that the P_i are not branch points, since the corresponding permutations are induced by Σ^2, and Σ is of period two. The permutations corresponding to Q_h are induced by Σ, under which the k-ads actually permuted are those and only those which contain one only of some pair $(t_{i1}t_{i2})$. Such a k-ad is interchanged with another as τ_h turns about Q_h. The total number of k-ads containing one and only one point from each of r pairs (t_{i1}, t_{i2}), $1 \leqq r \leqq j$, and $k-r-2l$ points τ_h, $0 \leqq l \leqq j-r$, is $C_{j,r}C_{j-r,l}C_{n-2j,k-r-2l}2^r$. Since each such k-ad contains $k-r-2l$ branch points, and since the k-ads are permuted in pairs by Σ, we have

$$\eta_s = \sum_{r=1}^{j}\sum_{l=0}^{j-r} C_{j,r}C_{j-r,l}C_{n-2j,k-r-2l}(k-r-2l)2^{r-1}. \tag{3}$$

A point $\pi(t, \tau)$ of V_k, for which the k-ad turning about it contains one and

only one point of a pair $(t_{i1}t_{i2})$ cannot be a branch point, since the corresponding permutation among the t's is induced by S^2. The k-ads containing both members of l such pairs and $k-2l$ points $\tau_h(1 \leqq l \leqq i)$ are $C_{j,l}\, C_{n-2j,k-2l}$ in number. Hence

$$\eta_s' = \sum_{l=1}^{j} C_{j,l} C_{n-2j,k-2l}\, l. \tag{4}$$

The decrease in p_k due to the presence of the special n-ic is therefore

$$\tilde{p} = \frac{1}{2k}\Big\{ j(n-k-1)C_{n-2,k-2} + \sum_{l=1}^{j} C_{j,l} C_{n-2j,k-2l}\, l - \sum_{r=1}^{j}\sum_{l=0}^{j-r} C_{j,r} C_{j-r,l} C_{n-2j,k-r-2l}(k-r-2l)2^{r-1} \Big\}, \tag{5}$$

and

$$p_k = (n-k)C_{n-1,k-1} - C_{n,k} + 1 - \tilde{p}. \tag{6}$$

3. **The multiple points of V_k.** The special n-ic introduces whatever multiple points exist. We determine what their nature must be to produce the value $\eta_s - \eta_s'$ obtained from §2, (3), (4). Let the special n-ic be $(ts_1)(ts_1) \cdots (ts_j)(ts_j)(tt_1)(tt_2) \cdots (tt_{n-2j})$. Then the point $P_l(s_1, s_1, s_2, s_2, \cdots, s_l, s_l, t_1, t_2, \cdots, t_{k-2l})$ is a simple point on V_k and contributes 0 to η_s and l to η_s'. There are $\sum_{l=1}^{j} C_{j,l}\, C_{n-2j,k-2l}$ such points. At the point

$$P_{rl}(s_1, s_2, \cdots, s_r, s_{r+1}, s_{r+1}, \cdots, s_{r+l}, s_{r+l}, t_1, t_2, \cdots, t_{k-r-2l})$$

there are 2^{r-1} or fewer branches, since $s_i (i \leqq r)$ meets V_k here in at most 2^{r-1} points. If there are 2^{r-1} branches, the contribution to η_s due to the t's and the s_i's $(i > r)$ at the point is $(k-r-l)\ 2^{r-1}$; the s_i's $(i \leqq r)$ contribute nothing. On the other hand, the contribution to η_s' due to P_{rl} is $l \cdot 2^{r-1}$. If the number of branches is smaller, the contributions to η_s due to this point increase, while that to η_s' decreases. There are $\sum_{r=1}^{j}\sum_{l=0}^{j-r} C_{j,r}\, C_{j-r,l}\, C_{n-2j,k-r-2l}$ such points. Hence only if there are 2^{r-1} branches at each point P_{rl}, can we have

$$(1) \quad \eta_s - \eta_s' = \sum_{r=1}^{j}\sum_{l=0}^{j-r} C_{j,r} C_{j-r,l} C_{n-2j,k-r-2l}(k-r-2l)2^{r-1} - \sum_{l=1}^{j} C_{j,l} C_{n-2j,k-2l}\, l,$$

the value obtained from §2, (3), (4).

THEOREM 1. *The point P_{rl} described above is a 2^{r-1}-fold point on V_k; in the calculation of the genus it is equivalent to $(r-1)\ 2^{r-2}$ ordinary double points.*

The first statement we have just proved. If the second holds, the decrease in the genus due to the special n-ic must be written:

$$(2) \qquad \tilde{p} = \sum_{r=2}^{j} \sum_{l=0}^{j-r} C_{j,r} C_{j-r,l} C_{n-2j,k-r-2l}(r-1)2^{r-2}.$$

Equating the right-hand members of (2) and §2, (5) leads to the following, after simplification:

$$(3) \quad j(n-k-1)C_{n-2,k-2} = \sum_{r=0}^{j} \sum_{l=0}^{j-r} C_{j,r} C_{j-r,l} C_{n-2j,k-r-2l}(kr-r-2l)2^{r-1}.$$

That (3) is satisfied under the conditions of our problem is proved in §5.

4. **The study of V_k from a projective view point; its multiple points and the manifolds associated with it.** Let the pencil of n-ics $[(a_1t)^n, (a_2t)^n]$ be contained in a linear system $\Sigma_{k-1} = [(a_1t)^n, (a_2t)^n, \cdots, (a_kt)^n]$. The k-ads of the pencil represent the points of a curve on the surface M^{n-k+1} of order $n-k+1$ given by the equation

$$(1) \qquad \begin{vmatrix} (a_1t_1)^n & (a_2t_1)^n \cdots (a_kt_1)^n \\ (a_1t_2)^n & (a_2t_2)^n \cdots (a_kt_2)^n \\ \cdots\cdots\cdots\cdots \\ (a_1t_k)^n & (a_2t_k)^n \cdots (a_kt_k)^n \end{vmatrix} \cdot \frac{1}{\pi(t_i t_j)} = 0,$$

where $t_1, t_2, \cdots, t_k$ are interpreted as Darboux coordinates in the space S_k.

Take the matrix

$$L_1 = \left\| \begin{matrix} (a_1t_1)^n & (a_2t_1)^n \cdots (a_{2k-2}t_1)^n \\ (a_1t_2)^n & (a_2t_2)^n \cdots (a_{2k-2}t_2)^n \\ \cdots\cdots\cdots\cdots \\ (a_1t_k)^n & (a_2t_k)^n \cdots (a_{2k-2}t_k)^n \end{matrix} \right\|,$$

where the n-ics are linearly independent. Hence $k \leqq (n+3)/2$; if not, we study instead the curve determined by the complementary $(n-k)$-ads. L_1 may be written briefly as $(a_1a_2 \cdots a_{2k-2})$. From L_1 select any set of $k-1$ determinants such that the only columns common to all are (a_1a_2), while each new determinant is obtained from that last found by dropping the third column and inserting as final column one not previously used. A typical set is $(a_1a_2 \cdots a_k)$, $(a_1a_2a_4 \cdots a_{k+1})$, $\cdots$, $(a_1a_2a_{k+1} \cdots a_{2k-2})$. Such a set we define as a set S of determinants.

The linearly independent n-ics $(a_1t)^n, \cdots, (a_mt)^n$, $m \leqq n+1$, determine $n+1$ linearly independent n-ics, $(c_1t)^n, (c_2t)^n, \cdots, (c_{n+1}t)^n$, such that $(c_it)^n$ is apolar to $(a_jt)^n$, $i \neq j$. In S_n the hyperplane $(a_jt)^n$ is on the point $(c_it)^n$, $i \neq j$. Thus the k-ad $(t_1, t_2, \cdots, t_k)$ making vanish the determinants of highest order of the matrix $(a_1a_2 \cdots a_m)$, $m \leqq k$, represents an S_{k-1} (on the points $t_1, \cdots, t_k$

of the norm curve N^k) joined to the $S_{n-m}[(c_{m+1}t)^n, (c_{m+2}t)^n, \cdots, (c_{n+1}t)^n]$ by an S_{n-1}. Hence the S_{k-1} meets the S_{n-m} in an S_{k-m}; that is, the k-ic $(qt)^k = (tt_1)(tt_2)\cdots(tt_k)$ is apolar to $k-m+1$ independent members of the linear system $[(c_{m+1}t)^n, \cdots, (c_{n+1}t)^n]$. If $\sum_{i=m+1}^{n+1}\lambda_i(c_iq)^k(c_it)^{n-k}\equiv 0$, we have $n-k+1$ equations in $n-m+1$ λ's. If there are $k-m+1$ independent solutions, the rank of the matrix of the system must be $n-k$. Thus the determinants of highest order in the matrix must vanish, which we indicate by writing $[c_{m+1}c_{m+2}\cdots c_{n+1}]\equiv 0$. The k-ad $(t_1, t_2, \cdots, t_k)$ making vanish the determinants of highest order of a matrix $(a_1a_2\cdots a_m)$, $m>k$, represents an S_{k-1} such that there are only $k-1$ conditions imposed for any hyperplane which is a linear combination of $(a_1t)^n, (a_2t)^n, \cdots, (a_mt)^n$ to be on the S_{k-1}. The S_{k-1} must therefore meet the $S_{n-m}[(c_{m+1}t)^n, (c_{m+2}t)^n, \cdots, (c_{n+1}t)^n]$ in a point; that is, $(qt)^k=(tt_1)(tt_2)\cdots(tt_k)$ is apolar to a member of the linear system $[(c_{m+1}t)^n, (c_{m+2}t)^n, \cdots, (c_{n+1}t)^n]$. As before, we have $n-k+1$ equations in $n-m+1$ λ's. For a solution to exist, the determinant must vanish for each set of $n-m+1$ equations. Thus again the matrix $[c_{m+1}c_{m+2}\cdots c_{n+1}]\equiv 0$. We note here that in either case the manifold represented by the matrix $(a_1a_2\cdots a_m)\equiv 0$, where the elements are binary n-ics, may likewise be represented by the matrix $[c_{m+1}c_{m+2}\cdots c_{n+1}]\equiv 0$, where the elements are linear in the coefficients of $(qt)^k$. Hence the formulas developed by Brill for the order of a manifold $(a_1a_2\cdots a_m)\equiv 0$ (cf. [2], pp. 391–395), may be more easily determined from the dual matrix $[c_{m+1}c_{m+2}\cdots c_{n+1}]\equiv 0$. For if we interpret the coefficients of $(qt)^k$ as variables, the latter becomes a matrix whose elements are linear forms. Such a matrix of i rows and j columns $(i\geqq j)$ is of order $C_{i,j-1}$ (cf. [3], p. 63).

We state without proof two preliminary theorems, the first of which is well known.

THEOREM 1. *If in a matrix of m rows and n columns $(m<n)$ every determinant of the mth order containing a certain set of i columns vanishes, while the i columns are not linearly dependent, then every determinant of the mth order in the matrix vanishes.*

THEOREM 2. *The determinants of a set S chosen as above from L_1, when divided by appropriate products and equated to zero, represent a set of hypersurfaces $M_{(s)}^{n-k+1}$ which intersect in a manifold Γ_s of dimension one.*

The truth of this theorem is implicitly assumed by Brill [2], though he does not use the geometric interpretation of this article. It may be checked by showing that the t's of a k-ad which represents a point on Γ_s are the roots of a k-ic apolar to a point n-ic in each of the S_{n-k}'s represented by the determinants of S; the $k-1$ conditions imposed on the coefficients of the k-ic

can be proved linearly independent. One part of Γ_s is the curve V_k. Another is the curve V_Δ whose points are represented by the k-ads making vanish all the determinants of highest order in L_1.

We now proceed to the fundamental theorems of this section.

THEOREM 3. *No part of V_k (except for a finite number of points) can appear multiply in any Γ_s.*

If a point is of multiplicity two or more for the intersection curve of $k-1$ surfaces in S_k, while it is an ordinary point for each surface, the tangent hyperplanes of the surfaces at that point must have at least an S_2 in common (cf. [4], p. 191); that is, at most only $k-2$ tangent hyperplanes can be linearly independent. It may then be readily proved that at least one surface in the linear system determined by the $k-1$ $M_{(s)}^{n-k+1}$ must have a double point here. However, no surface which is a linear combination of the $M_{(s)}^{n-k+1}$ can have a multiple point at any point determined from a non-singular n-ic of the pencil.

THEOREM 4. *There is no point on V_k common to the curves residual to V_k in all possible Γ_s's determined from every matrix L having the first two columns in common with L_1.*

We show that given V_k determined by the pencil $[(a_1t)^n, (a_2t)^n]$, and $(qt)^k=(tt_1)(tt_2)\cdots(tt_k)$, any k-ic factor of $(a_1t)^n$, we can find a matrix L such that the point $T(t_1, t_2, \cdots, t_k)$ is on Γ_s for every set $M_{(s)}^{n-k+1}$, but is on no curve residual to V_k.

If $k=2$, Γ_s and V_k are identical, so that the theorem is obvious. If $k=3$, L is of the form $(a_1a_2a_3a_4)$. Γ_s is given by the simultaneous vanishing either of $(a_1a_2a_3)$, $(a_1a_2a_4)$, or of the dual c-matrices, $[c_4c_5\cdots c_{n+1}]$, $[c_3c_5\cdots c_{n+1}]$. From the latter representation its order is found to be $(n-2)^2$. V_k is represented by $(a_1a_2)\equiv 0$, or $[c_3c_4\cdots c_{n+1}]\equiv 0$; V_Δ by $(a_1a_2a_3a_4)\equiv 0$, or $[c_5c_6\cdots c_{n+1}]\equiv 0$. Their orders are respectively $C_{n-1,n-3}$, $C_{n-2,n-4}$. Since the order of Γ_s is precisely the sum of those of V_k, V_Δ, our theorem is proved for this case if we can show that a matrix can be found for which T is not on V_Δ. For $k>3$ we determine the conditions to be satisfied if T is on a curve residual to V_k and V_Δ in Γ_s. Γ_s is given by the vanishing of $k-1$ a-matrices, or the $k-1$ dual c-matrices:

$$(2a)\qquad (a_1a_2a_3\cdots a_k), (a_1a_2a_4\cdots a_{k+1}), \cdots, (a_1a_2a_{k+1}\cdots a_{2k-2}) \equiv 0,$$

$$(2b)\qquad [c_{k+1}\cdots c_{n+1}], [c_{k+2}\cdots c_{n+1}c_3], \cdots, [c_{2k-1}\cdots c_{n+1}c_3\cdots c_k] \equiv 0.$$

It is made up of V_k, given by $[c_3c_4\cdots c_{n+1}]\equiv 0$, and of irreducible curves each of which is on at least one of the manifolds:

$$(3)\qquad [c_{k+2}\cdots c_{n+1}], [c_{k+3}\cdots c_{n+1}c_3], \cdots, [c_{2k-1}\cdots c_{n+1}c_3\cdots c_{k-1}] \equiv 0.$$

For consider an irreducible curve V' which is part of Γ_s, and for which $[c_{k+2} \cdots c_{n+1}] \not\equiv 0$. By Theorem 1, $[c_{k+1} \cdots c_{n+1} c_3] \equiv 0$ for V'. If none of (3) is satisfied, repeated applications of Theorem 1 prove that for the points of V', $[c_{k+1} \cdots c_{n+1} c_3 \cdots c_k] \equiv 0$; thus V' coincides with all or part of V_k. Since no part of V_k can figure multiply in Γ_s [cf. Theorem 3], a point T on a curve residual to V_k must therefore make vanish one of the matrices of (3), say $[c_{k+2} \cdots c_{n+1}]$. Let us further suppose that T is not on V_Δ, so that $[c_{2k-1} \cdots c_{n+1}] \not\equiv 0$. Then the polar with respect to $(qt)^k$ of a linear combination of $(c_{k+2}t)^n, \cdots, (c_{2k-2}t)^n$ is either identically zero or is the polar of a linear combination of $(c_{2k-1}t)^n, \cdots, (c_{n+1}t)^n$. This means that in S_n the S_{n-k+2} $[(tt_1)^n, \cdots, (tt_k)^n, (c_{2k-1}t)^n, \cdots, (c_{n+1}t)^n]$ meets the S_{k-4} $[(c_{k+2}t)^n, \cdots, (c_{2k-2}t)^n]$ in a point.

We now build a matrix L of the desired type. The hyperplane $(a_1t)^n$ intersects the norm-curve N^k in $t_1, \cdots, t_k$, and is intersected by $(a_2t)^n$ in an S_{n-2}. In this S_{n-2} we choose $n-2k+3$ points $(c_{2k-1}t)^n, \cdots, (c_{n+1}t)^n$, independent of each other and of $(tt_1)^n, \cdots, (tt_k)^n$. This insures that T is not on V_Δ, and if $k=3$, we may choose for $(c_3t)^n$, $(c_4t)^n$ any two further independent points in the S_{n-2}. If $k>3$, we proceed as follows: The points $(tt_1)^n, \cdots, (tt_k)^n$, $(c_{2k-1}t)^n, \cdots, (c_{n+1}t)^n$, determine an S_{n-k+2}. In the S_{n-2} of $(a_1t)^n$ we take an S_{2k-5} meeting the S_{n-k+2} in an S_{k-2}. In the S_{2k-5} but outside the S_{k-2} we choose $2k-4$ independent points $(c_3t)^n, \cdots, (c_{2k-2}t)^n$, such that the S_{k-4} determined by any $k-3$ will not intersect the S_{k-2}. Then T cannot be on any curve residual to V_k in a Γ_s. We take for $(c_1t)^n$, $(c_2t)^n$ any two further independent n-ics, and from these c's the set of a's (including a_1, a_2) may be determined.

THEOREM 5. *If a point is of multiplicity m for all Γ_s's determined from every matrix L, it is of multiplicity m for V_k.*

THEOREM 6. *If an n-ic of the pencil contains k pairs of double roots $t_1=t_2=s_1$, $t_3=t_4=s_2, \cdots, t_{2k-1}=t_{2k}=s_k$, then the point $(s_1, s_2, \cdots, s_k)$ is a double point for the surface M^{n-k+1}.*

Taking the particular n-ic for $(a_1t)^n$ and recalling the equation (1) for M^{n-k+1}, we see that the k lines of type $t_1=s_1$, $t_2=s_2, \cdots, t_{k-1}=s_{k-1}$ meet the surface here in two coincident points. If this is not a double point of the surface, the k lines must lie in a common tangent hyperplane. A point of such a line is $(ts_1)(ts_2) \cdots (ts_{k-1})(tt')$, where t' may vary. The k-ic giving the intersections of the common tangent hyperplane with the normcurve must be apolar to this k-ic and so may be written

$$(\eta t)^k \equiv \sum_{i=1}^{k-1} \lambda_i (ts_i)^k + \lambda_k (tt')^k.$$

But if $(\eta t)^k$ may also be written

$$(\eta t)^k \equiv \sum_{i=1}^{k-1} \mu_i (ts_i)^k + \mu_k (tt')^k,$$

we must have $\lambda_k = \mu_k = 0$, since there is no identity connecting kth powers of $k+1$ linear forms. Thus the hyperplane is a linear combination of the hyperplanes $(ts_1)^k, (ts_2)^k, \cdots, (ts_{k-1})^k$. Similarly it is a linear combination of any $k-1$ of the hyperplanes $(ts_1)^k, \cdots, (ts_k)^k$. Thus

$$\sum_{i=1}^{k-1} \lambda_i (ts_i)^k \equiv \sum_{j=2}^{k} \mu_j (ts_j)^k,$$

and $\lambda_1 = \mu_k = 0$. Likewise all the other λ's are zero. Hence no such hyperplane exists, and $(s_1, s_2, \cdots, s_k)$ is a double point of the surface.

THEOREM 7. *An S_{k-1} of the normcurve N^k meets the M^{n-k+1} in an $\overline{M}^{n-k+1}$ of similar type determined by a linear system Σ_{k-2} of $(n-1)$-ics related to the normcurve N^{k-1} in S_{k-1}.*

Let the hyperplane S_{k-1} have the parameter t_1 on N^k. It meets M^{n-k+1} in a locus of points with fixed coordinate t_1. The $(k-1)$-ads giving the other coordinates are obtained after factoring (tt_1) from a system $(\lambda_1, \cdots, \lambda_k)$, where $\sum_{i=1}^{k} \lambda_i (a_i t_1)^n = 0$. Due to this relation we have a linear system Σ_{k-2} determined by $k-1$ $(n-1)$-ics. Using also the theorem that the S_{k-1}'s of a normcurve N^k cut out on a fixed hyperplane of N^k the S_{k-2}'s of an N^{k-1} in the hyperplane, we may set up for $\overline{M}^{n-k+1}$ an equation similar to (1). By continuing this process, we arrive at the general theorem:

THEOREM 8. *The S_{k-r} $(r = 1, 2, \cdots, k-2)$ which is the intersection of r hyperplanes $t_1, t_2, \cdots, t_r$ of N^k meets the M^{n-k+1} in an $\overline{M}^{n-k+1}$ of similar type determined by a system Σ_{k-r-1} of $(n-r)$-ics related to the normcurve N^{k-r} in the S_{k-r}.*

THEOREM 9. *If the pencil contains an n-ic $(ts_1)(ts_1)(ts_2)(ts_2) \cdots (ts_j)(ts_j)(tt_1)(tt_2) \cdots (tt_{n-2j})$, then at any point P_{rl} $(s_1, s_2, \cdots, s_r, s_{r+1}, s_{r+1}, \cdots, s_{r+l}, s_{r+l}, t_1, t_2, \cdots, t_{k-r-2l})$, the space S_r determined by the intersection of hyperplanes $s_{r+1}, s_{r+1}, \cdots, s_{r+l}, s_{r+l}, t_1, t_2, \cdots, t_{k-r-2l}$ is tangent to M^{n-k+1} (the intersection of s_i with s_i is the S_{k-2} of N^k at s_i).*

For by Theorems 7 and 8 this S_r meets M^{n-k+1} in an $\overline{M}^{n-k+1}$ with a double point at P_{rl}. Hence any line in S_r through P_{rl} is tangent to M^{n-k+1} at that point.

THEOREM 10. *The point P_{rl} of Theorem 9 is a point of multiplicity 2^{r-1} at least for all Γ_s's determined from every matrix L.*

If $r=k$, the point is a double point for all M^{n-k+1}'s [cf. Theorem 6], and is therefore of multiplicity 2^{k-1} at least for the curve determined by any $k-1$ of the M^{n-k+1}'s. If $r<k$, P_{rl} must be a simple point for the general M^{n-k+1}. However, the tangent hyperplanes to the M^{n-k+1}'s at this point have a common S_r. Hence we may substitute for the original M^{n-k+1}'s of a set $M^{n-k+1}_{(s)}$ a system of $k-1$ independent M^{n-k+1}'s in which $r-1$ have double points at P_{rl}. Their intersection has therefore at P_{rl} a point of multiplicity 2^{r-1} at least ([4], p. 191).

THEOREM 11. *The point P_{rl} above described is a point of multiplicity exactly 2^{r-1} for V_k.*

That the multiplicity is at least 2^{r-1} is evident from Theorems 5 and 10. That V_k has no higher multiplicity at P_{rl} is shown by the fact that any hyperplane s_i $(i \leqq r)$ meets it here in only 2^{r-1} points.

We finally want to prove

THEOREM 12. *The restriction $k \leqq (n+3)/2$ imposed on the curve V_k in this section is valid for the generic curves $[W_p, S_p]$ of W_p.*

The curves meant are those cut out on W_p by $(p+1)$-secant S_p's of N^{2p-1} as defined in [1], §5. They are related to a pencil of $(2p-j)$-ics containing one member with $p-j-1$ double roots $(-2 \leqq j \leqq p-1)$. For $j \geqq 0$, they are involution curves determined by $(p-j)$-ics of the pencil (cf. [1], §§4, 6, (4), (7)); for $j=-2, -1$, they are projections of such curves (cf. [1], §6, (9)). Since $p-j \leqq (2p-j+3)/2$ for $j \geqq -3$, the restriction is satisfied. Furthermore, for $j \leqq 3$, the restriction is also satisfied for the involution curve determined from the p-ics of the pencil, which is a Cremona transform of the generic curve (cf. [1], §6, (11)).

5. **Proof of the identity** (3) in §3. The identity is obviously true for $j=0$. In the following proof we assume $n \geqq 2j$, $j \geqq 1$, $n \geqq k \geqq 1$. Wherever they occur, we take $C_{0,0}=1$ and $C_{a,b}=0$, if $a<b$. These conditions insure that the terms of the right-hand member involve true binomial coefficients and are restrictions obviously imposed by the geometric aspects of our problem. We have

$$\sum_{r=0}^{j}\sum_{l=0}^{j-r} C_{j,r}C_{j-r,l}C_{n-2j,k-r-2l}(kr-r-2l)2^{r-1}$$

$$(1)\qquad = j\sum_{r=0}^{j-1} 2^r \sum_{l=0}^{j-r-1} \frac{(j-1)!}{r!l!(j-r-l-1)!}\left[C_{n-2j,k-r-2l-1}(k-1) - C_{n-2j,k-r-2l-2}\right].$$

If in

$$\sum_{r=0}^{j-1} 2^r \sum_{l=0}^{j-r-1} \frac{(j-1)!}{r!l!(j-r-l-1)!} C_{n-2j,k-r-2l-1}$$

we put $j-1=j'$, $n-2=n'$, $k-1=k'$, we get

$$(2) \qquad \sum_{r=0}^{j'} 2^r \sum_{l=0}^{j'-r} \frac{j'!}{r!l!(j'-r-l)!} C_{n'-2j',k'-r-2l}.$$

Expanding and rearranging (2) in the form $\sum_{r=0}^{2j'} c_r C_{n'-2j',k'-r}$, we have

$$(3a) \qquad c_r = \sum_i 2^{r-2i} C_{j',r-i} C_{r-i,i}, \text{ for } r \leqq j',$$

where $i=0, 1, 2, \cdots, (r-1)/2$ or $r/2$, according as r is odd or even;

$$(3b) \qquad c_r = \sum_i 2^{2j'-r-2i} C_{j',2j'-r-i} C_{2j'-r-i,i}, \text{ for } r > j',$$

where $i=0, 1, 2, \cdots, (2j'-r-1)/2$ or $(2j'-r)/2$, according as r is odd or even. Since $2j'-r<j'$ if $r>j'$, both forms are of similar type. Using the identity $C_{j',r-i} C_{r-i,i} = C_{j',i} C_{j'-i,r-2i}$ (cf. [5], 2^0, p. 178), we transform (3a) into

$$(3a)' \qquad c_r = \sum_i 2^{r-2i} C_{j',i} C_{j'-i,r-2i}, \quad i = 0, 1, 2, \cdots, (r-1)/2, \text{ or } r/2.$$

We now prove that (3a)′ gives us the binomial coefficient $C_{2j',r}$. Let $2j'$ objects be divided into j' pairs. To a set of r taken, let i pairs ($i \leqq r/2$) contribute both their members, while $r-2i$ pairs contribute one member each. The i pairs may be chosen in $C_{j',i}$ ways, the $r-2i$ pairs in $C_{j'-i,r-2i}$ ways; the particular member from each of the latter in 2 ways. The total number of ways as i varies from 0 to $(r-1)/2$ or $r/2$ is the c_r given by (3a)′. Similarly the c_r of (3b) can be proved equal to $C_{2j',2j'-r} = C_{2j',r}$. Thus (2) becomes (cf. [5], 6^0, p. 178)

$$(4) \qquad \sum_{r=0}^{2j'} C_{2j',r} C_{n'-2j',k'-r} = C_{n',k'} = C_{n-2,k-1}.$$

By a similar process we prove

$$(5) \qquad \sum_{r=0}^{j-1} 2^r \sum_{l=0}^{j-r-1} \frac{(j-1)!}{r!l!(j-r-l-1)!} C_{n-2j,k-r-2l-2} = C_{n-2,k-2}.$$

Hence

$$(6) \qquad \sum_{r=0}^{j} \sum_{l=0}^{j-r} C_{j,r} C_{j-r,l} C_{n-2j,k-r-2l} (kr - r - 2l) 2^{r-1} = j[C_{n-2,k-1}(k-1) - C_{n-2,k-2}]$$

$$= j(n-k-1) C_{n-2,k-2},$$

which was to be proved.

6. **Special involution curves for $k=2$.** We want to know what situations arise when some or all of the j double roots in the special n-ic [§§2, 3, 4]

coincide, as may occur for the involution curves of W_p. Definite results can easily be obtained for $k=2$. In the special n-ic let $2k_1$, of the roots coincide in t_1, $2k_2$ in t_2, $\cdots$, $2k_m$ in t_m, while $n-2\sum_{i=1}^{m}k_i$ of the roots remain simple. If x_0 corresponds to the special n-ic, then as x, very near x_0, turns about x_0, the roots of the n-ic corresponding to x are permuted according to the permutation

$$(1)\quad S = (t_{11}t_{12}\cdots t_{1k_1})(t_{21}t_{22}\cdots t_{2k_2})\cdots(t_{m_1}t_{m_2}\cdots t_{mk_m})(\tau_1)(\tau_2)\cdots(\tau_{n-2\Sigma k_i}).$$

A corresponding permutation Σ takes place among the $C_{n,2}$ pairs of t's and τ's chosen from the roots of the n-ic. As x approaches x_0, a letter t_i chosen from the ith cycle approaches the point P_i of N^2, and τ_h approaches the point Q_h of N^2. Likewise a pair (t_it_i') chosen from the same cycle approaches the point π_i of V_2, a pair (t_it_l) from different cycles approaches the point π_{il} of V_2, a pair $(t_i\tau_h)$ approaches the point ρ_{ih} of V_2, and a pair $(\tau_h\tau_s)$ approaches the point σ_{hs} of V_2.

We first seek what contribution P_i makes to η_s due to permutations among the k-ads (t_it_l). Let d_{il} be the lowest common multiple of $2k_i$, $2k_l$. The permutation S^{2k_i} carries t_i around P_i. The permutation induced by this among the t's of the lth cycle is of period $d_{il}/2k_i$, and so consists of $4k_ik_l/d_{il}$ cycles of $d_{il}/2k_i$ letters each. Hence the pairs (t_it_l) on t_i are permuted likewise in $4k_ik_l/d_{il}$ cycles of $d_{il}/2k_i$ pairs each. The same argument is followed when we consider a point Q_h and pairs containing τ's. Hence

$$(2)\quad \eta_s = \sum_{i,l}[4k_ik_l/d_{il}][d_{il}(1/2k_i + 1/2k_l) - 2] + \left[n - 2\sum_i k_i\right]\left[\sum_i (2k_i - 1)\right],$$

where i, l take all integral values from 1 to m, and where $i<l$ in each term of the first summation. We next determine η_s'. No points π_{il}, ρ_{ih}, σ_{hs} of V_2 can be branch points, since the corresponding permutations are powers of S which leave fixed the t's and τ's involved in each case. If π_i is a branch point, the point (t_it_i') moving around it must reach its original position by an interchange of t_i, t_i'. The corresponding permutation among the letters of the ith cycle must therefore be of period two and consist of k_i cycles. As (t_it_i') moves around π_i, it passes successively through the points represented by these cycles. Hence each π_i counts once as a branch point and

$$(3)\quad \eta_s' = m.$$

Finally there are in the pencil of n-ics $2(n-1)-\sum_{i=1}^{m}(2k_i-1)$ members with a double root each. Their contribution is

$$\eta_0 - \eta_0' = \left[2(n-1) - \sum_{i=1}^{m} (2k_i - 1)\right][n-3]. \tag{4}$$

From (2), (3), and (4), we may easily determine p_2 by Zeuthen's formula.

In concrete cases it is of greater advantage to know at once the effect on V_2's genus of the points π_i, π_{il}, ρ_{ih}, σ_{hs}. Since a σ_{hs} is an ordinary simple point of V_2, and a ρ_{ih} is a simple point with a $2k_i$-point contact tangent τ_h, our attention is concentrated on points π_i and π_{il}. The effect of either type is independent of the roots of the n-ic not appearing in the pair determining the point. Thus π_i is best studied by considering a pencil of $2k_i$-ics which contains one member with a $2k_i$-fold root. By (2), (3), and (4), $\eta_s=0$, $\eta_s'=1$, $\eta_0-\eta_0'=(2k_i-1)(2k_i-3)$. Hence

$$\eta - \eta' = 4k_i^2 - 8k_i + 2, \tag{5}$$

and

$$p_2 = k_i^2 - 3k_i + 2. \tag{6}$$

Since $C_{2k_i-2,2}$ is the maximum value of the genus for curves of C_2's order we have proved the theorem:

THEOREM 1. *The point π_i introduced into V_2, if the pencil contains an n-ic with a $2k_i$-fold root, is equivalent to $(k_i-1)^2$ ordinary double points in the calculation of the genus.*

THEOREM 2. *If the pencil of n-ics contains a member with one $2k_i$-fold root, and another $2k_l$-fold root, the point π_{il} introduced into V_2 decreases the genus by the number $M=2k_ik_l-k_i-k_l+f(k_i, k_l)$, where $f(k_i, k_l)$ is the greatest common divisor of k_i and k_l. Since M may be written $C_{2k_l,2}+[2k_l(k_i-k_l)-k_i+f(k_i, k_l)]$, the effect of π_{il} on the genus is that of a $2k_l$-fold point into which is absorbed an extra number of double points equal to the number N in the brackets.*

Note the symmetry of N in k_l and k_i-k_l. To prove the theorem, we take a pencil of $2(k_i+k_l)$-ics containing one member $(ts_i)^{2k_i}(ts_l)^{2k_l}$. Substituting $2k_ik_l/f(k_i, k_l)$ for d_{il} in (2), and using also (3), (4), we have

$$\eta - \eta' = 4k_i^2 + 8k_ik_l + 4k_l^2 - 4k_i - 4k_l - 4f(k_i, k_l) - 2. \tag{7}$$

Zeuthen's formula gives us

$$p_2 = k_i^2 + 2k_ik_l + k_l^2 - 2k_i - 2k_l - f(k_i, k_l) + 1. \tag{8}$$

The maximum genus for a curve of V_2's order is $C_{2k_i+2k_l-2,2}$. Allowing for the reductions $(k_i-1)^2$, $(k_l-1)^2$ in p_2 due to π_i, π_l respectively, we get $2k_ik_l-k_i-k_l+f(k_i, k_l)$ as the reduction due to π_{il}.

We now construct several curves in order to give concrete examples of the

multiple points. Let us first consider the pencil $\lambda_0(tt_1)^8\lambda_1(\alpha t)^8$, where $t_1=0$, $(\alpha t)^8=\alpha_0 t^8+\alpha_1 t^7+\alpha_2 t^6+\alpha_3 t^5+\alpha_4 t^4+\alpha_5 t^3+\alpha_6 t^2+\alpha_7 t+\alpha_8$. The equation of V_2^7 is

$$\begin{vmatrix} t^8 & (\alpha t)^8 \\ t'^8 & (\alpha t')^8 \end{vmatrix} \cdot \frac{1}{t-t'} = 0. \tag{9}$$

If we put $x_0=tt'$, $x_1=t+t'$, $x_2=1$, the point (t_1t_1) on V_2^7 has coordinates $(0, 0, 1)$; the line t_1 has the equation $x_0=0$; and the equation of V_2^7, arranged in powers of x_2, becomes

$$\begin{aligned} &x_0^3x_2^3(\alpha_7x_0+4\alpha_8x_1) - x_0^2x_2^2(\alpha_5x_0^3+3\alpha_6x_0^2x_1+6\alpha_7x_0x_1^2+10\alpha_8x_1^3) \\ &+ x_0x_2(\alpha_3x_0^5+2\alpha_4x_0^4x_1+3\alpha_5x_0^3x_1^2+4\alpha_6x_0^2x_1^3+5\alpha_7x_0x_1^4+6\alpha_8x_1^5) \\ &- \alpha_1x_0^7-\alpha_2x_0^6x_1-\alpha_3x_0^5x_1^2-\alpha_4x_0^4x_1^3-\alpha_5x_0^3x_1^4-\alpha_6x_0^2x_1^5 \\ &- \alpha_7x_0x_1^6-\alpha_8x_1^7=0. \end{aligned} \tag{10}$$

The point (t_1t_1) is shown to be a fourfold point on V_2^7. The line t_1, counting three times as a tangent line, meets V_2^7 in seven points here. If we put $x_0=y$, $x_1=x$, $x_2=1$, and use Newton's method of approximation, we find that in the neighborhood of the origin (t_1t_1), the curve has three branches of type $y=Ax^2$ and one of type $y=Bx$. On a slight dislocation of the configuration every pair of the three former branches intersect in two points, while the fourth intersects the others in one point each. This accounts for the nine double points absorbed at (t_1t_1).

We next consider the pencil $\lambda_0(tt_1)^8(tt_9)^2+\lambda_1(\beta t)^{10}$, where $t_1=0$, $t_9=\infty$. If we again transform to homogeneous coordinates, the point (t_1t_9) has coordinates $(0, 1, 0)$; the equations of t_1, t_9 are $x_0=0$, $x_2=0$ respectively; and the equation of V_2^9, arranged in powers of x_1, becomes

$$\begin{aligned} &\beta_{10}x_1^7x_2^2+\beta_9x_0x_1^6x_2^2+x_1^5x_2^2(\beta_8x_0^2-6\beta_{10}x_0x_2)+x_1^4x_2^2(\beta_7x_0^3-5\beta_9x_0^2x_2) \\ &+ x_1^3x_2^2(\beta_6x_0^4-4\beta_8x_0^3x_2+10\beta_{10}x_0^2x_2^2)+x_1^2x_2^2(\beta_5x_0^5-3\beta_7x_0^4x_2+6\beta_9x_0^3x_2^2) \\ &- x_1(\beta_0x_0^8-\beta_4x_0^6x_2^2+2\beta_6x_0^5x_2^3-3\beta_8x_0^4x_2^4+4\beta_{10}x_0^3x_2^5) \\ &- \beta_1x_0^8x_2+\beta_3x_0^7x_2^2-\beta_5x_0^6x_2^3+\beta_7x_0^5x_2^4-\beta_9x_0^4x_2^5=0. \end{aligned} \tag{11}$$

The point (t_1t_9) is a double point; the two tangents coincide in t_9, which meets V_2^9 here in eight points. Putting $x_0=x$, $x_1=1$, $x_2=y$, we find by Newton's approximation that the two branches at the origin (t_1t_9) are of types $y=Ax^4$, $y=-Ax^4$, respectively. Each branch touches the tangent at a point of undulation. If the coefficients of V_2^9's equation were very slightly changed, the two branches would intersect in the four double points which are absorbed at (t_1t_9).

We finally take the pencil $\lambda_0(tt_1)^6(tt_7)^4+\lambda_1(\gamma t)^{10}$, where $t_1=0$, $t_7=\infty$. The

point (t_1t_7) has coordinates (0, 1, 0); the equations for t_1, t_7 are $x_0=0$, $x_2=0$ respectively, and $V_2{}^9$'s equation, arranged in powers of x_1, reads

$$
\begin{aligned}
&\gamma_{10}x_1{}^5x_2{}^4 + \gamma_9x_0x_1{}^4x_2{}^4 - x_0x_1{}^3(\gamma_0x_0{}^5 - \gamma_8x_0x_2{}^4 + 4\gamma_{10}x_2{}^5)\\
&- x_0{}^2x_1{}^2x_2(\gamma_1x_0{}^4 - \gamma_7x_0x_2{}^3 + 3\gamma_9x_2{}^4) + x_0{}^2x_1x_2(2\gamma_0x_0{}^5 - \gamma_2x_0{}^4x_2\\
&+ \gamma_6x_0{}^2x_2{}^3 - 2\gamma_8x_0x_2{}^4 + 3\gamma_{10}x_2{}^5) + \gamma_1x_0{}^7x_2{}^2 - \gamma_3x_0{}^6x_2{}^3 + \gamma_5x_0{}^5x_2{}^4\\
&- \gamma_7x_0{}^4x_2{}^5 + \gamma_9x_0{}^3x_2{}^6 = 0.
\end{aligned}
\tag{12}
$$

The point (t_1t_7) is a fourfold point on $V_2{}^9$, at which the four tangents coincide in t_7, meeting $V_2{}^9$ in six points here. If we put $x_0=x$, $x_1=1$, $x_2=y$, we find by Newton's method that the curve has at the origin (t_1t_7) two branches of types $y^2=Ax^3$, $y^2=-Ax^3$ respectively. If we allowed these two cusps to change into interlacing loops, we should bring into evidence the eight double points absorbed at (t_1t_7).

For higher values of k the same general method may be employed to determine η_s, η_s', but the application will naturally involve difficulties due to the greater complexities of these cases.

Bibliography

1. A. B. Coble and J. H. Chanler, *The geometry of the Weddle manifold W_p*, American Journal of Mathematics, vol. 57 (1935), pp. 183–218.

2. A. Brill. *Über Elimination aus einem gewissen System von Gleichungen*, Mathematische Annalen, vol. 5 (1872), pp. 378–396.

3. M. Pieri, *Sull'ordine della varietà generata da più sistemi lineari omografici*, Rendiconti del Circolo Matematico di Palermo, vol. 11 (1897), pp. 58–63.

4. E. Bertini, *Einführung in die projective Geometrie Mehrdimensionaler Räume*, 1924.

5. A. B. Coble, *Restricted systems of equations*, American Journal of Mathematics, vol. 36 (1914), pp. 167–186.

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RELATIVE LINEAR SETS AND SIMILARITY OF MATRICES WHOSE ELEMENTS BELONG TO A DIVISION ALGEBRA*

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It is the purpose of this paper to develop the theory of the similarity transformation for matrices whose elements belong to a division algebra. In order to get a basis for generalization, the theory of the similarity transformation for matrices whose elements belong to a field is sketched in what seems to the authors a more suggestive method than those used heretofore.‡

L. A. Wolf's paper entitled *Similarity of matrices in which the elements are real quaternions*§ treats the case of the quaternion algebra over any subfield of the real field, by passing to the equivalent square matrices of order $2n$ with elements in the subfield.

Some of the results of the present paper are given in an abstract and a subsequent paper by N. Jacobson.‖ Jacobson's results are to a certain extent more general. In the present paper a usable rational process is given for determining the equivalence or non-equivalence of matrices whose elements belong to a division algebra, and a theorem is developed concerning the rank of a polynomial in a matrix, which is not indicated by Jacobson. Some of the results contained herein were given by Ingraham at the summer meeting of the Society in 1935.¶

I. The similarity transformation for the commutative case

1. **Review of certain known theory.** If M is an $m \times n$ matrix (i.e., a matrix with m rows and n columns) with elements in a field F, the rank of M is

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‡ For somewhat similar treatment see F. Gantmacher, *La théorie géométrique des diviseurs élémentaires d'après M. Krull*, Transactions, University of Odessa, Mathematics 1, 1935, pp. 89–108 (Russian) avec Franz, resumé.

§ L. A. Wolf, Bulletin of the American Mathematical Society, vol. 42 (1936), pp. 737–743.

‖ N. Jacobson, *On pseudo-linear transformations*, Proceedings of the National Academy of Sciences, vol. 21 (1935), pp. 667–670. N. Jacobson, *Pseudo-linear transformations*, Annals of Mathematics, vol. 38 (1937), p. 485.

¶ M. H. Ingraham, *Characteristic spaces associated with the matrix whose elements belong to a division algebra*, Bulletin of the American Mathematical Society, vol. 41 (1935), p. 622.

equal to the maximum number of linearly independent rows and (which is the same) the maximum number of linearly independent columns. Also the rank of M is $n-s$ where s is the maximum number of linearly independent vectors ξ satisfying the equation $M\xi=0$. The equivalence of the two definitions of rank is essentially the theorem that the order of the linear extension of the columns of M plus the order of the maximal set right orthogonal to M is equal to the number of columns of M.

The set U of all vectors ξ such that $M\xi=0$ is a linear set over F. ξ and U are both said to be *right orthogonal* to M. Let $L(\xi_1, \xi_2, \cdots, \xi_n)$ denote the least linear set containing $\xi_1, \xi_2, \cdots, \xi_n$, i.e., the totality of all vectors of the form $\sum \xi_i a_i$ where the a_i are in the field F. $L(\xi_1, \xi_2, \cdots, \xi_n)$ is said to be the *linear extension* of $\xi_1, \xi_2, \cdots, \xi_n$.

2. **Definition of relative linear independence, relative linear extensions, and connection with similarity transformation.** Let M be an $n\times n$ square matrix with elements in a field F. A set of vectors $\xi_1, \xi_2, \cdots, \xi_k$ is defined to be *linearly independent relative to* M if no one of the ξ_i is the zero vector and if whenever a set of polynomials $h_1, h_2, \cdots, h_k$ with coefficients in F exists such that $\sum_{i=1}^{k} h_i(M)\xi_i=0$, it implies $h_i(M)\xi_i=0$, (i). A set U of vectors is said to be *linear relative to* M if (1) whenever ξ_1 is in U, $h(M)\xi$ is in U for every polynomial h in F; (2) if ξ_1 and ξ_2 are in U, then $\xi_1+\xi_2$ is in U. Denote by $L_M(\xi_1, \xi_2, \cdots, \xi_k)$ the *linear extension of* $\xi_1, \xi_2, \cdots, \xi_k$ *relative to* M, i.e., the least set U containing $\xi_1, \xi_2, \cdots, \xi_k$ which is linear relative to M. It is the totality of vectors of the form $\sum_{i=1}^{k} h_i(M)\xi_i$, where the h_i are polynomials in F.

It should be noted that if $M=I$, these definitions reduce to the usual ones for linear independence, linear sets, and linear extensions.

The polynomial h and the vector ξ are said to be *associated relative to the matrix* M if $h(M)\xi=0$. As in the ordinary theory of the minimum equation of a matrix there is associated with every vector ξ relative to M a unique polynomial h of lowest degree with leading coefficient unity. This polynomial divides all others associated with the vector ξ and is said to be *minimally associated* with the vector ξ *relative to* M. Clearly, the minimum polynomial such that $h(M)=0$ is the least common multiple of all those associated with the various vectors ξ of the $n\times n$ vector space with elements in F.

Consider two similar matrices M and N. Then there exists a non-singular matrix T such that $T^{-1}MT=N$. Let $\xi_1, \xi_2, \cdots, \xi_k$ be linearly independent relative to M, and if h_i is minimally associated with ξ_i relative to M, then the vectors $\zeta_i=T^{-1}\xi_i$ are linearly independent relative to N, and h_i is the minimum polynomial associated with ζ_i relative to N.

Conversely, if M and N are two $n\times n$ matrices, if $\xi_1, \xi_2, \cdots, \xi_k$ is a set of vectors linearly independent relative to M, if $\zeta_1, \zeta_2, \cdots, \zeta_k$ is a set of vectors linearly independent relative to N such that $L_N(\zeta_1, \zeta_2, \cdots, \zeta_k) = L_M(\xi_1, \xi_2, \cdots, \xi_k)$ is the total vector space, and if $h_1, h_2, \cdots, h_k$ are minimally associated with $\xi_1, \xi_2, \cdots, \xi_k$ respectively relative to M and are also minimally associated with $\zeta_1, \zeta_2, \cdots, \zeta_k$ respectively relative to N, then M is similar to N. If $P=(\xi_1, M\xi_1, \cdots, M^{m_1-1}\xi_1, \xi_2, M\xi_2, \cdots, M^{m_2-1}\xi_2, \cdots, \xi_k, M\xi_k, \cdots, M^{m_k-1}\xi_k)$, and

$$Q = (\zeta_1, N\zeta_1, \cdots, N^{m_1-1}\zeta_1, \zeta_2, N\zeta_2, \cdots, N^{m_2-1}\zeta_2, \cdots, \zeta_k, N\zeta_k, \cdots, N^{m_k-1}\zeta_k), \tag{1}$$

where m_i is the degree of h_i, then P and Q are non-singular. If $T=PQ^{-1}$ then $T^{-1}MT=N$.

3. **Theory of relative linear independence and extension. Invariants of the similarity transformation.** If U is a linear set relative to M, and if $\xi_1-\xi_2$ is in U, then it is said that $\xi_1\equiv\xi_2 \bmod U$. The usual processes associated with congruences pertain to this definition. Moreover, given the vector ξ and U, there exists a minimum polynomial h with leading coefficient unity such that $h(M)\xi\equiv 0 \bmod U$ and this polynomial divides all other polynomials satisfying this congruence. The polynomial h is said to be *minimally associated* with the vector $\xi \bmod U$ *relative to* M. If h is associated with $\xi \bmod U$, then h is associated with $\xi \bmod U_1$ for all relative linear sets U_1 containing U.

The following five theorems are given for two vectors but clearly, by iteration, lead to obvious generalizations for more than two vectors. In all of these the equalities may be replaced by congruences with a relative linear set as modulus.

THEOREM 1. *If h_1 and h_2 are minimally associated with the vectors ξ_1 and ξ_2 respectively relative to M, and if h_1 and h_2 are relatively prime, then the vectors ξ_1 and ξ_2 are relatively linearly independent with respect to the matrix M.*

Let $g_1(M)\xi_1=g_2(M)\xi_2=\xi_3$. The polynomial minimally associated with ξ_3 must be a divisor of both h_1 and h_2 and is therefore 1. Hence if $g_1(M)\xi_1 = g_2(M)\xi_2$, then $g_1(M)\xi_1=g_2(M)\xi_2=0$, and the theorem is proved.

THEOREM 2. *If h_1 and h_2 are two relatively prime polynomials whose product $h=h_1h_2$ is minimally associated with the vector ξ relative to M, and if $\xi_1=h_2(M)\xi$ and $\xi_2=h_1(M)\xi$, then ξ_1 and ξ_2 are relatively linearly independent with respect to M and $L_M(\xi_1, \xi_2)=L_M(\xi)$. Moreover, the polynomials h_1 and h_2 are minimally associated with the vectors ξ_1 and ξ_2 respectively relative to M.*

By Theorem 1 these vectors, ξ_1 and ξ_2, are relatively linearly independent with respect to M. Let $p_1h_1+p_2h_2=1$. Hence

$$\xi = p_2(M)h_2(M)\xi + p_1(M)h_1(M)\xi = p_2(M)\xi_1 + p_1(M)\xi_2.$$

Hence $L_M(\xi_1, \xi_2)$ contains $L_M(\xi)$. Obviously $L_M(\xi)$ contains $L_M(\xi_1, \xi_2)$, and therefore $L_M(\xi) = L_M(\xi_1, \xi_2)$.

By a similar method one can prove

THEOREM 3. *If h_1 and h_2 are relatively prime polynomials and are minimally associated with ξ_1 and ξ_2 respectively relative to M, then the polynomial minimally associated with $\xi_1+\xi_2$ relative to M is h_1h_2; moreover $L_M(\xi_1+\xi_2) = L_M(\xi_1, \xi_2)$.*

Suppose h_1 and h_2 are any two polynomials. One may express $h_1 = \prod p_i^{s_i}\prod q_i^{t_i}$ and $h_2 = \prod p_i^{v_i}\prod q_i^{u_i}$ where the p_i's and q_i's are polynomials which are relatively prime, pair by pair, and where $s_i \geqq v_i$ while $t_i < u_i$. If h_1 and h_2 are minimally associated with the vectors ξ_1 and ξ_2, then by Theorem 2 vectors ξ_3 and ξ_4 exist with which are minimally associated the polynomials $g_1 = \prod p_i^{s_i}$ and $g_2 = \prod q_i^{u_i}$ respectively, and by Theorem 3 the polynomial minimally associated with $\xi_3+\xi_4$ is g_1g_2, the least common multiple of h_1 and h_2. One may therefore state

THEOREM 4. *If h_1 and h_2 are minimally associated with the vectors ξ_1 and ξ_2 respectively relative to M, then there exists a vector ξ contained in $L_M(\xi_1, \xi_2)$ such that the least common multiple of h_1 and h_2 is the polynomial minimally associated with ξ relative to M.*

THEOREM 5. *If h_1 and h_2 are minimally associated with the vectors ξ_1 and ξ_2 respectively relative to M, if $h_1 = p_1h_2$, and if h_3 is minimally associated with $\xi_2 \bmod L_M(\xi_1)$ relative to M so that $h_3(M)\xi_2 = h_4(M)\xi_1$, then p_2 and p_3 exist such that $h_2 = p_2h_3$ and $h_4 = p_3h_3$; furthermore the vectors ξ_1 and $\xi_3 = \xi_2 - p_3(M)\xi_1$ are linearly independent relative to M, and h_3 is minimally associated with both ξ_3 and $\xi_3 \bmod L_M(\xi_1)$ relative to M.*

Since $h_2(M)\xi_2 = 0$, it follows that $h_2(M)\xi_2 \equiv 0 \bmod L_M(\xi_1)$. Hence p_2 exists such that $h_2 = p_2h_3$. Since $h_3(M)\xi_2 = h_4(M)\xi_1$, $0 = h_2(M)\xi_2 = p_2(M)h_3(M)\xi_2 = p_2(M)h_4(M)\xi_1$, and hence p_2h_4 must be divisible by h_1. Moreover $h_1 = p_1h_2 = p_1p_2h_3$, hence h_4 is divisible by p_1h_3 and hence by h_3, and there exists a p_3 such that $h_4 = p_3h_3$. Hence $h_3(M)\xi_2 = p_3(M)h_3(M)\xi_1$, and so if $\xi_3 = \xi_2 - p_3(M)\xi_1$, then $h_3(M)\xi_3 = 0$. Suppose a polynomial $h_5 \neq 0$ of lower degree than h_3 existed such that $h_5(M)\xi_3 = 0$, then $h_5(M)\xi_2 = h_5(M)p_3(M)\xi_1$, that is, $h_5(M)\xi_2 \equiv 0 \bmod L_M(\xi_1)$ where h_5 is of lower degree than h_3. Since this is impossible, h_3 is minimally associated with ξ_3 and with $\xi_3 \bmod L_M(\xi_1)$. From

this it can be proved at once that ξ_1 and ξ_3 are linearly independent relative to M.

From the discussion in the preceding part of this section it follows that given any set of vectors $(\zeta_1, \zeta_2, \cdots, \zeta_k)$, a set $(\xi_1, \xi_2, \cdots, \xi_m)$ may be found such that $\xi_1, \xi_2, \cdots, \xi_m$ are linearly independent relative to M and such that $L_M(\xi_1, \xi_2, \cdots, \xi_m) = L_M(\zeta_1, \zeta_2, \cdots, \zeta_k)$. Moreover by repeated use of Theorems 2 and 5 and their extensions to more than two vectors and to congruences this may be done in such a way that the polynomials h_i minimally associated with the ξ_i may be taken as powers of irreducible polynomials, or by repeated use of Theorems 4 and 5 and their extensions, the h_i may be chosen such that h_{i+1} divides h_i. The only limitations on the h_i are given by the following theorem.

THEOREM 6. *If $\xi_{11}, \xi_{12}, \cdots, \xi_{1s}$ and $\xi_{21}, \xi_{22}, \cdots, \xi_{2t}$ are two sets of vectors each of which is linearly independent relative to M such that $L_M(\xi_{11}, \xi_{12}, \cdots, \xi_{1s}) = L_M(\xi_{21}, \xi_{22}, \cdots, \xi_{2t})$, and if h_{1i} and h_{2i} are minimally associated with ξ_{1i} and ξ_{2i} respectively relative to M, and if h is an irreducible polynomial, and k any positive integer, then the number of the polynomials h_{1i} divisible by h^k is equal to the number of the polynomials h_{2i} divisible by h^k.*

Let $S_1 = L_M(\xi_{11}, \xi_{12}, \cdots, \xi_{1s})$ and $S_2 = L_M(\xi_{21}, \xi_{22}, \cdots, \xi_{2t})$. Since $S_1 = S_2$, the numbers of linearly independent vectors in S_1 and S_2 respectively which are orthogonal to $h(M)$ are equal. The number of linearly independent vectors of S_1 orthogonal to $h(M)$ is equal to the sum of the numbers of such vectors in the sets $L_M(\xi_{1i})$. If $h_{1i} = p_{1i}h$, then the m vectors $p_{1i}(M)\xi_{1i}$, $Mp_{1i}(M)\xi_{1i}, \cdots, M^{m-1}p_{1i}(M)\xi_{1i}$, where m is the degree of h, are linearly independent and form a base for the set of vectors orthogonal to $h(M)$ in $L_M(\xi_{1i})$. If h_{1i} is not divisible by h, then no vector of $L_M(\xi_{1i})$ other than the zero vector is orthogonal to $h(M)$. Hence the number of polynomials h_{1i} or h_{2i} divisible by h is n_1/m, where n_1 is the order of the set of vectors in S_1 (equal to the order in S_2) orthogonal to $h(M)$. This proves the theorem for $k = 1$.

Let n_2 be the number of linearly independent vectors ξ in $L_M(h(M)\xi_{11}, \cdots, h(M)\xi_{1s}) = L_M(h(M)\xi_{21}, \cdots, h(M)\xi_{2t})$ which satisfy the equality $h(M)\xi = 0$, then by reasoning as above, n_2/m is the number of polynomials h_{1i} or the number of polynomials h_{2i} which are divisible by h^2, etc.

As an immediate consequence follows

THEOREM 7. *If $\xi_1, \xi_2, \cdots, \xi_k$ are linearly independent relative to M such that $L_M(\xi_1, \xi_2, \cdots, \xi_k)$ is the total vector space, and if the polynomial h_i minimally associated with ξ_i relative to M is a power of an irreducible polynomial, then these h_i are the same for all sets of vectors satisfying these conditions.*

The polynomials of such a set are called the *characteristic divisors* of $M-\lambda I$, and if the irreducible polynomials are linear, the characteristic divisors of $M-\lambda I$ are identical with the elementary divisors as usually defined.

The invariance of the invariant factors also follows at once. The Dickson rational canonical form is a matrix N such that the h_i are the invariant factors when $\zeta_1, \zeta_2, \cdots, \zeta_k$ in Q of equation (1) are $\delta_1, \delta_{m_1+1}, \delta_{m_1+m_2+1}, \cdots, \delta_{m_1+m_2+\cdots+m_{k-1}+1}$, where the vector δ_j is the Kronecker delta vector, 1 in the jth place and zero elsewhere, and m_i is the degree of h_i. It is also easy to characterize similarly the Jordan canonical form based on elementary divisors, as well as forms based on the highest power of irreducible factors occurring in the invariant factors, namely, the characteristic divisors.

The above treatment has two distinct advantages. First, starting with any n linearly independent vectors, a set of vectors $\xi_1, \xi_2, \cdots, \xi_k$ linearly independent relative to M, such that $L_M(\xi_1, \xi_2, \cdots, \xi_k)$ is the total vector space, may be found by rational means, usually without great effort. Secondly, all of the canonical forms are derived by the same theoretical development.

II. Outline of the problem for the case where the elements of the matrix belong to a quasi-field A

The chief reason that the extension of the above theory to the case where the elements of M belong to a quasi-field instead of a field is not trivial is that the rank of a polynomial in M is not invariant under the similarity transformation. Moreover, one cannot say that if $g(M)\xi=0$ then $g(T^{-1}MT)T^{-1}\xi=0$.

In order to obviate these difficulties, it was found that a new operation instead of ordinary multiplication should be defined. Given the polynomial $g(\lambda)=\sum\lambda^i a_i$, the matrix M, and the vector ξ, we define $g(M)\odot\xi$ to be the vector $\sum M^i\xi a_i$. In the case that all the elements involved belong to a field, $g(M)\odot\xi=g(M)\xi$. From this definition it follows that if $g(M)\odot\xi=0$, $g(T^{-1}MT)\odot T^{-1}\xi=0$. Whenever $g(M)\odot\xi=0$, ξ is said to be $\odot$-orthogonal (tentatively read as dot-orthogonal) to $g(M)$. It is seen that the order of the set of vectors $\odot$-orthogonal to $g(M)$ is an invariant of the similarity transformation. It does not follow if $g(M)\odot\xi=0$ that $g(M)\odot\xi a=0$, if a is a non-commutative element of A. Secondly, the operation $\odot$ cannot be performed upon ξ with only a knowledge of the matrix $g(M)$ but requires that one knows both M and the polynomial g. Moreover, if $g_1=\sum\lambda^i a_{1i}$, $g_2=\sum\lambda^i a_{2i}$, $g_1\odot g_2$ is defined to be $\sum_i\lambda^i g_2 a_{1i}=\sum_k\lambda^k\sum_{i+j=k}a_{2i}a_{1j}$. This last definition is nothing but the usual one for the product g_2g_1 of two polynomials in a commutative indeterminate with coefficients in A. If $g=g_1\odot g_2$, then g_2 is said to be an interior factor of g, and g_1 an exterior factor of g.

Dickson* has shown the existence of polynomials p_1 and p_2 such that $p_1 \odot g_1 + p_2 \odot g_2$ is the greatest common interior divisor, (g_1, g_2), of g_1 and g_2, and Ore† has proved the existence of the least common exterior multiple, $[g_1, g_2]$, of g_1 and g_2. The degree of $[g_1, g_2]$ is equal to the sum of the degrees of g_1 and g_2 less the degree of (g_1, g_2).

In a quasi-field A the total set of elements each of which is commutative with every other element of A is a field containing 0 and 1 and is called the *centrum* C.

THEOREM 8. *If f_1 and f_2 are two polynomials such that $f_1 \odot f_2 = h$, where h is a polynomial with coefficients in the centrum, then $f_1 \odot f_2 = h = f_2 \odot f_1$.*

Let $h = f_1 \odot f_2$, then $f_1 \odot f_2 \odot f_1 = h_1 \odot f_1 = f_1 \odot h$. Hence from the uniqueness of division, $h = f_2 \odot f_1$.

In terms of the $\odot$-process relative linear independence and the extension process L_M may readily be defined and it is not difficult to see that the theory of similarity transformation may be made to depend on the existence of sets of vectors $\xi_1, \xi_2, \cdots, \xi_k$ and $\eta_1, \eta_2, \cdots, \eta_k$ such that $L_M(\xi_1, \xi_2, \cdots, \xi_k) = L_N(\eta_1, \eta_2, \cdots, \eta_k)$ is the whole vector space, and such that both sets of vectors are minimally associated with the same set of polynomials, as in the last section.

III. VECTORS ORTHOGONAL TO POLYNOMIALS IN A MATRIX WITH ELEMENTS IN A FINITE DIVISION ALGEBRA A

A quasi-field A is a division algebra over the centrum C if there exists a finite number m of elements α_i in A such that every element a of A can be expressed as

$$a = \sum_{i=1}^{m} \alpha_i c_i,$$

where the coefficients c_i are in C.

If M is an $n \times n$ matrix and $\xi_1, \xi_2, \cdots$ a set of vectors with elements in a division algebra, the existence of a polynomial g such that $g(M) \odot \xi_i = 0$ can readily be established from the fact that not more than n of the vectors $M^j \xi_i$ $(j = 1, 2, \cdots)$ are right linearly independent. If $g(M) \odot \xi_i = 0$, then g and ξ_i are said to be *associated relative to* M. There is a unique polynomial g_i of lowest degree with leading coefficient unity associated with ξ_i. This g_i is an

* L. E. Dickson, *Algebren und ihre Zahlentheorie*, Zurich, 1927, p. 256. The product $g_1 \odot g_2$ of this paper should be interpreted as $g_2 g_1$ of Dickson and interior factor as left-hand factor, etc.

† O. Ore, *Theory of non commutative polynomials*, Annals of Mathematics, (2), vol. 34 (1933), pp. 480–508.

interior factor of every other polynomial g associated with the vector ξ_i and it is said to be *minimally associated with ξ_i relative to M.*

A set of vectors is said to be *right linear* if for every a_1 and a_2 in A $\xi_1 a_1 + \xi_2 a_2$ is in the set whenever ξ_1 and ξ_2 are in the set. In particular, if a_1 and a_2 are in the centrum C, the set of vectors is said to be *linear relative to the centrum.* It follows that

THEOREM 9. *The maximal set of vectors S associated with the polynomial g relative to M is a linear set relative to the centrum.*

If $g = \sum \lambda^i a_i$ and a is any element of A, let $g_a = \sum \lambda^i a^{-1} a_i a$, then g_a will be called the transform of g by a.* In most applications a will be one of the basal elements α_i of A relative to its centrum. Clearly the transform of a $\odot$-product of two polynomials is the $\odot$-product of their transforms. Moreover, if ξ is associated with g, then ξa is associated with g_a, and if g is minimally associated with ξ, then g_a is minimally associated with ξa.

THEOREM 10. *If h is a polynomial with coefficients in the centrum, the maximal space associated with h relative to M is a right linear space; conversely, if S is any right linear space, the polynomial minimally associated with the entire space S relative to M has coefficients in the centrum.*

To prove the converse, let a be any element of A different from zero. $Sa = S$. Let $g = \sum \lambda^i a_i$ be the polynomial of lowest degree associated with every vector of S, hence $g(M) \odot S = 0$. Then $g_a(M) \odot S = g_a(M) \odot (Sa) = 0$. The degrees of g_a and g are equal, and the leading coefficient of each is unity. As the polynomial minimally associated with S is unique, $g_a = g$ and $a^{-1} a_i a = a_i$, (i). Hence every a_i is in the centrum C.

THEOREM 11. *If g is any polynomial with leading coefficient unity, there exists a matrix M and a vector ξ such that g is minimally associated with ξ relative to M.*

If

$$g = \lambda^k - \sum_{i=0}^{k-1} \lambda^i a_i,$$

consider the matrix

$$M = \begin{pmatrix} 0 & 0 & 0 \cdots 0 & a_0 \\ 1 & 0 & 0 \cdots 0 & a_1 \\ 0 & 1 & 0 \cdots 0 & a_2 \\ \cdot & \cdot & \cdot \cdots \cdot & \cdot \\ 0 & 0 & 0 \cdots 1 & a_{k-1} \end{pmatrix} = (\delta_2, \delta_3, \cdots, \delta_k, \eta),$$

* This transform g_a is a special case of the transforms of a polynomial g by a second polynomial as used by Ore. In this paper, however, the only polynomials used for transforming others are constants.

where δ_i is the Kronecker δ and η is the kth column of M. Then δ_1 is effective as ξ.

THEOREM 12. *If g is any polynomial in the division algebra A, then the least common exterior multiple of g and its transforms by the basal elements α_i of A, i.e., $[g, g_{\alpha_2}, \cdots, g_{\alpha_m}]$, is a polynomial with coefficients in the centrum.*

From the preceding theorem there exists a matrix M and a vector ξ such that g is minimally associated with ξ relative to M. Any element in U, the right linear extension of ξ, is of the form $\xi(\sum \alpha_i c_i)$, where the c_i are in the centrum. Since g_{α_i} is minimally associated with $\xi\alpha_i$, $h=[g, g_{\alpha_2}, \cdots, g_{\alpha_m}]$ is minimally associated with U, and hence by Theorem 10, h has coefficients in the centrum.

If $h=[g, g_{\alpha_2}, \cdots, g_{\alpha_m}]$, then h is the polynomial of lowest degree with coefficients in the centrum which has g as an interior factor, and g is said to *define* h. If h is the polynomial of lowest degree with coefficients in the centrum which is associated with ξ relative to a matrix M, then h is said to be minimally associated with ξ over C relative to M. It follows that if h_1 is a polynomial in C and is associated with ξ and if h is minimally associated with ξ over C, then h is a factor of h_1. Also, if any polynomial g is a factor of h_1, and if g defines h, then h is a factor of h_1.

THEOREM 13. *If g is minimally associated with ξ relative to M, and if g defines h, then h is minimally associated with ξ over the centrum relative to M.*

THEOREM 14. *If g is an irreducible polynomial in A, then*

$$h=[g, g_{\alpha_2}, \cdots, g_{\alpha_m}]$$

is a polynomial irreducible in the centrum C.

Suppose $h=h_1 \odot h_2$, where h_1 and h_2 are in C, then g divides either h_1 or h_2 for if this were not the case, polynomials p_1, q_1, p_2, q_2 exist such that $p_1 \odot h_1=1+q_1 \odot g$ and $p_2 \odot h_2=1+q_2 \odot g$, and hence

$$p_1 \odot h_1 \odot p_2 \odot h_2 = p_1 \odot p_2 \odot h_1 \odot h_2 = 1 + (q_1 + q_2 + q_1 \odot g \odot q_2) \odot g$$

which would necessitate g and $h_1 \odot h_2$ being relatively prime, contrary to hypothesis.

THEOREM 15. *If g defines h but no interior factor of g of lower degree than the degree of g defines h, and if h is irreducible in C, then g is irreducible in A.*

THEOREM 16. *If g_1 defines h_1, if g_2 defines h_2, and if $g_1 \odot g_2$ defines h, then h is a factor of $h_1 \odot h_2$ and $[h_1, h_2]$ is a factor of h.*

Let $h_1=f_1 \odot g_1$ and $h_2=f_2 \odot g_2$. By Theorem 8

$$h_1 \odot h_2 = f_1 \odot g_1 \odot f_2 \odot g_2 = f_2 \odot f_1 \odot g_1 \odot g_2,$$

and hence h is a factor of $h_1 \odot h_2$. If $h = f \odot g_1 \odot g_2$, then also from Theorem 8, $[h_1, h_2]$ is a factor of h.

The following two theorems are corollaries of the above theorem.

THEOREM 17. *If $g = g_m \odot g_{m-1} \odot \cdots \odot g_1$ where the g_i are irreducible, and if $h_1, h_2, \cdots, h_t$ are distinct polynomials in C where s_j of the g_i each define h_j, $s_j > 0$, and if g defines h, then h divides $\prod h_j^{s_j}$ and is divisible by $\prod h_j$.*

THEOREM 18. *If g_1 defines h_1 and g_2 defines h_2, and if $(h_1, h_2) = 1$, then $g_1 \odot g_2$ defines $h_1 \odot h_2$.*

THEOREM 19. *If g_1 defines h_1, if h_2 divides h_1, and if $g_2 = (g_1, h_2)$, then g_2 defines h_2.*

Suppose that g_2 defines h_3. By hypothesis g_2 is an interior factor of h_2, hence h_3 is a factor of h_2. Let $h_2 = h_4 \odot h_3$, $h_1 = h_5 \odot h_2 = h_5 \odot h_4 \odot h_3$, and $g_1 = g_3 \odot g_2$. From Theorem 16, g_3 must define a multiple of $h_5 \odot h_4$, call it $h_6 \odot h_5 \odot h_4$. Let $h_6 \odot h_5 \odot h_4 = g_4 \odot g_3$. If h_4 is of degree greater than zero, g_3 and h_4 must have a common interior divisor other than 1, for if $(g_3, h_4) = 1$ then there exist polynomials p_1 and p_2 such that $p_1 \odot g_3 + p_2 \odot h_4 = 1$. Consequently, g_3 is an interior divisor of $h_6 \odot h_5$ which is impossible. Let g_5 be the common factor of g_3 and h_4. From this it follows that $g_5 \odot g_2$ is an interior factor of g_1 and of h_2, contrary to hypothesis.

If in Theorems 1–4 of parts I and II it is understood that all coefficients of the polynomials h_i are in the centrum, these theorems apply to the case in which the basic number system is a division algebra rather than a field. These same modifications apply to the proofs of these theorems except that the polynomials g_1 and g_2 in the proof of Theorem 1 may not have coefficients in the centrum. Furthermore, the polynomial h_i minimally associated with ξ_i relative to a matrix M is interpreted to be the polynomial with coefficients in the centrum minimally associated with ξ_i relative to M throughout these theorems.

THEOREM 20. *If g, any polynomial in A, defines h, and if h is associated with ξ over the centrum relative to M, then there exists a set of vectors ξ_i, where ξ_i is associated with g_{α_i}, such that $\xi = \sum_{i=1}^{m} \xi_i$.*

Let $h = f_{\alpha_i} \odot g_{\alpha_i}$. The polynomials $f, f_{\alpha_2}, \cdots, f_{\alpha_m}$ have no common exterior factor other than 1, for if there were such, let it be f_1. Then $h = f_1 \odot h_1$ where h_1 is a common multiple of g and its transforms and is of lower degree than h. Hence there exist polynomials p_i such that $\sum_i f_{\alpha_i} \odot p_i = 1$. Let $\xi_i = f_{\alpha_i}(M) \odot p_i(M) \odot \xi$ and the theorem follows at once. Moreover, $\xi_i \alpha_i^{-1}$ is associated with g.

THEOREM 21. *If U is a relative linear set and if h is minimally associated with a vector ξ* mod *U over the centrum C relative to M, and if g defines h, but no interior factor of g of lower degree than the degree of g defines h, then there exists a vector η in $L_M(\xi)$ such that*: (1) *g is minimally associated with η* mod *U, and* (2) *there exists a polynomial p such that $g(M) \odot \eta = p(M) \odot h(M) \odot \xi$.*

Consider $\xi_{1i} = \xi_i \alpha_i^{-1}$ where the ξ_i is defined as in the proof of Theorem 20. Let h_i be minimally associated with ξ_{1i} mod U over C. Since h, which is minimally associated with ξ mod U, is associated with each ξ_i mod U, h both divides and is divisible by $[h_1, h_2, \cdots, h_m]$, hence $[h_1, h_2, \cdots, h_m] = h$. For any polynomial q and vector ξ,

$$(q(M) \odot \xi)\alpha_i^{-1} = q_{\alpha_i^{-1}}(M) \odot (\xi\alpha_i^{-1}).$$

Also there exists a polynomial q_1 such that

$$q_{\alpha_i^{-1}}(M) \odot (\xi\alpha_i^{-1}) = q_1(M) \odot \xi.$$

Hence for some q_2,

$$\begin{aligned} \xi_{1i} &= (f_{\alpha_i}(M) \odot p_i(M) \odot \xi)\alpha_i^{-1} \\ &= f(M) \odot p_{i\alpha_i^{-1}}(M) \odot (\xi\alpha_i^{-1}) \\ &= f(M) \odot q_2(M) \odot \xi. \end{aligned}$$

Hence

$$g(M) \odot \xi_{1i} = h(M) \odot q_2(M) \odot \xi,$$

so that any relative linear combination η of the ξ_{1i} whose coefficients are polynomials in C will satisfy condition (2) of the theorem. The existence of such a combination η with which h is minimally associated is guaranteed by Theorem 4. This η will also satisfy condition (1).

THEOREM 22. *If g_1 and g_2 each defines h but no interior divisor of g_1 or of g_2 of degree less than the degree of g_1 or g_2 respectively defines h, then g_1 and g_2 are of the same degree.**

Let $\langle g \rangle$ represent the degree of any polynomial g. Suppose $\langle g_1 \rangle \leqq \langle g_2 \rangle$. By Theorem 11 a matrix M and a vector ξ_1 can be chosen such that the order of M is equal to $\langle g_1 \rangle$ and g_1 is minimally associated with ξ_1 relative to M. By Theorem 21 the existence of a vector ξ_2 is established such that g_2 is minimally associated with ξ_2 relative to M, but the polynomial minimally associated with any vector relative to M is of degree not more than the order of M. Hence $\langle g_1 \rangle = \langle g_2 \rangle$.

Polynomials such as g_1 and g_2 of Theorem 22 are said to be *polynomials of minimum degree defining h*. The degree of g_1 and g_2 is dependent upon h

* This proof, independent of any theorem of unique factorization, seems of interest.

alone, and the degree of such polynomials shall be denoted by the symbol $\langle\langle h\rangle\rangle$, that is, $\langle\langle h\rangle\rangle = \langle g_1\rangle = \langle g_2\rangle$.

THEOREM 23. *If h is irreducible in C and if g is a polynomial of minimum degree defining h, then the degree of h is an integral multiple of the degree of g.*

THEOREM 24. *If g_1 defines h_1 and g_2 defines h_2, then there exists some transform $g_{2\alpha_i}$ of g_2 such that $g_{2\alpha_i} \odot g_1$ defines h_3 where h_1 is a factor of h_3 but $h_3 \neq h_1$ and h_3 is a factor of $h_2 \odot h_1$.*

Suppose for every i, $g_{2\alpha_i} \odot g_1$ defines h_1 and therefore is an interior factor of h_1. Then $[g_2 \odot g_1, g_{2\alpha_2} \odot g_1, \cdots, g_{2\alpha_m} \odot g_1] = h_2 \odot g_1$ is an interior factor of h_1, and therefore there exists a polynomial f such that $h_1 = f \odot h_2 \odot g_1 = h_2 \odot f \odot g_1$. Hence $f \odot g_1$ is in the centrum, which contradicts the hypothesis.

One can readily obtain

THEOREM 25. *If h is an irreducible polynomial and if g_1 is a polynomial of minimum degree defining h, there exists a set of transforms g_i $(i = 1, 2, \cdots, t)$ of g_1 by the basal elements α_i such that $g_t \odot g_{t-1} \odot \cdots \odot g_1$ is a polynomial of minimum degree defining h^t, and hence $\langle\langle h^t\rangle\rangle = t\,\langle\langle h\rangle\rangle$.*

THEOREM 26. *If $\xi_1, \xi_2, \cdots, \xi_k$ is a set of vectors linearly independent relative to M such that $L_M(\xi_1, \xi_2, \cdots, \xi_k)$ is the whole space, and if g_i is the minimum polynomial associated with ξ_i relative to M, then the rank of $h(M)$, where h is any polynomial in the centrum, is equal to $m - \sum_{i=1}^{k} \langle g_{1i}\rangle$ where m is the order of M and $g_{1i} = (h, g_i)_{ex}$, the greatest common exterior divisor of h and g.*

Since $g_{1i} = (h, g_i)_{ex}$, let $g_i = g_{1i} \odot g_{2i}$ and $h = g_{1i} \odot g_{3i} = g_{3i} \odot g_{1i}$. Hence $h(M) \odot g_{2i}(M) \odot \xi_i = 0$ and also $h(M) \odot M^j \odot g_{2i}(M) \odot \xi_i = 0$. Since g_i is the minimum polynomial associated with ξ_i, the number of vectors $M^j \odot g_{2i}(M) \odot \xi_i$ right linearly independent and orthogonal to $h(M)$ is equal to the degree of g_{1i}, $\langle g_{1i}\rangle$. Furthermore, since $\xi_1, \xi_2, \cdots, \xi_k$ are linearly independent relative to M by hypothesis, all vectors of the form $M^j \odot g_{2i}(M) \odot \xi_i$ for $i = 1, 2, \cdots, k$ and $j = 0, 1, 2, \cdots, \langle g_{1i}\rangle - 1$ are right linearly independent and orthogonal to $h(M)$. Consequently it follows that the rank of $h(M)$ is $m - \sum_{i=1}^{k} \langle g_{1i}\rangle$; for if any other vector ξ were orthogonal to $h(M)$ it would be dependent upon those orthogonal to $h(M)$ in each of $L_M(\xi_1), L_M(\xi_2), \cdots, L_M(\xi_k)$.

IV. EXISTENCE FOR n-SPACE OF A BASE, LINEARLY INDEPENDENT RELATIVE TO AN $n \times n$ MATRIX

If M is an $n \times n$ matrix, and if the minimum polynomial in C of M is $\prod h_i^{k_i}$, where the h_i are distinct irreducible polynomials, if a linearly independent base relative to M were found for the space associated with $h_i^{k_i}$, then a line-

arly independent base relative to M for the whole space is the totality of vectors in the bases for the separate spaces associated with each of the $h_i^{k_i}$, according to Theorem 2.

The rank of $\prod_{i\neq j} h_i^{k_i}(M)$ is equal to the order of the space orthogonal to $h_j^{k_j}(M)$. Hence the columns of $\prod_{i\neq j} h_i^{k_i}(M)$ form a base for the space associated with $h_j^{k_j}$. At least one of these columns is a vector such that $h_j^{k_j}$ is minimally associated with it over C relative to M or else h_j appears to a lower power than the k_jth in the minimum polynomial of M in C.

To simplify the notation, let h^{k_1} be the highest power of an irreducible polynomial h in C occurring in the minimum polynomial of M in C, and let g_1 be a polynomial of minimum degree defining h. By Theorem 24 there exist transforms, $g_{k_1}, g_{k_1-1}, \cdots, g_1$, of g_1 by certain basal elements α_i of A such that $g_{k_1}\odot g_{k_1-1}\odot \cdots \odot g_1$ is a polynomial of minimum degree defining h^{k_1}. By the preceding paragraph and Theorem 21 there exists a vector ξ_1 such that $g_{k_1}\odot g_{k_1-1}\odot \cdots \odot g_1$ is the minimum polynomial associated with ξ_1, and h^{k_1} is minimally associated over C with ξ_1. Since $h^t(M)\odot\xi_1$ and $g_t(M) \odot g_{t-1}(M) \odot \cdots \odot g_1(M) \odot \xi_1$ are minimally associated with $g_{k_1-t}\odot g_{k_1-t-1}\odot \cdots \odot g_1$ and $g_{k_1}\odot g_{k_1-1}\odot \cdots \odot g_{t+1}$ respectively, the order of $L_M(h^t(M)\odot\xi_1)$ is equal to the order of $L_M(g_t(M)\odot g_{t-1}(M)\odot \cdots \odot g_1(M)\odot\xi_1)$. These two spaces are identical since the first is contained in the second. Therefore, if f_1 is any polynomial such that $h^{k_1-t}(M)\odot f_1(M)\odot\xi_1=0$ there exist polynomials p_1 and p such that

$$(2)\qquad \begin{aligned} f_1(M) \odot \xi_1 &= p_1(M) \odot h^t(M) \odot \xi_1 \\ &= g_t(M) \odot g_{t-1}(M) \odot \cdots \odot g_1(M) \odot p(M) \odot \xi_1; \end{aligned}$$

i.e., any vector in $L_M(\xi_1)$ orthogonal to $h^{k_1-t}(M)$ is of the form given above.

Suppose $k_2 \leqq k_1$ is the largest integer such that there exists a vector ξ_{21} in the space associated with h^{k_1} for which h^{k_2} is the minimum polynomial in C associated with ξ_{21} mod $L_M(\xi_1)$. This vector ξ_{21} may be found among the columns of $\prod_{i\neq j} h_i^{k_i}(M)$ where $h=h_j$. Since $f=g_{k_2}\odot g_{k_2-1}\odot \cdots \odot g_1$ is a polynomial of minimum degree defining h^{k_2}, by Theorem 21 the existence of a vector ξ_{22} is established such that f is minimally associated with ξ_{22} mod $L_M(\xi_1)$, and also the existence of p_2 is assured such that $f(M)\odot\xi_{22}=h^{k_2}(M)\odot p_2(M)\odot\xi_{21}$ and hence the vector $f(M)\odot\xi_{22}$ is in $L_M(\xi_1)$ and orthogonal to $h^{k_1-k_2}(M)$. If equation (2) is applied with $t=k_2$, there exists a polynomial p_3 such that $f(M)\odot\xi_{22}=f(M)\odot p_3(M)\odot\xi_1$. Take $\xi_2=\xi_{22}-p_3(M)\odot\xi_1$. It follows that $f(M)\odot\xi_2=0$, and since $\xi_2\equiv\xi_{22}$ mod $L_M(\xi_1)$, f is minimally associated both with ξ_2 and with ξ_2 mod $L_M(\xi_1)$.

Let k_3 be the largest integer such that there exists a vector ξ_{31} in the space associated with h^{k_1} such that h^{k_3} is minimally associated over C with

ξ_{31} mod $L_M(\xi_1, \xi_2)$. The polynomial h^{k_1} is associated with ξ_{31}, moreover h^{k_2} is associated with ξ_{31} mod $L_M(\xi_1)$ because of the maximal property of k_2. Then in $L_M(\xi_1, \xi_2, \xi_3)$ there will be a ξ_3 minimally associated with $g_{k_3} \odot g_{k_3-1} \odot \cdots \odot g_1$ mod $L_M(\xi_1, \xi_2)$ and hence minimally associated with h^{k_3} mod $L_M(\xi_1, \xi_2)$ over C, and such that $h^{k_3}(M)\xi_3 = 0$.

This process may be continued to complete the base for the total vector space.

It has been proved by Theorem 26 that

$$rk(h^k(M)) = m - \sum_i \langle (h^k, g_{k_i} \odot g_{k_i-1} \odot \cdots \odot g_1) \rangle.$$

Hence

$$rk(h^k(M)) = m - k\langle\langle h \rangle\rangle \text{ (number of } k_i \geqq k) - \sum_i k_i \langle\langle h \rangle\rangle$$

where the sum runs for values $k_i < k$. From this it follows that the polynomials associated with the above canonical proper base of the relative set equal to the total vector space are completely determined by a knowledge of the rank of the powers of h_i, where the h_i are the irreducible factors of the minimum equation of M. Since in part I it was shown that the matrices M and N are similar if relative to M and N there exists a proper base for the total vector space associated with the same set of minimum polynomials, Theorem 27 follows.

THEOREM 27. *If $h = \prod_{i=1}^{l} h_i^{t_i}$ is the minimum polynomial in the centrum for each of two matrices M and N, where h_i is irreducible, a necessary and sufficient condition that the matrix M is similar to the matrix N is that the rank of $h_i^j(M)$ is equal to the rank of $h_i^j(N)$ for every $i = 1, 2, \cdots, l$ and $j = 1, 2, \cdots, t_i$.*

If ξ_{ij}, $(i = 1, 2, \cdots)$, $(j = 1, 2, \cdots)$, is a set of linearly independent vectors whose linear extension relative to M is the whole space, if g_{ij} is the minimum polynomial associated with ξ_{ij}, and if g_{ij} is a polynomial of minimum degree defining $h_i^{k_{ij}}$, such an $h_i^{k_{ij}}$, where h_i is irreducible, is called a characteristic divisor of M. Hence the necessary and sufficient condition that two matrices be similar is that the characteristic divisors of the two matrices be equal. It is useful to note that the above implies that two matrices M and N are similar if for every polynomial h with coefficients in the centrum the rank of $h(M)$ is equal to the rank of $h(N)$.

Since two matrices are equivalent only if $rk(h^k(M)) = rk(h^k(N))$ whenever h is irreducible and k is a positive integer, a necessary condition that for a system of polynomials g_i there exists a set of vectors ξ_i linearly independent relative to M and such that the minimum polynomial associated with ξ_i is g_i and

$L_M(\xi_1, \xi_2, \cdots, \xi_k)$ is the whole space, is that the numbers $\sum\langle(h^k, g_i)\rangle$ are those specified by Theorem 27. This is sufficient because a diagonal block matrix N may be constructed with blocks of the form of the matrix in Theorem 11, having g_i as the minimum polynomial of each block.

The remainder of this section is devoted to describing a process which furnishes a method of testing whether two matrices are similar, and if M and N are similar, it furnishes a method of finding a set of polynomials g_i and sets of vectors ξ_i and η_i, each k in number, such that the ξ_i are linearly independent relative to M and the η_i are linearly independent relative to N; such that $L_M(\xi_1, \xi_2, \cdots, \xi_k) = L_N(\eta_1, \eta_2, \cdots, \eta_k) = S$, the total space; and such that g_i is minimally associated with ξ_i relative to M, and g_i is minimally associated with η_i relative to N. This process is wholly rational and does not involve the factorization of polynomials in the centrum into irreducible factors nor knowledge of a polynomial of minimum degree defining h; any factorization into powers of relatively prime factors in the centrum C of the minimum polynomial in C of M and N is suitable for an initial start and for each of these factors it is sufficient to have some polynomial g defining it. Either the required sets of vectors are found or one of the known factors of the minimum polynomial in C of M and N is factored with coefficients in C, or else a polynomial g of lower degree than the degree of the initial g is found which is associated with some factor of this minimum polynomial. The method therefore either yields the required vectors and polynomials or it reduces in a finite number of steps to the case studied in the beginning of this section. It should be borne in mind that this process is to be carried out simultaneously for M and N in practice, but the following treatment is for a matrix M only.

Let the minimum polynomial with coefficients in C for both M and N be $\prod h_i^{l_i}$, where the h_i are relatively prime. The whole vector space is the sum of the relative linear spaces orthogonal to $h_i^{l_i}$ and as above there is at least one vector such that $h_i^{l_i}$ is minimally associated with it. These spaces are linearly independent relative to M. To simplify notation, consider h to be any of the h_i, and g_i a transform of g_1. If, without assuming that the polynomials involved are irreducible, the method used in the beginning of this section is applied in constructing a proper base relative to a matrix M and a proper base relative to a matrix N, these bases having the properties described in the preceding paragraph, then the following situations may arise, and these are the only difficulties in applying this reasoning and process.

1. A vector ξ and a linear set U may be found such that $h^k(M) \odot \xi \neq 0 \bmod U$ and $h^{k+1}(M) \odot \xi = 0 \bmod U$, but the polynomial minimally associated with $\xi \bmod U$ over C is not h^{k+1}. This leads to a factorization of h.

2. g_1 and a set of transforms $g_2, g_3, \cdots, g_t$ exist such that $g_t \odot g_{t-1} \odot \cdots \odot g_1$ defines h^t but there exists a transform g_{t+1} such that $g_{t+1} \odot g_t \odot \cdots \odot g_1$ defines a multiple of h^t, not equal to h^t, and not equal to h^{t+1}. This also leads to a factorization of h.

3. A vector ξ and a linear set U exist such that $g_t \odot g_{t-1} \odot \cdots \odot g_1$ defines h^t, and h^t is minimally associated with ξ over C but ξ is minimally associated with a proper divisor, g_{11}, of $g_t \odot g_{t-1} \odot \cdots \odot g_1$. In this case the polynomial g_{11} defines h^t where $\langle g_{11} \rangle < t\langle g_1 \rangle$. By Theorem 19 (h, g_{11}) defines h. If $\langle (h, g_{11}) \rangle < \langle g_1 \rangle$ then one arrives immediately at a new start with a polynomial of lower degree than $\langle g_1 \rangle$ defining h. If $\langle (h, g_{11}) \rangle \geqq \langle g_1 \rangle$, let $g_{11} = g_{12} \odot (h, g_{11})$. By Theorem 16, g_{12} defines a power of h, less than or equal to the $(t-1)$st power, whereas $\langle g_{12} \rangle < (t-1)\langle g_1 \rangle$. Continuing with (h, g_{12}) defining h and by repeated use of the preceding argument, in a finite number of steps a polynomial is found, of degree lower than $\langle g_1 \rangle$ defining h. This case may arise in trying to find a vector ξ minimally associated with $g_t \odot g_{t-1} \odot \cdots \odot g_1$, and also if for such a ξ, $h^{t_1}(M) \odot \xi$ is minimally associated with a proper divisor of $g_{t-t_1} \odot g_{t-t_1-1} \odot \cdots \odot g_1$.

In carrying out this process simultaneously for M and N, one is led either to the proof of the non-similarity of M and N or the construction of proper bases for the whole space relative to M and N respectively, such that the polynomials minimally associated with the one base are equal to those minimally associated with the other.

This procedure which for the general case seems to be both long and involved is for such usual cases as third- and fourth-order matrices with quaternionic elements not very difficult to carry through. For practical operation it should be noted that if $\prod h_i^{l_i}$ is the minimum polynomial of M, and the h_i are relatively prime, then the columns of $\prod_{i \neq j} h_i^{l_i}$ form a base for the space orthogonal to $h_j^{l_j}(M)$.

It would seem that other problems could be advantageously attacked by methods of this paper. As an example, given two similar matrices M and N, all non-singular matrices S such that $S^{-1}MS = N$ consist of matrices S that transform $(\xi_i, M\xi_i, M^2\xi_i, \cdots, M^{\langle g_i \rangle - 1}\xi_i)$ into $(\eta_i, N\eta_i, N^2\eta_i, \cdots, N^{\langle g_i \rangle - 1}\eta_i)$, (i), where the ξ_i and η_i are sets of vectors whose relative linear extensions are the whole space and where g_i is minimally associated with ξ_i relative to M and with η_i relative to N. In particular, the non-singular matrices commutative with M are those which transform any set ξ_i into a set ξ_{1i}, having the same property.

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THE BINET OF QUADRICS IN S_3*

BY

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1. **Introduction.** The concept of a linear system of quadrics was introduced by Dupin in 1808.† He considered the pencil determined by two concentric central quadrics. Ten years later, Lamé‡ defined the net of quadrics. The web of quadrics was defined and considered first by de Jonquières§ in 1862. These systems of quadrics have since been treated by many mathematicians.‖

A binet of quadrics is an ∞^4 linear system of quadric surfaces in S_3. The term binet is suggested for this system merely because it contains twice the number of essential parameters contained in a net. The system is not in any other sense a "double net."

The binet has not been treated as such, but only as implied in certain polar and apolar relations associated with a general linear system of surfaces.

The purpose of this paper is to determine the characteristics of a binet of quadrics in S_3.

2. **The binet.** The equation of a binet of quadrics in S_3 is

$$\sum \lambda_i f_i = 0, \qquad i = 1, 2, 3, 4, 5,$$

wherein the $f_i = 0$ are the equations of five quadrics in S_3. These five quadrics are not restricted or related in any way.

The binet contains ∞^3, ∞^2, ∞^1, finite systems of surfaces, both linear and non-linear, satisfying one, two, three, four conditions respectively. Linear systems will be discussed in §4 and non-linear systems in §8 and §9.

The two non-linear ∞^3 systems, however, will be defined now because their locus is to be used in developing the theory. One is the system of quadric cones and the other, the system of pencils of quadrics with one contact.

Since it is one condition for a quadric to be a cone, the binet contains ∞^3 cones; that is, any point in S_3 is the vertex of a finite number of cones of the

* Presented to the Society, September 3, 1936; received by the editors July 16, 1936.

† Dupin, Journal de l'École Polytechnique, vol. 14 (1808), p. 80.

‡ Lamé, Examen (1818), pp. 29 and 37. On pp. 28 and 35 of the same paper, Lamé also wrote the equation of a general pencil of quadrics in the form $f+\lambda g=0$.

§ de Jonquières, Journal de Mathématiques, (2), vol. 7 (1862), p. 412.

‖ Encyklopädie der Mathematischen Wissenschaften, vol. III_2, pp. 212–223, 246–250, 250–254. Pascal, Repertorium der höhere Mathematik, vol. II_2 (1922), pp. 616–626, 626–628, 629–631. (In both references, the three sets of pages refer to pencils, nets, webs, respectively.)

binet. Also, since it is one condition for two quadrics to have one contact and since these two quadrics determine a pencil of quadrics, all of which have contact with each other at that point, the binet contains ∞^3 such pencils and any point of S_3 is the point of contact of the quadrics in a finite number of pencils. The jacobian of a binet of quadrics in S_3, i.e., the locus of the vertices of cones and the locus of simple contacts of the binet, is, then, the space S_3 containing the binet.

3. **The associated correspondence.** The (1, 1) correspondence between the primes of S_4 and the quadrics of the binet is defined by the equations,

$$\rho y_i = f_i, \qquad i = 1, 2, 3, 4, 5,$$

in which $y_i=0$ are the equations of the primes of the coordinate pentaprime in S_4 and $f_i=0$ are any five quadrics of the binet.

To a plane of S_4 corresponds an elliptic quartic curve, the intersection of two quadrics of the binet and the basis curve of a pencil of quadrics of the binet. To a line of S_4 corresponds 8 points of intersection of three quadrics or the basis points of a net of quadrics of the binet.

In the transformation from S_3 to S_4, the image of a plane of S_3 is a rational quartic surface of S_4 and the image of a line of S_3 is a conic of S_4.

4. **Linear systems of the binet.** To the ∞^6 planes of S_4, each considered as bearing a pencil of primes, correspond ∞^6 pencils of quadrics of the binet. Each pencil has an elliptic quartic as basis curve and contains four quadric cones of the binet.

To the ∞^6 lines of S_4, each considered as bearing a bundle of primes, correspond ∞^6 nets of quadrics of the binet. Each net of quadrics has 8 basis points. Each net also has a jacobian curve γ_i of order 6, rank 16 and genus 3 with only apparent double points. This jacobian curve is the locus of nodes and contacts of quadrics of the net.

Each line of S_4 also bears a bundle of planes, the intersections of the primes of the bundle. To the ∞^6 lines of S_4, therefore, considered as bearing bundles of planes, correspond ∞^6 nets of elliptic quartic space curves, each net of curves being the intersections of pairs of quadrics of the associated net of quadrics.

If two quadrics of a net F_1 are tangent at a point P_1, their curve of intersection has a node at P_1. The locus of P_1, considered as contacts of pencils of quadrics of F_1, is the jacobian curve γ_1 of F_1. The locus of the same point P_1, considered as a node on the quartic curves associated with F_1, is the jacobian curve of the net of quartic curves. Therefore each γ_i is the jacobian, both of a net of quadric surfaces and of the net of associated quartic curves.

To the ∞^4 points of S_4, each point considered as bearing ∞^3 primes, cor-

respond ∞^4 webs of quadrics of the binet. Each web of quadrics has a jacobian surface J of order 4, genus 1 and containing ten lines, the axes of the ten pairs of planes of the web.

Each point of S_4 also bears ∞^4 planes and ∞^3 lines, the intersections of the ∞^3 primes through that point. To the ∞^4 planes corresponds a binet of elliptic quartic curves and to the ∞^3 lines correspond ∞^3 sets of 8 points, belonging to ∞^3 nets of quadrics of the binet. Thus there are ∞^4 binets of quartic curves contained in the binet of quadrics. However, the jacobians of all the webs of quartic curves in the binet of curves associated with the point P_1 are one and the same jacobian J_1, the jacobian of the web W_1 of quadrics associated with P_1, since J_1 is the locus of contacts of quadrics of W_1, and a curve of the associated binet can have a node when and only when two quadrics of W_1 have a contact.

Since two points of S_4 have a line in common, any two webs of primes in S_4 have a bundle of primes in common and, correspondingly, in S_3, any two webs of quadrics have, in common, a net of quadrics. The Jacobian surfaces of two webs intersect in a curve of order 16 consisting of the curve γ of order 6 and genus 3, the jacobian of the net common to the two webs, and a curve ϕ of order 10 which is common to the ∞^4 jacobians J_i of the webs of the binet. The jacobian quartic surfaces J_i, therefore, form a binet with ϕ as basis curve. This basis curve ϕ is of order 10, genus 11, rank 40 and has 25 apparent double points. Each jacobian curve γ intersects ϕ in 20 points.

Since three points of S_4 determine a plane, any three webs of quadrics in S_3 have a pencil of quadrics in common. The three J_i of the three webs intersect in 64 points. Of these points, 4 are the vertices of the four cones of the common pencil and the other 60 are basis points common to all the J_i of the binet of jacobians. It has been shown that these J_i have a basis curve ϕ of order 10 and rank 40. The equivalence of ϕ on three J_i is 60. Therefore the basis curve ϕ absorbs the 60 basis points and the binet of quartic surfaces J_i has the curve ϕ as its only basis element.

5. **The locus of axes of composite quadrics.** Let f_0 be a quartic of the binet that consists of two planes intersecting in the line l. Through l passes a web of jacobian quartics J_i, the jacobians of the ∞^3 webs of quadrics which have f_0 in common. Since this web of jacobian quartics J_i has a net in common with any other web of jacobian quartics, it follows that the locus of the axes of composite quadrics of the binet is a ruled surface which is a fixed component of the jacobians of all the webs of J_i.

The jacobian of a web of quartic surfaces J_i with ϕ as basis curve, is a surface of order 12 containing ϕ as a triple curve. This surface consists of a quadric of the binet and a ruled surface R of order 10 containing ϕ as a triple

curve. R is the locus of the ∞^1 lines which are axes of composite quadrics of the binet. The genus of a plane section of R is 6.

Each jacobian J intersects R in a curve of order 40 consisting of ϕ counted three times and the 10 lines lying on that jacobian.

The two systems of surfaces, the ∞^4 webs of quadrics with no basis elements and the ∞^4 webs of J_i with the basis curve ϕ, are thus so related that, aside from the common surface R, the jacobian of a web of either system is a surface of the other system.

6. **The branch-point primal L in S_4.** In nets and webs of quadrics, the branch-point primal is the image of the jacobian of the system of quadrics. This is also true in the case of a binet. The jacobian of the binet is the linear space S_3 containing the binet. L is, therefore, in (1, 1) correspondence with S_3 and is rational. L is of order 8, the number of intersections of three quadrics of S_3.

The complete image of L is S_3. Since the primes of S_4 are in (1, 1) correspondence with the quadrics of the binet, the birational image of a prime section of L is a quadric of the binet. Then the prime sections of L are also rational.

To a tangent prime of L corresponds a cone of the binet, the point of contact in S_4 corresponding to the vertex of the quadric cone in S_3. To any point P_0 in S_4, considered as bearing a web of primes, corresponds a web W_0 of quadrics in S_3. The locus of the vertices of the cones of W_0 is J_0. The locus of contacts of the tangent primes to L from P_0 is the contour surface Λ_0 on L of the tangent cone from P_0. Λ_0 is in (1, 1) correspondence with J_0 and is, therefore, the image of J_0 on L. Since J_0 is of order 4, Λ_0 is of order 16 and therefore the order of the tangent cone to L from a point of S_4 is 16. Also, since the J_i form a binet in S_3, the image surfaces Λ_i form a binet on L. Each J_i is of genus 1. Then each Λ_i is of genus 1 and the tangent cones to L from a point are of genus 1.

A section by any prime π_0 of the tangent cone to L from P_0 is a surface L_0, the branch-point surface of the web of quadrics corresponding to the web of primes on P_0. The tangent planes to L_0 are sections by π_0 of the tangent primes to L from P_0.

The following symbols will be used for the characteristics of a surface, the subscripts 3 and 0 referring to the prime section of L and of the tangent cone to L respectively: n, order; n', class; a', class of plane section; a, order of tangent cone; b, order of nodal curve; c, order of cuspidal curve; δ', number of bitangents of plane section; κ', number of inflections of plane section; δ, number of nodal lines of tangent cone; κ, number of cuspidal lines of tangent cone; q, rank of nodal curve; r, rank of cuspidal curve; b', class of

bitangential developable; c', class of parabolic developable; γ, number of intersections of nodal and cuspidal curve, cusps on nodal curve; β, number of intersections of nodal and cuspidal curve, cusps on cuspidal curve; t, number of triple points of nodal curve; β', number of tacnodal tangent planes; γ', number of nodo-cuspidal tangent planes; t', number of triple tangent planes; ρ, class of nodal developable; σ, class of cuspidal developable; ρ', order of bitangential curve; σ', order of parabolic curve; C', number of conic tropes; D, genus.

Since the prime section L_0 of the tangent cone to L from any point P_0 of S_4 is the branch-point surface associated with the web W_0, L_0 has the following characteristics:*

$$n_0 = 16, \quad n_0' = 4, \quad a_0' = a_0 = 12, \quad D_0 = 1, \quad C_0' = 10, \quad \delta_0' = 22, \quad \kappa_0' = 24,$$
$$b_0 = 60, \quad q_0 = 40, \quad \gamma_0 = 120, \quad t_0 = 80,$$
$$c_0 = 36, \quad r_0 = 68, \quad \beta_0 = 80,$$
$$\delta_0 = 28, \quad \kappa_0 = 24, \quad \sigma_0 = 32, \quad \rho_0 = 80,$$
$$b_0' = c_0' = \beta_0' = \gamma_0' = t_0' = \sigma_0' = \rho_0' = 0.$$

Each 4-space surface Λ_0 is the projection on L from P_0 of the 3-space surface L_0.

To the bundle of primes through any line l_1 of S_4 corresponds a net of quadrics F_1 of the binet. The locus of the vertices of cones in F_1 is the curve γ_1. Since the contacts of the ∞^1 tangent primes to L from l_1 are in (1, 1) correspondence with the vertices on γ_1 of the cones of F_1, the contour curve Γ_1 of the cone (of the second species) from l_1 to L is in (1, 1) correspondence with γ_1. These curves Γ_i form an ∞^6 linear system (or biweb) on L.

The plane section of the tangent cone to L from l_1 made by any plane of S_4 is the branch-point curve G_1 in that plane, corresponding birationally to the jacobian curve γ_1 of F_1. The plane curve G_1 has the characteristics of the plane section of the 3-space tangent cone to L_0, the branch-point surface of a web. Then G_1 has the characteristics:

order $n_1 = 12$,	nodes $\delta_1 = 28$,	cusps $\kappa_1 = 24$,
class $m_1 = 4$,	bitangents $\tau_1 = 0$,	inflections $\iota_1 = 0$.

Each 4-space curve Γ_i on L is the projection on L from l_i of the plane curve G_i.

The plane section of L made by any plane of S_4 is the birational image of an elliptic quartic curve, the intersection of two quadrics of the binet. This plane section is of order 8 and genus 1. Its class equals the order of the tan-

* T. R. Hollcroft, *The general web of algebraic surfaces of order n and the involution defined by it*, these Transactions, vol. 35 (1933), p. 868.

gent cone to L from a point, which is 16. The remaining characteristics of the plane section are obtained by Plücker's equations. The characteristics are:

order $n_2=8$, class $m_2=16$, genus $p_2=1$, nodes $b_2=20$,
cusps $c_2=0$, bitangents $\delta_2=80$, inflections $\kappa_2=24$.

The prime section of L made by any prime of S_4 is the birational image of a quadric of the binet and is, therefore, of order 8 and genus 0. The class of an S_3 section equals the order of the tangent cone to L from a line, which is 12. The characteristics of a plane section of the S_3 section are the same as the characteristics of a plane section of L.

Since the nodal curve of the prime section is of order 20, the nodal surface b of L is of order 20. This nodal surface has a pinch curve C_j whose image in S_3 is the basis curve ϕ of the jacobian binet. The curves C_j and ϕ are in (1, 1) correspondence. Since ϕ is of order 10 and genus 11, C_j is of order 20 and genus 11. The curve C_j is cut by an S_3 in 20 points which are j_3 pinch points of the nodal curve of the prime section.

The remaining characteristics of the prime section of L are obtained from the above characteristics by means of the Cayley-Zeuthen equations.

The characteristics of the prime section of L are:

$$
\begin{aligned}
&n_3 = 8, \quad n_3' = 12, \quad a_3 = a_3' = 16, \quad D_3 = 0,\\
&\delta_3' = 80, \quad \kappa_3' = 24, \quad c_3 = r_3 = \beta_3 = \gamma_3 = 0,\\
&b_3 = 20, \quad q_3 = 50, \quad k_3 = 105, \quad t_3 = 20, \quad j_3 = 20,\\
&b_3' = 22, \quad c_3' = 24, \quad \delta_3 = 60, \quad \kappa_3 = 36,\\
&\sigma_3' = 32, \quad \sigma_3 = 0, \quad \rho_3' = 72, \quad \rho_3 = 60,\\
&\gamma_3' = 0, \quad \beta_3' = 44, \quad t_3' = 24,\\
&q_3' = 50, \quad r_3' = 60, \quad k_3' = 134.
\end{aligned}
$$

In the above, k_3 is the number of apparent double points of the nodal curve and q_3', r_3', k_3' are the reciprocals of the respective unaccented symbols already defined.

The remaining characteristics of the primal L will now be derived.

Since a quadric cannot have two or more distinct nodes or a singular point of higher order than a node, L does not have tangent primes with contact at more than one distinct point or with higher contact at one point. This is also revealed by the characteristics of the tangent cones to L in that there are no stationary or bitangent primes to L from a point or from a line. Thus, for L, the orders of the bitangential surface, the parabolic surface and their associated curves are all zero as are also the numbers of tangent primes

satisfying four conditions whose contacts would occur at intersections of these surfaces and curves.

Also, since $c_3=0$ for the prime section, L has no cuspidal curve.

A prime section L_0 of the tangent cone to L from any point P_0 has $C_0'=10$ conic tropes, i.e., ten planes, each tangent to L_0 along a conic. Each conic trope corresponds uniquely to a composite quadric of the web W_0. The image of the plane α_0 of a conic trope of L_0 is a composite quadric f_0 of W_0. The image of the axis of f_0, which is also a line of J_0, is the contact conic c_0' of the conic trope.

The prime section of L by the prime π_1 determined by P_0 and α_0 corresponds to the composite quadric f_0 considered as a quadric of the binet. The prime π_1 is tangent to L along a conic c_0 and intersects L in two rational quartic surfaces which are tangent to each other along c_0. The two quartic surfaces are the images respectively of the two planes of f_0 and c_0 is the image of the axis of f_0.

The image of R, the locus of axes of composite quadrics of the binet, is the surface T on L. T is of order 20 and has the pinch curve C_j of order 20 as a triple curve. T is a surface generated by the singly infinite non-linear system of conics which are in $(1,1)$ correspondence with the axes of composite quadrics of the binet. Through each point of S_4 pass ten primes, each of the nature of π_1 described in the preceding paragraph. T counts doubly in the intersection of L and its hessian.

The jacobian of the web of prime sections of L from a point P_0 is the contour surface Λ_0 of order 16 of the tangent cone from P_0 to L. Λ_0 contains C_j as a simple curve. Since S_4 contains ∞^4 points , the contour surfaces Λ_i form a binet on L for which C_j is a simple curve. The jacobian of the web of Λ_i associated with the points of any prime of S_4 consists of the surface T and the prime section of L made by that same prime. T is, then, common to the jacobians of all the ∞^4 webs of Λ_i of the binet of Λ_i on L. T is the locus of contacts of the contour surfaces Λ_i.

From the characteristics of L already found, the others are obtained by means of formulas for primals in four dimensions derived by Roth.* The characteristics of L, other than those of its cones and sections, are, therefore, as follows:

Order $N=8$; rank $a=16$; first class $m=12$, class $w=4$; order of nodal surface b, $\mu_0=20$; order of tangent cone of b, $\mu_1=50$; class of b, $\mu_2=20$; order apparent double curve† of b, $k=105$; number of apparent triple points of b

* L. Roth, *Some formulae for primals in four dimensions*, Proceedings of the London Mathematical Society, (2), vol. 35 (1933), pp. 549–552, and vol. 39 (1935), pp. 334–338.

† The order of the apparent double curve of a surface in S_4 is here defined as the order of the double curve of the projection of this surface on S_3.

and of its apparent double curve, $T_1=140$; order of triple curve C_t of b, $t=20$; rank of C_t, $r_t=60$; class of immersion of C_t in L, $t_1=80$; class of immersion of C_t in b, $t_0=140$; number of quadruple points of C_t and L (6-fold on b), $\xi=5$; order of pinch curve C_j, $j=20$; rank of C_j, $r_j=60$; class of immersion of C_j in L, $j_1=40$; class of immersion of C_j in b, $j_0=80$; number of pinch points of b (actual intersections of C_t and C_j), $\tau=40$; number of apparent pinch points of b (ceto of L), $\nu_2=20$; order of curve C_ρ, the locus of contacts of tangent primes from any point whose contacts lie on the nodal surface b, $\rho=60$; order of surface T on L (T is the image of R in S_3), $T=20$.

7. **Images in S_3 of L and its singularities.** The image of L is S_3. The images of the prime sections of L are the quadrics of the binet in S_3.

The birational image of the pinch curve C_j (of order 20 and genus 11) of the nodal surface b of L is the basis curve ϕ (of order 10 and genus 11) of the binet of jacobian surfaces J_i in S_3.

The image of the ruled surface R of order 10, the locus of axes of composite quadrics of the binet, is T, a surface generated by conics of order 20 on L. R contains ϕ as a triple curve and T contains C_j as a triple curve.

The image of the nodal surface b (of order 20) of L is a surface B of order 10. B contains ϕ as a simple curve. B also contains the double curve C_t', a curve of order 30 and the image of the triple curve C_t. The surfaces b and B are in (1, 2) point correspondence.

The curves ϕ and C_t' in S_3 intersect in $\tau=40$ points, images of the 40 intersections of C_j and C_t on b, which are actual pinch points of b.

The surfaces B and R intersect in a curve of order 100 consisting of ϕ counted three times and a C_{70}, the image of the curve of intersection of order 140 (in addition to C_j) of b and T on L.

R and any J_i intersect in a curve of order 40 consisting of ϕ counted three times and the ten lines of J_i. Correspondingly, on L, T and any Λ_i intersect in a curve of order 80, consisting of C_j counted three times and ten conics, images of the ten lines of J_i.

B and any J_i intersect in a curve of order 40 consisting of ϕ and a C_{30}, corresponding respectively to C_j and C_ρ (of order $\rho=60$) and comprising the total intersection of b and the associated Λ_i on L. C_ρ is the locus of points on b at which primes through the point associated with the given J_i are tangent to L.

8. **Singular quadrics of the binet.** The only singularities a quadric may have are a node and a binode. One condition is necessary for the quadric to have a node, in which case the quadric becomes the nodal cone. Three conditions are necessary for the quadric to break up into two planes, i.e., be the planes and axis of a binode.

The binet, therefore, contains ∞^3 quadric cones, the locus of whose ver-

tices (nodes) is S_3; and ∞^1 composite quadrics, the locus of whose axes (line of nodes) is the ruled surface R of order 10 containing ϕ as a triple curve.

9. **Loci of contacts of quadrics of the binet.** Every point of S_3 is a simple contact of the quadrics of a pencil of the binet.

The images in S_3 of singular surfaces, curves and points of L are defined in §7. With the exception of R, these images are contact loci of the binet.

The surface B is the locus of contacts of pencils of quadrics of the binet in each of which the quadrics have contact at two distinct points. B, the image of b, is of order 10 and contains ϕ as a simple curve and C_t' as a double curve. The curve ϕ is the locus of contacts of pencils of quadrics such that in each pencil the quadrics have four-point contact at one point. C_t' is the locus of contacts of pencils of quadrics in each of which the quadrics have contact at three distinct points.

C_t' and ϕ intersect in 40 points at each of which two of the three contacts of a pencil of quadrics coincide. The third contact associated with each such coincidence is on C_t' but not on ϕ. Then, in the binet, there are 40 pencils of quadrics in each of which the quadrics have one simple and one four-point contact.

There are $\xi=5$ quadruple points of C_t and of L. To each of these correspond four distinct points on C_t', which are nodes of C_t', such that the quadrics of a pencil have simple contact at each of these four points. Therefore, the binet of quadrics contains five pencils in each of which the quadrics have four distinct contacts.

R and B intersect in ϕ counted three times and in a C_{70}, the locus of contacts of pencils of quadrics with two contacts and such that one quadric of each pencil is composite and also such that the two planes of the composite quadric are the common tangent planes respectively of the quadrics at the two contacts.

The binet has no locus of stationary contacts of quadrics and no locus (surface, curve, or set of points) of contacts that involve stationary contacts.* Every point of S_3, however, is a stationary contact of a pencil of quadrics belonging to a web of quadrics of the binet.

* A somewhat similar situation occurs in a net of conics. The net of conics contains nine pencils in each of which the conics osculate each other, but no pencils with pairs of contacts.

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CONCERNING ESSENTIAL CONTINUA OF CONDENSATION*

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A collection G will be said to be of class 1 if (1) it is an upper semi-continuous collection of mutually exclusive continua, and (2) with respect to its elements it is a continuum with no continuum of condensation.

The subcontinuum K of the continuum M is said to be an *essential continuum of condensation* of M provided it is true that (1) K is a continuum of condensation of M, and (2) if G is a collection of class 1 filling up M, then some continuum of G contains K.

THEOREM 1. *If the non-degenerate subcontinuum K of the compact continuum M has no continuum of condensation and for every collection G of class* 1 *filling up M there is a continuum of G that contains K, then K is a continuum of condensation of M.*

Proof. Suppose K is not a continuum of condensation of M. Then there exists a point set D which is a domain both with respect to M and to K. Since K has no continuum of condensation, D contains a free† segment with respect to K, that is to say, a segment t of some arc lying in K such that no point of t is a limit point of $K-t$, in other words, such that t is a domain with respect to K. The point set t is also a domain with respect to M. If $M-t$ is a continuum, let G denote the collection whose elements are $M-t$ and the points of t. If $M-t$ is not connected, it is the sum of two mutually exclusive continua g_A and g_B containing the endpoints A and B respectively of t. In this case, let G denote the collection whose elements are g_A, g_B and the points of t. In the first case G is a simple closed curve, and in the second case it is an arc with respect to its elements. In each case it is a collection of class 1 and no continuum of G contains K. This involves a contradiction.

THEOREM 2. *If K is a subcontinuum of the compact continuum M, and for every collection G of class* 1 *filling up M there is a continuum of G that contains K, then K contains a continuum of condensation of M.*

Proof. Suppose K contains no continuum of condensation of M. Then it

* Presented to the Society, January 2, 1936; received by the editors August 16, 1936.

† Cf. R. L. Moore, *Foundations of Point Set Theory*, Colloquium Publications of the American Mathematical Society, vol. 13, 1932, p. 144. This book will be referred to as P.S.T.

contains no continuum of condensation of itself. Hence, by Theorem 1, K is itself a continuum of condensation of M.

THEOREM 3. *If the arc-wise connected continuum K is a subset of the compact continuum M, and for every collection G of class 1 filling up M there is a continuum of G that contains K, then K contains a continuum of condensation of M of diameter equal to that of K.*

Theorem 3 is a consequence of Theorem 1 and the fact that no arc has a continuum of condensation. That it does not remain true on the omission of the stipulation that K be arc-wise connected may be seen from the following example.

EXAMPLE 1. Let A, B, C, D, and E denote five points lying on a straight line in the order $ABCDE$ and such that each of the straight line intervals AB, BC, CD, and DE is of length 1. Let M_1, M_2, and M_3 denote three indecomposable continua of diameter 1 containing the intervals BC, CD, and DE respectively and such that no proper subcontinuum of M_2 contains both C and D. Let M denote the point set which is the sum of the interval AB and the continua M_1, M_2, and M_3. If G is a collection of class 1 filling up M, some continuum of G contains the continuum $M_1+M_2+M_3$ and therefore is of diameter not less than 3. But there is no continuum of condensation of M of diameter as great as 2.

THEOREM 4. *If the compact continuum M has no essential continuum of condensation, then for every non-degenerate subcontinuum K of M there exists an upper semi-continuous collection G of mutually exclusive continua filling up M such that G is, with respect to its elements, either an arc or a simple closed curve and such that no continuum of the collection G contains K.*

Proof. By hypothesis and Theorem 2 there exists a collection H of class 1 filling up M and such that no continuum of H contains K. Let W denote the collection whose elements are the continua of the collection H that intersect K. There exists a free segment t with respect to H such that every element of t belongs to W. By an argument analogous to that used in a similar connection in the above proof of Theorem 1, it may be shown that there exists an upper semi-continuous collection G of mutually exclusive continua filling up M such that G is either an arc or a simple closed curve with respect to its elements and such that every element of t is an element of G. The continuum K is not a subset of any continuum of the collection G.

NOTATION. If G is a collection of point sets, the notation G^* will be used to denote the sum of all the point sets of the collection G.

THEOREM 5. *If M is a compact regular curve, n is a positive integer and, for each i less than or equal to n, A_i and B_i are points of M and A_{n+1} is a point of*

M and B_{n+1} is a subset of M and there exist collections H and K of class 1 filling up M and such that (1) if i is not greater than n, no continuum of H contains both A_i and B_i, and (2) the continuum of K that contains A_{n+1} contains no point of B_{n+1}; then there exists a collection G of class 1 filling up M such that, if $i \leqq n+1$, no continuum of G intersects both A_i and B_i.

Proof. For each i not greater than n let a_i denote the continuum of H containing A_i and let b_i denote the one that contains B_i. Let a_{n+1} denote the continuum of H that contains A_{n+1}. For each i less than or equal to n, either a_i or b_i is distinct from a_{n+1}. Let L denote the set of all continua x distinct from a_{n+1} such that, for some i not more than n, x is identical either with a_i or with b_i. There exists a finite set T of subsets of H such that (1) each element of T is a free segment with respect to H, that is to say, a segment t of elements of H such that no element of t is a limit element of $H-t$, (2) the set $H-T$ is the sum of two mutually separated sets W_1 and W_2 of elements of H such that W_1 contains L and W_2 contains a_{n+1}. There exists a set T' of points such that (1) each point of the set T' is an element of H belonging to some segment of the set T, (2) for each segment t of the set T there is only one point of T' which is an element of t. The set $H-T'$ is the sum of two mutually separated sets Q_1 and Q_2 of elements of H such that W_1 and W_2 are subsets of Q_1 and Q_2 respectively. For each point P of the point set $T'+Q_2^*$ let x_P denote the continuum of the collection K that contains P, and let g_P denote the component of $x_P \cdot (T'+Q_2^*)$ that contains P. Let G denote the collection whose elements are the elements of Q_1 and the continua g_P for all points P of $T'+Q_2^*$.

THEOREM 6. *If M is a compact regular curve, n is a positive integer and, for each i not greater than n, A_i and B_i are points of M and, for each such i, there exists a collection H_i of class 1 filling up M and such that no continuum of H_i contains both A_i and B_i, then there exists a collection G of class 1 filling up M such that, if $i \leqq n$, no continuum of G contains both A_i and B_i.*

Proof. Suppose m is a positive integer for which there exists a collection G_m of class 1 filling up M and such that, if $i \leqq m$, no continuum of G_m contains both A_i and B_i. If $m<n$ then, by hypothesis and Theorem 5, there exists a collection G_{m+1} of class 1 filling up M such that, if $i \leqq m+1$, no continuum of this collection contains both A_i and B_i. The conclusion of Theorem 6 follows by mathematical induction.

THEOREM 7. *If M is a compact regular curve and d is a positive number and for every collection G of class 1 filling up M there is some continuum of G of diameter greater than or equal to d, then M has an essential continuum of condensation of diameter d.*

Proof. Suppose n is a positive integer such that $1/n$ is $<d$. There exists a finite set K of points of M such that every subcontinuum of M of diameter equal to $1/(3n)$ contains at least one point of K. Suppose N is a subcontinuum of M of diameter d. There exist two points A and B of N at a distance apart equal to d. There exist two subcontinua a and b of N containing A and B respectively and each of diameter $1/3n$. The continua a and b contain points A' and B' belonging to K. The distance from A' to B' is greater than $d-1/n$. It follows that there exist finite sequences $A_1, A_2, \cdots, A_j$ and $B_1, B_2, \cdots, B_j$ such that (1) for each i less than j, A_i and B_i are points of K at a distance apart greater than $d-1/n$, and (2) if N is a subcontinuum of M of diameter d, then, for some i, A_i and B_i are points of N. Hence, by hypothesis and Theorem 6, there exists a number m_n such that if G is any collection of class 1 filling up M, then some continuum of the collection G contains both A_{m_n} and B_{m_n}. Let H denote the set of all continua h such that h contains both A_{m_n} and B_{m_n} and belongs to some collection of class 1 filling up M. Let T denote the common part of all the continua of the set H. Suppose T contains no continuum that contains both A_{m_n} and B_{m_n}. Then T is the sum of two mutually exclusive closed point sets T_A and T_B containing A_{m_n} and B_{m_n} respectively. There exists a finite point set L that separates T_A from T_B in M. Every continuum of the set H contains both A_{m_n} and B_{m_n} and therefore some point of L. Hence, by Theorem 6, there exists a point of L that belongs to every continuum of H and therefore to T. This is impossible. Hence T contains a continuum containing both A_{m_n} and B_{m_n}. Thus for every positive integer n there exists a subcontinuum K_n of M of diameter greater than $d-1/n$ and such that if G is any collection of class 1 filling up M, then some continuum of G contains K_n. There exists an ascending sequence of positive integers $n_1, n_2, n_3, \cdots$ such that the sequence $K_{n_1}, K_{n_2}, K_{n_3}, \cdots$ has a sequential limiting set E. The point set E is a continuous curve of diameter d. If G is any collection of class 1 filling up M, some continuum of G contains E. There exist two points X and Y belonging to E and at a distance apart equal to d. There exists an arc XY lying in E. The arc XY is an essential continuum of condensation of M of diameter d.

It is clear from Example 1 that Theorem 7 does not remain true if the requirement that M be a compact regular curve is replaced by the weaker requirement that it be a compact and webless* continuum.

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* The compact continuum M is said to be *webless* provided it is true that if G_1 and G_2 are two upper semi-continuous collections of mutually exclusive continua filling up a subcontinuum of M and such that each of them is an acyclic continuous curve with respect to its elements, then there does not exist an uncountable subcollection H of G_1 such that no continuum of H is a subset of any continuum of G_2 (cf. P.S.T., p. 355).

THEOREM 8. *If M is a compact and webless continuum and there exists a positive number d such that, if G is any collection of class 1 filling up M, some continuum of G is of diameter more than d, then M has an essential continuum of condensation.*

Proof. Suppose M has no essential continuum of condensation. Then, by Theorem 36 of page 361 of P.S.T. together with Theorem 2 of the present paper, M is a regular curve. Hence, by Theorem 7, it has an essential continuum of condensation of diameter d.

In P.S.T., a subcontinuum K of a compact continuum M was called an essential continuum of condensation of M of type 1 if there exists a non-degenerate subcontinuum T of K such that, if G is a collection of class 1 filling up M, then some continuum of G contains T; and the subcontinuum K of M was called an essential continuum of condensation of M of type 2 if there exist a point P of K and a point O of M distinct from P such that, if G is a collection of class 1 filling up M, then O belongs to the continuum of G that contains P. The first of these definitions is open to the objection* that an essential continuum of condensation of type 1 of a continuum M is not necessarily a continuum of condensation of M. However, by Theorem 2, every essential continuum of condensation of type 1 (in the sense of P.S.T.) of a compact continuum M *contains* a continuum which is a continuum of condensation of M and which is therefore an essential continuum of condensation of M in the sense of the present treatment; and if an essential continuum of condensation of type 1 of the compact continuum M is itself a continuum with no continuum of condensation, then it *is* an essential continuum of condensation of M in the present sense.

It is easy to see that there exists a compact continuum M containing a continuum K which is an essential continuum of condensation of M of type 2 in the sense of P.S.T. but which does not contain more than one point of any continuum of condensation of M. However, by Theorem 8, if the compact and webless continuum M has an "essential continuum of condensation of type 2," then it has an essential continuum of condensation in the present sense.

THEOREM 9. *If M is a compact continuum such that every non-degenerate subcontinuum of M contains uncountably many local separating points† of M,*

* Cf. R. L. Moore, *Fundamental Theorems of Point Set Theory*, The Rice Institute Pamphlet, vol. XXIII (1936), p. 58. The word "non-degenerate" is to be omitted in the statement of Axiom 2 on p. 3.

† A point P is a *local separating point* of M provided some neighborhood V of P exists such that $M \cdot \overline{V} - P$ is separated between some pair of points belonging to the component of $M \cdot \overline{V}$ which contains P. See G. T. Whyburn, *Local separating points of continua*, Monatshefte für Mathematik und Physik, vol. 36 (1929), p. 305, and *Sets of local separating points of a continuum*, Bulletin of the American Mathematical Society, vol. 39 (1933), p. 97.

and H and K are two mutually exclusive closed subsets of M, then there do not exist infinitely many mutually exclusive subcontinua of M each containing both a point of H and a point of K.

Proof. Suppose there exists an infinite sequence α of mutually exclusive subcontinua of M each intersecting both H and K. There exists a subsequence β of α converging to some continuum L that intersects both H and K. There exists a non-degenerate subcontinuum T of L containing no point of $H+K$. By hypothesis L contains an uncountable set Q of local separating points of M belonging to T. By a theorem of Whyburn's,* there exists a point P of Q of Menger order† 2 of M. There exists a domain D with respect to M containing P such that $\overline{D}$ contains no point of H or of K and such that γ, the boundary of D with respect to M, consists of only two points. Since P belongs to the sequential limiting set of β, infinitely many continua of β contain points of D. But each of them contains points belonging to $H+K$ and therefore to $S-D$. Hence each of them intersects γ. But the continua of β are mutually exclusive and γ is a finite point set. This involves a contradiction.

THEOREM 10. *If AB is an arc lying in the compact continuum M and containing uncountably many local separating points of M, then there exists an open subset D of M containing AB such that there are uncountably many points each separating A from B in $\overline{D}$.*

Proof. By the above mentioned theorem of Whyburn's there exists an uncountable subset K of $AB-(A+B)$ such that every point of K is of Menger order 2 of M. Suppose P is a point of K. There exists an open subset I_P of M containing P such that (1) I_P contains neither A nor B, (2) the boundary of I_P consists of two points, (3) I_P-P is the sum of two connected and mutually separated point sets containing A_P and B_P respectively of the boundary, with respect to M, of I_P such that A_P lies between A and P, and B_P lies between B and P, on the arc AB. There exist open subsets H_{AP} and H_{BP} of M containing AA_P and BB_P respectively and such that $\overline{H}_{AP}$ and $\overline{H}_{BP}$ are mutually exclusive. The point set $H_{AP}+I_P+H_{BP}$ is an open subset T_P of M containing AB, and P separates A from B in T_P.

There exists a sequence $D_1, D_2, D_3, \cdots$ of open subsets of M closing down on AB. For each point P of K there exists a number n_P such that $\overline{D}_{n_P}$ is a subset of T_P. The point set K is uncountable. Hence there exist a number m and an uncountable subset L of K such that, if P is any point whatsoever

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* G. T. Whyburn, *Concerning the cut points of continua*, these Transactions, vol. 30 (1928), p. 606.

† Cf. K. Menger, *Grundzuge einer Theorie der Kurven*, Mathematische Annalen, vol. 95 (1925), pp. 272-306.

of L, $n_P=m$. For every point P of L, $\bar{D}_m$ is a subset of T_P and therefore P separates A from B in D_m.

THEOREM 11. *If every non-degenerate subcontinuum of the compact continuum M contains uncountably many local separating points of M, then M has no essential continuum of condensation.*

Proof. Suppose K is a non-degenerate subcontinuum of M. By Theorem 9, and Theorem 8 of Chapter 2 of P.S.T., every subcontinuum of M is a continuous curve. Hence K contains an arc AB. By Theorem 10 there exists a domain D, with respect to M, containing AB and such that there are uncountably many points each separating A from B in D. Let N denote the set of all points that separate A from B in D. The point set N contains a perfect set.* Hence it contains a perfect point set H having no non-degenerate component. Let Q denote the set of all subcontinua q of AB such that (1) q does not contain more than two points of H, but (2) every subcontinuum of AB of which q is a proper subset contains more than two points of H. Let W denote the set of all components x of $M-AB$ such that there are at least two continua of the set Q each containing a limit point of x. Each point set of the set W contains at least one point of $M-D$. Hence, by Theorem 9, W is a finite set. Furthermore if x is a point set of the set W, there do not exist infinitely many continua of the set Q each containing a limit point of x. Hence if L denotes the set of all continua q of Q such that q contains a limit point of some point set of the set W, the set L is finite. Hence there exist two points E and F such that (1) E and F are both continua of the set Q, and (2) the interval EF of AB contains no limit point of any point set of the set W. Let T denote the set of all continua of the set Q which are subsets of EF. For each continuum t of the set T, let g_t denote the component of $M-(EF-t)$ that contains t. Let G_1 denote the set of all continua g_t for all continua t of the set T except E and F. If $\bar{g}_E+\bar{g}_F$ is not a continuum, let G denote the collection whose elements are g_E, g_F and the elements of G_1. If $\bar{g}_E+\bar{g}_F$ is a continuum C, let G denote the collection whose elements are C and the elements of G_1. In the first case G is an arc, and in the second case it is a simple closed curve, with respect to its elements. In neither case does any continuum of G contain the continuum K. Hence M has no essential continuum of condensation.

THEOREM 12. *If the compact and webless continuum M has no essential continuum of condensation, then every non-degenerate subcontinuum of M contains uncountably many local separating points of M.*

Proof. Suppose K is a non-degenerate subcontinuum of M. By hypothesis

* G. T. Whyburn, *Cut points of connected sets and of continua*, these Transactions, vol. 32 (1930), p. 151.

and Theorem 2 there exists a collection G of class 1 filling up M and such that no continuum of the collection G contains K. Let H denote the collection of all the continua of G that intersect K. There exists a free segment t with respect to G whose elements belong to H. But there exists no uncountable set of mutually exclusive non-degenerate subcontinua of M. Hence uncountably many of the elements of t are points of K. But clearly every point of K which is an element of t is a local separating point of M.

Theorem 12 does not remain true on the omission from its hypothesis of the requirement that M be webless. For a plane continuum consisting of a square together with its interior has no essential continuum of condensation and no one of its points separates it locally.

Neither is it true that if the subcontinuum K of the compact and webless continuum M contains no essential continuum of condensation of M, then K contains uncountably many local separating points of M. Consider the following example.

EXAMPLE. In a Cartesian plane S let OX and OY denote the axes of coordinates. If h is a positive number and E and F are points of OX such that x_E, the abscissa of E, is less than x_F, the abscissa of F, then by the *R-set of altitude h whose base is EF* will be meant the continuous curve M described in Example 2 on page 347 of P.S.T. except that A, B, A', and B' denote (x_E, h), (x_F, h), E, and F, respectively, instead of $(0, 1)$, $(1, 1)$, $(0, 0)$, and the point which was erroneously labelled $(0,1)$ instead of $(1,0)$ in the first sentence of the description of Example 1 on page 346. In accordance with this new meaning of the letters A, B, A', and B', $A_1'B_1'$, $A_2'B_2'$, $A_3'B_3'$, $\cdots$ of course denote subintervals of EF. For each n, the interval $A_n'B_n'$ will be called the nth Cantor interval associated with the R-set of altitude h whose base is EF.

Let M_1 denote the R-set of altitude 1 whose base is the interval whose endpoints are $(0, 0)$ and $(1, 0)$. For each n, let r_{1n} denote the nth Cantor interval associated with M_1, and let M_{1n} denote the R-set of altitude $1/(n+2)$ whose base is r_{1n}. Let W_1 denote the collection of R-sets M_{11}, M_{12}, M_{13}, $\cdots$. For each m, let M_{1nm} denote the R-set of altitude $1/(n+2)(m+2)$ whose base is the mth Cantor interval associated with M_{1n}. Let W_2 denote the collection of all R-sets M_{1nm} for all sensed pairs of positive integers n and m. This process may be continued indefinitely. Thus there exists an infinite sequence $W_1, W_2, W_3, \cdots$ such that (1) for each k, W_k is the collection of all R-sets $M_{1n_1n_2\cdots n_k}$ for all sensed sets of k positive integers $n_1, n_2, \cdots, n_k$, (2) for each k, and sensed set of k positive integers $n_1, n_2, n_3, \cdots, n_k$, $M_{1n_1n_2n_3\cdots n_kn_{k+1}}$ is the R-set of altitude $1/(n_1+2)(n_2+2)\cdots(n_{k+1}+2)$ whose base is the (n_k+1)th Cantor interval associated with $M_{1n_1n_2\cdots n_k}$, (3) for each n, M_{1n} is as described above.

Let M denote the point set which is the sum of M_1 and all the R-sets of the collections W_1, W_2, W_3, $\cdots$. Let K denote the straight line interval whose extremities are (0, 0) and (1, 0). The continuum K contains no local separating point of M. Nevertheless, though it is a continuum of condensation of M, it contains no essential continuum of condensation of M.

THEOREM 13. *If M is a compact continuum and W is a set of collections of class* 1 *each filling up M, and G is the collection of all continua g such that, for some point P of M, g is the component containing P of the common part of all continua x such that x contains P and belongs to some collection of the set W, then G is itself a collection of class* 1.

Proof. The collection G is an upper semi-continuous collection of mutually exclusive continua. Suppose there exists a continuum K of elements of G which is an essential continuum of condensation of G. There exists a collection Q of the set W such that no continuum of the collection Q contains every continuum of the set K. For each continuum q of the collection Q let t_q denote the set of all continua of G which are subsets of q. Each set t_q is a continuum of elements of G and the collection of all sets t_q for all continua q of the collection Q is an upper semi-continuous collection T of mutually exclusive continua of elements of G. But, with respect to its elements, T is a continuum with no continuum of condensation and K is not a subset of any set of the collection T. This involves a contradiction. Hence G belongs to class 1.

With the help of Theorem 13 the following theorem may be easily established.

THEOREM 14. *If Q is the collection of all continua that have no essential continua of condensation, then every compact continuum has Q-atomic* subsets.*

DEFINITION.† The continuum M is said to have *property N* if for every positive number e there exists a finite set G of mutually exclusive non-degenerate subcontinua of M such that every subcontinuum of M of diameter greater than e contains some continuum of the set G.

THEOREM 15. *If a compact continuum has property N, it is a regular curve.*

Proof. Suppose M is a compact continuum with property N. It is clear

* If M is a continuum and Q is a topological collection of continua, then the subcontinuum K of M is said to be a *Q-atomic subset* of M provided there exists an upper semi-continuous collection G of mutually exclusive continua filling up M such that, with respect to its elements, G belongs to Q and such that (1) K is an element of G and (2) if H is any other upper semi-continuous collection of mutually exclusive continua filling up M such that H is, with respect to its elements, a continuum of the collection Q, then every continuum of the collection G is a subset of some continuum of the collection H, or in other words, each continuum of H is either an element of G or the sum of two or more elements of G. (Cf. The Rice Institute Pamphlet, loc. cit.)

† The Rice Institute Pamphlet, loc. cit.

that there do not exist a positive number d and infinitely many mutually exclusive subcontinua of M all of diameter greater than d. Hence M is a continuous curve. Suppose A and B are two distinct points of M. There exists a finite set H of mutually exclusive non-degenerate subcontinua of M such that every subcontinuum of M of diameter greater than one half the distance from A to B contains some continuum of the set H. There exists a finite point set T such that each continuum of the set H contains only one point of T and each point of T belongs to some continuum of the set H and is distinct from A and from B. Every subcontinuum of M that contains A and B contains some continuum of the set H and therefore some point of T. Hence, since M is a continuous curve, T separates A from B in M. Therefore M is a regular curve.

THEOREM 16. *In order that the regular curve M should fail to have property N it is necessary and sufficient that it should contain an arc AB such that every segment of AB contains an endpoint of an arc in M which has only its endpoints in common with AB.*

Proof. That this condition is sufficient was shown in The Rice Institute Pamphlet. The necessity will now be shown. Suppose the regular curve M does not have property N. Then there exists a positive number e such that if H is any finite set of mutually exclusive non-degenerate subcontinua of M, there exists a subcontinuum of M with diameter more than e but containing no continuum of the set H. There exists, in M, an arc A_1B_1 of diameter greater than e. There exists a finite set H_1 of mutually exclusive intervals of A_1B_1 such that every interval of A_1B_1 of diameter more than 1 contains an arc of the set H_1. There exists in M an arc A_2B_2 of diameter more than e and containing no arc of the set H_1. There exists a set H_2 of mutually exclusive intervals of A_2B_2 such that every interval of A_2B_2 of diameter more than $1/2$ contains an arc of the set H_2. Let K denote the set of all arcs of the set H_2 which are not subsets of any arc of the set H_1. For each arc k of the set K there exists an interval h_k of k intersecting no arc of the set H_1. Let K' denote the set of all arcs h_k for all arcs k of the set K, and let T_1 denote the set of all arcs of H_1 which contain no arc of H_2. Let H_2' denote the set $T_1+(H_2-K)+K'$. The set H_2' is a finite set of mutually exclusive arcs. Hence there exists in M an arc A_3B_3 of diameter more than e and containing no arc of the set H_2'. The arc A_3B_3 contains no arc either of the set H_1 or of the set H_2. This process may be continued indefinitely. It follows that there exist two infinite sequences $A_1B_1, A_2B_2, \cdots$ and $H_1, H_2, \cdots$ such that, for each n, (1) A_nB_n is an arc, of diameter more than e, lying in M, (2) H_n is a finite set of mutually exclusive intervals of A_nB_n such that every interval of A_nB_n of length more than $1/n$ contains an interval of the set H_n, (3) the arc $A_{n+1}B_{n+1}$ contains no in-

terval of any one of the sets $H_1, H_2, \cdots, H_n$. There exists an infinite sequence of positive integers $n_1, n_2, n_3, \cdots$ such that the sequence $A_{n_1}B_{n_1}, A_{n_2}B_{n_2}, \cdots$ has a non-degenerate sequential limiting set K. Since M is a regular curve, the continuum K is a regular curve and therefore it contains an arc AB. Suppose E and F are two points of AB. There exist points E_1, E_2, E_3, E_4, E_5, and E_6 lying on AB in the order $EE_1E_2E_3E_4E_5E_6F$. There exist two mutually exclusive connected point sets D and R containing E_3 and E_6 respectively and such that (1) D and R are both domains with respect to M, (2) $D\cdot(AB)$ is a subset of the segment of AB whose endpoints are E_2 and E_4 and $R\cdot(AB)$ is a subset of the one whose endpoints are E_5 and F. There exists a number i such that $A_{n_i}B_{n_i}$ contains neither of the intervals E_1E_2 and E_4E_5 of AB but contains points X and Y belonging to D and R respectively and distinct from E_3 and E_6 respectively. There exist arcs XE_3 and YE_6 lying in D and R respectively. The continuum $XE_3+A_{n_i}B_{n_i}+YE_6$ contains an arc E_3ZE_6 with endpoints at E_3 and E_6. There exist points O_1 and O_2 lying respectively on the segments E_1E_2 and E_4E_5 of AB and such that E_3ZE_6 contains neither O_1 nor O_2. Let W denote the common part of E_3ZE_6 and the point set which is the sum of the intervals AO_1 and O_2B of AB. The point set W is closed and contains E_6. Hence there is a first point of the set W in the order from E_3 to E_6 on E_3ZE_6. Let L denote this point, and let Y denote the last point of the interval O_1O_2 of AB in the order from E_3 to L on the interval E_3L of E_3ZE_6. The interval YL of E_3ZE_6 is an arc in M with one endpoint on the segment EF of AB and having only its endpoints in common with AB.

THEOREM 17. *If AB is an arc and, for each positive integer n, H_n is a finite set of points $X_{1n}, X_{2n}, \cdots, X_{m_nn}$ lying on AB in the order indicated, X_{1n} and X_{m_nn} being A and B respectively, and H_n being a subset of H_{n+1}, and Q_n is a set of arcs $X_{1n}Z_{1n}X_{2n}$, $X_{2n}Z_{2n}X_{3n}, \cdots, X_{m_n-1,n}Z_{m_n-1,n}X_{m_nn}$ each having only its endpoints in common with AB, and every point of AB is a limit point of the point set obtained by adding together all the arcs of the sets $Q_1, Q_2, Q_3, \cdots$ and M is a regular curve containing that point set, then AB is an essential continuum of condensation of M.*

Theorem 17 may be established by an argument similar to that employed on pages 61 and 62 of vol. XXIII of The Rice Institute Pamphlet to prove that a certain straight line interval AB is an essential continuum of condensation of a certain continuum M.

THEOREM 18. *If the compact and webless continuum M does not have property N, there exists an upper semi-continuous collection G of mutually exclusive continua filling up M such that, with respect to its elements, G is a continuum with an essential continuum of condensation.*

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Proof. Suppose M has no essential continuum of condensation. Then, by Theorem 36 of Chapter V of P.S.T., it is a regular curve. Hence, by Theorem 16 of this paper and Theorem 7 of page 67 of vol. XXIII of The Rice Institute Pamphlet, there exists an arc AB lying in M such that if t is any segment of AB and K is a closed subset of $M-t$, there is an arc lying in M and having in common with AB only its endpoints which belong to t. It easily follows that there exist two infinite sequences $H_1, H_2, H_3, \cdots$ and $Q_1, Q_2, Q_3, \cdots$ such that, for each n, (1) H_n is a set of 2^n points $X_{1n}, X_{2n}, \cdots, X_{2^n,n}$ lying on AB in the order indicated, (2) for each odd number i less than 2^n there are only four points of the set H_{n+1} between the points X_{in} and $X_{i+1,n}$ of the set H_n, (3) Q_n is a set of arcs $X_{1n}Z_{1n}X_{2n}, X_{3n}Z_{2n}X_{4n}, X_{5n}Z_{3n}X_{6n}, \cdots, X_{2^n-1,n}Z_{2^n-1,n}X_{2^n,n}$ each lying in M and having only its endpoints in common with AB and each of diameter less than $1/n$. For each n, let I_n denote the set of all points X such that X lies on AB between the endpoints of some arc of the set H_n. The set I_n is the sum of 2^{n-1} segments of AB and I_{n+1} is a subset of I_n. Let T denote the common part of all the point sets of the infinite sequence $I_1, I_2, I_3, \cdots$. The point set T is closed, totally disconnected, and uncountable. For each point P of $AB-T$, let k_P denote the component of $AB-T$ that contains P. Let K denote the set of all the point sets k_P for all points P of $AB-T$. Let G denote the collection whose elements are the continua of the collection K and the points of M that belong to no continuum of the collection K. Let L denote the collection whose elements are the continua of G which are subsets of the arc AB and, for each n and i ($i \leqq 2^n$), let X'_{in} denote the continuum of the collection G that contains X_{in}. For each n and odd number i less than 2^n, let $X'_{in}Z_{in}X'_{i+1,n}$ denote the set whose elements are X'_{in}, $X'_{i+1,n}$ and the points of $X_{in}Z_{in}X_{i+1,n}$ which do not belong to AB. For each n, let H'_n denote the collection whose elements are $X_{1n}', X_{2n}', \cdots, X'_{2^n,n}$ and let Q'_n denote the collection whose elements are $X_{1n}'Z_{1n}X_{2n}', X_{3n}'Z_{2n}X_{4n}', \cdots, X'_{2^n-1,n}Z_{2^n-1,n}X'_{2^n,n}$. Let A' and B' denote the continua of G that contain A and B respectively. In the space whose "points" are the elements of G, (1) L is an arc $A'B'$ with endpoints at A' and B', (2) for each n, H'_n is a set of points $X_{1n}', X_{2n}', \cdots, X'_{2^n,n}$ lying on $A'B'$ in the order indicated, X_{1n}' and $X'_{2^n,n}$ being A' and B' respectively, (3) H'_n is a subset of H'_{n+1}, (4) Q'_n is a collection of arcs each having only its endpoints in common with $A'B'$ and, finally, G is a regular curve and each point of $A'B'$ is a limit point of the point set $H'_1+H'_2+H'_3+\cdots$. It follows, by Theorem 17, that $A'B'$ is an essential continuum of condensation of G.

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CONCERNING CERTAIN TOPOLOGICALLY FLAT SPACES†

BY

F. BURTON JONES

In any space satisfying Axioms 0, 1, 2, 3, 4, and 5 of R. L. Moore's *Foundations of Point Set Theory*‡ a large body of Plane Analysis Situs theorems holds true. Nevertheless, Moore has an example§ of a completely separable space satisfying these axioms which because of its "hilly" character is not homeomorphic with a subset of a plane or a sphere. In this paper Axiom 5 is replaced by an axiom, Axiom 5_1^*, which is not satisfied by spaces of this "hilly" character. As a result, any completely separable (or metric) space satisfying Axioms 0–4 and 5_1^* is homeomorphic with a subset of a plane or a sphere. However, in the presence of Axioms 0–4 neither of the Axioms 5 and 5_1^* follows from the other.

Axiom 5_1‖ of *Foundations* suggests the following two axioms:

AXIOM 5_1^1. *If P is a point of region R, there exists in R a domain D containing P such that the boundary of D is compact.*¶

AXIOM 5_1^2. *If P is a point of a region R, there exists in R a domain D containing P such that the boundary of D is connected.*††

There is, however, a certain amount of similarity between these last two axioms. For suppose that a space satisfies Moore's Axioms 0, 1, and 2 and

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‡ American Mathematical Society Colloquium Publications, volume XIII, New York, 1932. Hereafter, in this paper, this book will be referred to as *Foundations*.

§ This example of Moore's, although unpublished, has been well known among his students for some four or five years.

‖ If P is a point of a region R, there exists a connected domain D containing P and bounded by a compact continuum T such that $D+T$ is a subset of R.

¶ A space satisfying Axiom 5_1^1 is said to be *locally peripherally compact.*

†† A space satisfying Axiom 5_1^2 is said to be *locally peripherally connected.*

Axiom 5_1^1.† Then if P is a point of a region R, there exists a region R' containing P and lying together with its boundary in R. And by Axiom 5_1^1 there exists in R' a domain D' containing P whose boundary β is compact. But β may be covered by a finite collection $D_1, D_2, D_3, \cdots, D_j$ of connected domains such that for each n, $n \leqq j$, $\overline{D}_n$ is a subset of $R-P$.‡ Let D denote the component of $R-\sum \overline{D}_n$ which contains P. Thus the following theorem holds true and will be used hereafter as

AXIOM 5_1^*. *If P is a point of a region R, there exists in R a domain D containing P whose boundary is a subset of the sum of a finite number of continua lying in $R-D$.*

In a sense Axiom 5_1^* is common to both Axioms 5_1^1 and 5_1^2 in that it follows as a theorem from either of them in the presence of Axioms 0, 1, and 2.

Now let S denote a space satisfying Moore's Axioms 0, 1, and 2.

THEOREM A. *If Axiom 5_1^* holds true at each of two distinct points A and B of S and no point of S separates A from B, then there exists in S a simple closed curve J containing A and B.*§

Proof. Let AB denote an arc from A to B.‖ If R is a region containing A but not B, suppose that there exists no point O of $AB \cdot R$ such that no point X of the interval AO of AB separates A from O in R. Then there exists a sequence α of points of AB which converge to A such that if X and Y are points of α and X follows Y in α, then X separates A from Y in R. Hence, for each point X of α except the first, $R-X$ is the sum of two domains U_X and V_X such that V_X contains A and U_X contains every point of α preceding X in α. But no point separates A from B; so for each point X of α, there exists

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† Axioms referred to are stated either in the text of this paper or in *Foundations*.

AXIOM 0. *Every region is a point set.*

AXIOM 1. *There exists a sequence $G_1, G_2, G_3, \cdots$ such that (1) for each n, G_n is a collection of regions covering S; (2) for each n, G_{n+1} is a subcollection of G_n; (3) if R is any region whatsoever, X is a point of R, and Y is a point of R either identical with X or not, then there exists a natural number m such that if g is any region belonging to the collection G_m and containing X then $\overline{g}$ is a subset of $(R-Y)+X$; (4) if $M_1, M_2, M_3, \cdots$ is a sequence of closed point sets such that, for each n, M_n contains M_{n+1} and, for each n, there exists a region g_n of the collection G_n such that M_n is a subset of $\overline{g}_n$, then there is at least one point common to all the point sets of the sequence $M_1, M_2, M_3, \cdots$.*

AXIOM 2. *If P is a point of a region R there exists a non-degenerate connected domain containing P and lying wholly in R.*

‡ If M denotes a point set, $\overline{M}$ denotes the point set consisting of M and all the limit points of M.

§ See Theorem 40 of Chapter II of *Foundations* and G. T. Whyburn, *On the cyclic connectivity theorem*, Bulletin of the American Mathematical Society, vol. 37 (1931), pp. 429–433.

‖ Theorem 1 of Chapter II of *Foundations*: *If A and B are distinct points of a connected domain D, there exists a simple continuous arc from A to B that lies wholly in D.*

an arc from A to B lying in $S-X$, and in this arc there exists a segment $\widehat{PE}$ lying in R such that $AB \cdot PE = P$ and E belongs to the boundary of R.† Therefore, if for no X of α is P identical with A, there exist a subsequence $X_1, X_2, X_3, \cdots$ of α and a sequence $P_1E_1, P_2E_2, P_3E_3, \cdots$ of arcs such that for each n (1) $AB \cdot P_nE_n = P_n$, (2) E_n belongs to the boundary of R, (3) $\widehat{P_nE_n}$ is a subset of R, and (4) X_{n+1} separates $\sum_{i=1}^{n} \widehat{P_iE_i}$ from A in R. But (1), (2), and (3) lead to a contradiction of (4), for since there exists in R a domain D containing A whose boundary is a subset of the sum of a finite collection Δ of continua lying in $R-D$, there exists a continuum H of Δ lying in $R-A$ such that H does not contain more than a finite number of the points $X_1, X_2, X_3, \cdots$ but for infinitely many values of n, H contains a point of P_nE_n. Therefore, either there exists an arc PE which is a subset of an arc from A to B such that $PE \cdot AB = P = A$, or if R is a region containing A but not B, there exists a point O of $AB \cdot R$ such that no point of the interval AO of AB separates A from O in R.

Suppose the latter is true. Let $R_1, R_2, R_3, \cdots$ be a monotonic sequence of regions closing down on A and $O_1, O_2, O_3, \cdots$ be a sequence of points of AB such that, for each n, (1) the interval AO_n of AB is a subset of R_n, (2) no point X of AO_n separates A from O_n in R_n. For each n and each point X of AO_n, there exists an arc T_{xn} lying in R_n and having only its end points in common with AB such that the segment S_{xn} of AB between these end points contains X and is a subset of R_n. For each n, there exists a finite collection of the segments S_{xn} covering the arc $O_{n+1}O_{n+2}$ of AB. But each $S_{xn}+T_{xn}$ is a simple closed curve lying in R_n. Hence there exists a sequence $J_1, J_2, J_3, \cdots$ of simple closed curves such that (1) for each n, J_n and J_{n+1} have at least two points in common, and (2) if R is a region containing A there exists a number δ such that if $n > \delta$, J_n is a subset of R. By Theorem 35 of Chapter II of *Foundations* there exists in $\sum J_n$ a simple closed curve J_A containing A and a segment of AB.

If, on the other hand, there exists an arc PE which is a subset of an arc PB from A to B such that $PE \cdot AB = P = A$, then there exists in PB an arc PX_1 such that P and X_1 are the only points PX_1 has in common with AB. But $PX_1 + AX_1$(of AB) forms a simple closed curve containing A and some segment of AB.

So in either case there exists a simple closed curve J_A containing A and a segment $\widehat{OF}$ of AB. Likewise, there exists a simple closed curve J_B containing B and a segment $\widehat{YE}$ of AB. Now for each point X of the interval FY of AB, there exists an arc T_x such that T_x has only its end points in common with AB, and the segment S_x of AB lying between these end points contains X.

† If AB is an arc with end points A and B, $\widehat{AB}$ is used to denote $AB-(A+B)$.

Some finite collection of these segments S_x covers the arc FY. Since every S_x+T_x is a simple closed curve, there exists a sequence of simple closed curves $J_1=J_A, J_2, \cdots, J_k=J_B$ such that for each n, $n<k$, J_n contains at least two points of J_{n+1}. By Theorem 35 of Chapter II of *Foundations* there exists in the sum of these simple closed curves a simple closed curve J containing A and B.†

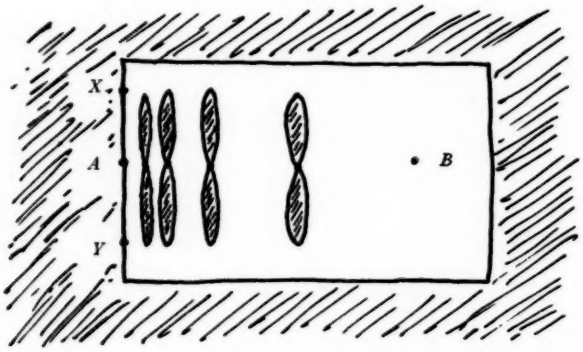

FIGURE 1

The example indicated in Figure 1 is a subset of a plane with the shaded portions removed. Except for those points of the arc XAY different from A, the boundary points of these shaded portions are not removed. It will be easily seen that the result is a connected, connected im kleinen inner limiting subset of the plane and hence satisfies Axioms 0, 1, and 2. Also Axiom 5_1* is satisfied at every point except A. As a matter of fact, the space is locally compact at every point except A. It is also true in this space that if P is a point and R is a region containing A, there exists in R an arc (or a simple closed curve) separating A from P. The space contains no cut points. Yet there exists no simple closed curve containing A and B.

† This argument may be slightly modified to establish a proposition somewhat more general than Theorem A.

THEOREM A'. *If* (1) *S is a space satisfying R. L. Moore's Axioms* 0, 1, *and* 2, (2) *neither of the distinct points A and B of S is a cut point nor a local end point of S, and* (3) *no point separates A from B in S, then A and B lie on a simple closed curve in S.* (See G. T. Whyburn, *The cyclic and higher connectivity of locally connected spaces*, American Journal of Mathematics, vol. 53 (1931), pp. 427–442.)

The modification may be outlined as follows: In the first paragraph of the proof, if A is not a local end point, $\prod V_X$ contains a point E' distinct from A which belongs to the component of R that contains A. Let AE' denote an arc from A to E' in R and let $E'B$ denote an arc from E' to B in $S-A$. Then $AE'+E'B$ contains an arc PE which is a subset of an arc from A to B such that $PE\cdot AB=P=A$. This takes us up to the last sentence of the first paragraph of the proof, which remains unchanged from there on.

Suppose now that, in addition to Axioms 0–2, S satisfies R. L. Moore's Axiom 4† and Axiom 5_1*.

THEOREM B. *The space S is either acyclic or contains no cut point.*

Proof. Suppose that S contains a simple closed curve J. Then it is evident from Axiom 4 that S is connected. Suppose that some point X separates S. The point X does not belong to J, for it is clear from Axiom 4 that no point of a simple closed curve separates S. Let AX denote an arc with end points A and X such that $AX \cdot J = A$. Let M denote the set of cut points of S that belong to AX and let B denote the first point of $\overline{M}$ in the order from A to X. The point B is a cut point of S. For if it is not, then B is a sequential limit point of a sequence $B_1, B_2, B_3, \cdots$ of M and at the same time lies on a simple closed curve C containing a segment of AX which contains B; hence, C contains points of the sequence $B_1, B_2, B_3, \cdots$ which contradicts Axiom 4. But since the segment AB of AX contains no point of M, no point separates A from B in X and by Theorem A there exists a simple closed curve containing A and B which again contradicts Axiom 4.

It is not the purpose of this paper to treat the acyclic case;‡ so we shall at this point assume R. L. Moore's Axiom 3.§

1. CONSEQUENCES OF AXIOMS 0−4 AND 5_1*.

THEOREM 1. *If A and B are distinct points, there exists a simple closed curve containing A and B.*

THEOREM 2. *If A and B are distinct points, there exists a simple closed curve separating A from B.*

Proof. There exists a simple closed curve J such that J is the sum of two arcs AXB and AYB. Since $S-J$ is the sum of two connected domains I and E each having J for its boundary, there exist two arcs $X_1O_1Y_1$ and $X_2O_2Y_2$ such that (1) the points X_1 and X_2 lie on the segment AXB and the points Y_1 and Y_2 lie on the segment AYB, and (2) the segment $X_1O_1Y_1$ is a subset of I and the segment $X_2O_2Y_2$ is a subset of E. The simple closed curve formed by the sum of the four arcs $X_1O_1Y_1$, $X_2O_2Y_2$, X_1X_2(of AXB), and Y_1Y_2(of AYB) separates A from B in S.

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† If J is a simple closed curve, $S-J$ is the sum of two mutually separated connected point sets such that J is the boundary of each of them.

‡ Such a space if connected would be a *Regular (Menger) Curve*, and these curves have been studied considerably already. See *Foundations* for the definition and references on *Regular (Menger) Curves*.

§ If O is a point, $S-O$ is connected.

THEOREM 3. *If P is a point not belonging to the closed and compact point set H, there exists a simple closed curve separating P from H.*

Proof. For each point X of H there exists a simple closed curve separating P from X. Since H is closed and compact, there exists a finite collection of simple closed curves $J_1, J_2, \cdots, J_n$ such that if X is any point of H some one of them separates P from X. For each integer i, $i \leqq n$, let T_i denote an arc from J_1 to J_i lying in $S-P$ and let M denote the compact continuum $J_1+J_2+\cdots+J_n+T_1+T_2+\cdots+T_n$. The continuum M separates P from $H-H\cdot M$. Now for each point X of M there is a simple closed curve separating X from P. Let $C_1, C_2, \cdots, C_j$ denote a finite collection of these whose interiors with respect to P as the point at infinity† cover M. By Theorem 13 of Chapter III of *Foundations*, $C_1+C_2+\cdots+C_j$ contains a simple closed curve separating P from M and hence, separating P from H.

With the help of Theorem 3, the arguments of R. L. Moore with slight modifications prove that a number of the theorems of Chapter IV of *Foundations* hold true in this space. Of these the following five theorems will be used.

THEOREM 4. *If the points A and B belong to different components of the closed and compact point set M, there exists a simple closed curve separating A from B and containing no point of M* (Theorem 10 of Chapter IV of *Foundations*).

THEOREM 5. *If α and β are two connected point sets and neither of the two mutually exclusive closed and compact point sets H and K separates α from β, then $H+K$ does not separate α from β* (Theorem 16).

THEOREM 6. *If the common part of the closed and compact point sets H and K is a continuum and neither H nor K separates the point A from the point B, then $H+K$ does not separate A from B* (Theorem 18).

THEOREM 7. *No arc separates S* (Theorem 19).

THEOREM 8. *If the points A and B are separated from each other by the closed and compact point set M then they are separated from each other by a continuum which is a subset of M and which contains no proper subset that separates A from B* (Theorem 24).

THEOREM 9. *If the compact continuum K does not separate the point A from the point B and G is a finite collection of compact continua such that* (1) *the common part of any two elements of G is a subset of K, and* (2) *if H is any element of G, H does not separate A from B and $H\cdot K$ is either connected or vacuous, then $K+G^*$ does not separate A from B.*

† For the definition of certain terms and phrases used without definition in this paper the reader is referred to *Foundations*.

Theorem 9 may be established by a finite number of applications of Theorems 5 and 6.

THEOREM 10. *If AXB is an arc and J is a simple closed curve separating A from B, then $J+AXB$ contains a simple closed curve J_1 separating A from B such that $J_1 \cdot AXB$ is connected.*

Proof. Let A_1 and B_1 denote the first and last points respectively that the arc AXB has in common with J. Let AYB denote an arc from A to B not containing any point of the interval A_1B_1 of AXB. Let G denote the collection of all simple closed curves C in $J+AXB$ such that $C \cdot AXB$ is connected and $C \cdot AYB$ is not vacuous. The collection G is finite. Let K denote $A_1B_1+(J-J \cdot G^*)$.† The point set K is a continuum containing no point of AYB; hence, K does not separate A from B. Since the common part of any two elements of G is a subset of K and the common part of K with an element of G is connected, if no simple closed curve of the collection G separated A from B, then by Theorem 9, $K+G^*$ would not separate A from B. But $K+G^*$ contains J and separates A from B. Therefore some element J_1 of G separates A from B.

DEFINITIONS. Suppose that P is a point of a domain Q, and Δ is a collection of continua whose sum separates P from the boundary of Q such that each continuum of Δ lies in a component of $Q-P$ whose boundary contains P but no component of $Q-P$ contains more than one element of Δ. Then (1) Δ is said to be *minimal*‡ with respect to Q and P, (2) Δ is said to *minimally separate* P from the boundary of Q, and (3) if D is a domain containing P whose boundary is a subset of Δ^*, Δ is said to *surround D minimally* with respect to Q and P.

THEOREM 11. *If Δ is a collection of continua which surrounds a connected domain D minimally with respect to a domain Q and a point P, then* (a) *each component of $Q-P$ whose boundary contains P, contains one and only one element of Δ, and* (b) *no component of $D-P$ has boundary points in more than one element of Δ.*

Proof. Consider (a). By definition, if C is a component of $Q-P$ whose boundary contains P, then C contains not more than one element of Δ. But since $S-P$ is connected, the boundary of C also contains a point of the boundary of Q; hence C must contain at least one element of Δ. Therefore C contains one and only one element of Δ.

† If G denotes a collection of point sets, G^* will be used to denote the sum of the elements of G.

‡ The term *minimal* is used here because if Δ were finite no collection of fewer elements would separate P from the boundary of Q.

Next consider (b). Suppose that there exists a component I of $D-P$ which has boundary points in more than one element of Δ. Let C denote the component of $Q-P$ which contains I. Then C contains more than one element of Δ. But since D is connected, the boundary of C contains P. This is a contradiction of (a).

THEOREM 12. *If P is a point of a domain Q, not S, there exist in Q a finite collection Δ of continua and a connected domain D containing P such that Δ surrounds D minimally with respect to Q and P.*

Proof. Let D_1 denote a domain lying in Q and containing P whose boundary is a subset of the sum of the elements of a finite collection Δ_1 of continua lying in $Q-D_1$. Let N denote the sum of all the continua of Δ_1 which lie in components of $Q-P$ whose boundaries contain P. If H and K are components of N lying in the same component C of $Q-P$, let T denote an arc in C from a point of H to a point of K. Let Δ denote the collection of all components of the point set obtained by adding N to the sum of the arcs T. The collection Δ minimally separates P from the boundary of Q. Let D denote the component of $Q-\Delta^*$ which contains P. Then the boundary of D is a subset of Δ^* and hence Δ surrounds D minimally with respect to Q and P.

DEFINITION. If a domain is connected and contains one of the complementary domains of each simple closed curve lying in it, then it is said to be *simply connected.*

DEFINITION. If D is a connected domain and D contains one of the complementary domains of each simple closed curve lying in $\overline{D}$, then $\overline{D}$ is said to be *simply connected* and D is said to be *strongly simply connected.*

THEOREM 13. *Space is simply connected.*

THEOREM 14. *If D is a connected domain and there exist a collection Δ of continua, a domain Q, and a point P of D, such that Δ surrounds D minimally with respect to Q and P, then D is simply connected.*

Proof. Suppose, on the contrary, that there exists in D a simple closed curve J such that neither of its complementary domains, I and E, is a subset of D. Since J lies in D, both I and E contain points of D and, therefore, points of the boundary β of D. Let T denote an arc lying in $S-P$ irreducible from $\beta \cdot I$ to $\beta \cdot E$. The segment T intersects J and, hence, is a subset of D. Therefore T lies in $Q-P$ and contains points of two different continua of Δ. This contradicts Theorem 11.

THEOREM 15. *If P is a point of a domain Q, there exists in Q a simply connected domain D containing P.*

Theorem 15 is a consequence of Theorems 12, 13, and 14.

THEOREM 16. *If D is a connected domain, every component of $S-\overline{D}$ is strongly simply connected.*

Proof. Let C denote a component of $S-\overline{D}$. Let J denote a simple closed curve lying in $\overline{C}$, and let I denote the interior of J with respect to a point ω of D. Suppose that X is a point of $\overline{D}$ and belongs to I. Then $X+I$ is a connected point set containing ω and a point of I but no point of J. This is a contradiction.

THEOREM 17. *If D is a complementary domain of a locally compact continuous curve M and P is a point of the boundary of D, then $K=D+P$ is a connected, connected im kleinen inner limiting set.*

Proof. It is evident that K is a connected inner limiting set. It is furthermore evident that K is connected im kleinen at every one of its points with the possible exception of P. Suppose that K is not connected im kleinen at P. Then there exists a simply connected domain Q containing P such that (1) the component C of $Q\cdot K$ that contains P is not open with respect to K at the point P, and (2) $Q\cdot M$ is compact. Hence there exists a sequence α of points of K converging to P such that each point of α belongs to some component of $Q\cdot K$ but not to C and no two points of α belong to the same component of $Q\cdot K$. There exists a point O of K not belonging to Q, for otherwise $Q\cdot K=K$ would be connected and open with respect to K at P. For each point X of α let XO denote an arc from X to O lying in D, and let Y denote the first point in the order from X to O that XO has in common with the boundary of Q.

Now let Q_1 denote a domain lying in Q and containing P such that β_1, the boundary of Q_1, is a subset of the sum of a finite collection Δ of continua lying in $Q-Q_1$. For infinitely many points X of the sequence α the interval XY of the arc XO contains a point of some one of the continua of the finite collection Δ. Hence there exists a continuum H of Δ such that for infinitely many points X of α the arc XY contains a point of H. Now let D_H and D_P denote connected domains lying in Q and containing H and P respectively such that $\overline{D}_H\cdot\overline{D}_P=0$.

For each point X of α such that X lies in D_P and the arc XY contains some point of H, let TX denote an arc lying in D_P such that T belongs to M but the segment TX is a subset of D, and let Z denote a point of $XY\cdot H$. Now since D_H lies in Q and contains points of more that one component of $Q\cdot K$, D_H contains some point of M. For each Z let ZW denote an arc lying in D_H such that the segment ZW is a subset of Q and W is a point of M. Thus for some infinite subsequence $x_1, x_2, x_3, \cdots$ of α there correspond sequences

of points $T_1, T_2, T_3, \cdots$ and $W_1, W_2, W_3, \cdots$ such that (1) the points $T_1, T_2, T_3, \cdots$ belong to $D_P \cdot M$, (2) the points $W_1, W_2, W_3, \cdots$ belong to $D_H \cdot M$, (3) for each n, $T_nX_n+X_nZ_n+X_nW_n$ contains an arc T_nW_n lying in Q, and (4) if $m \neq n$, $\widehat{T_mW_m}$ and $\widehat{T_nW_n}$ are mutually exclusive and lie in different components of $Q \cdot K$. Since M is a continuous curve, $Q \cdot M$ is compact and $\overline{D}_H \cdot \overline{D}_P = 0$, there exist two mutually exclusive connected subsets d_P and d_H of $Q \cdot M$ which are open with respect to M such that for infinitely many values of n and in particular for the three different values a, b, and c of n, T_n and W_n lie in d_P and d_H respectively. Let T_aT_b and T_aT_c denote arcs lying in d_P and W_aW_b and W_aW_c denote arcs lying in d_H. There exist three arcs AO_aB, AO_bB, and AO_cB lying in $T_aT_b+T_aW_a+W_aW_b$, $T_aT_b+T_bW_b+W_aW_b$, and $T_aT_c+T_cW_c+W_aW_c$ respectively and having only their end points A and B in common. By Theorem 5 of Chapter III of *Foundations*, the sum of one pair of these three arcs, say AO_aB and AO_cB, forms a simple closed curve J lying in Q and whose interior I with respect to a point ω of the boundary of Q contains the other segment, $\widehat{AO_bB}$. Hence I contains the component C of $Q \cdot K$ which contains $\widehat{T_bW_b}$, for C contains no point of J. However, C has limit points in the boundary of Q. But since Q is simply connected, $\overline{I}$ is a subset of Q. This is a contradiction, and K is connected im kleinen at P.

THEOREM 18. *If D is a complementary domain of a locally compact continuous curve, and P is a point of the boundary of D, then P is accessible from D.*

Theorem 18 follows immediately from Theorems 1 and 10 of *Foundations* and Theorem 17.

A number of the intuitive propositions of the plane concerning abutting and crossing arcs hold true. Although some of these will be used in arguments to follow, they will not be stated and the reader is referred to Chapter IV, Theorems 28–32, of *Foundations* for their precise statement and proof. Some of these proofs must be modified, however, to be valid for the set of axioms used here.

THEOREM 19. *An arc is accessible from both sides at any interior point.*

THEOREM 20. *A ray separates space into at most two connected domains.*

Proof. Suppose, on the contrary, that there exists a ray $\overrightarrow{PB}$ such that $S-\overrightarrow{PB}$ contains three components. There exist three arcs A_1B_1, A_2B_2, and A_3B_3 no two of which contain points of the same component of $S-\overrightarrow{PB}$ such that (1) for each i, $(i=1, 2, 3)$, $A_iB_i \cdot \overrightarrow{PB} = B_i$, and (2) B_1 belongs to the interval PB_2, and B_2 to the interval PB_3, of $\overrightarrow{PB}$. By Theorems 7 and 9, $M = A_1B_1+A_2B_2+A_3B_3+PB_3$ does not separate space. By Theorem 18 the

points A_1, A_2, and A_3 are accessible from $S-M$. Thus for each i, $i=1, 2, 3$, there exists an arc A_iB_i' lying in $(S-M)+A_i$ and having only the point B_i' in common with $\overrightarrow{PB}$. For each i, $i=1, 2, 3$, let J_i denote the simple closed curve composed of A_iB_i, A_iB_i' and the interval B_iB_i' of $\overrightarrow{PB}$, and let I_i denote that complementary domain of J_i whose common part (if any) with $\overrightarrow{PB}$ is compact. Either (1) I_1 contains A_2, or (2) I_3 contains A_2, or (3) I_2 contains A_1, or (4) I_2 contains A_3. Any one of these four possibilities leads to a contradiction. For instance, suppose that I_1 contains A_2. Since $I_1 \cdot \overrightarrow{PB}$ is a subset of PB_1, it is clear, by Theorems 9 and 18, that I_1+A_1 contains an arc from A_1 to A_2 not intersecting $\overrightarrow{PB}$.

THEOREM 21. *If the arc AB is a subset of the connected domain D and $D-AB$ is connected, then D contains a simple closed curve separating A from B.*

Proof. Since the arc AB is accessible from both sides at an interior point O, there exist two arcs EO and FO abutting on AB from different sides such that neither EO nor FO has any point except O in common with the arc AB. The arcs EO and FO contain points E_1 and F_1, respectively, such that the intervals E_1O and F_1O of EO and FO respectively lie in D. In $D-AB$ there exists an arc E_1F_1, and in $E_1O+E_1F_1+F_1O$ there exists a simple closed curve J which lies in D and separates A from B.

THEOREM 22. *If P is a point of a connected domain Q, then there exists a region R containing P such that if X is a point of $R-P$, X lies in a simple domain which together with its boundary is a subset of Q.*

Proof. If Q is S, the theorem is evident from Theorem 2. Suppose that the boundary of Q is not vacuous and that the theorem is false.

Then there exists a sequence α of points X of $Q-P$ converging to P such that X does not lie in a simple domain which together with its boundary is a subset of Q. Let ρ denote a monotonic sequence of simply connected domains D closing down on P. Let D_1 denote an element of ρ lying in Q. It is clear from Axioms 3 and 5_1^* that Q_1-P contains only a finite number of components. One of these components C_1 contains two points X_1 and X_2 of α. Let T_1 denote an arc from X_1 to X_2 in C_1. Let D_2 denote the first element of ρ which follows D_1 and contains no point of T_1. One of the components of D_2-P contains two points X_3 and X_4 of α and an arc T_2 from X_3 to X_4. Let D_3 denote the first element of ρ not containing a point of T_1+T_2. Then D_3-P contains an arc T_3 whose end points are points of α. This process may be continued. Thus there exists a sequence of mutually exclusive arcs $T_1, T_2, T_3, \cdots$ converging to P such that for each n, T_n is a subset of D_n-P and the end points of T_n are points of α. By the preceding theorem, for each n, T_n separates D_1.

For each n, one of the components of D_1-T_n does not contain P and in this component there exists a segment W_n having one end point in T_n and containing no point of $T_1+T_2+\cdots+T_{n-1}$. By Theorems 7 and 9, $P+T_1+T_2+\cdots+T_n$ does not separate space. Hence there exists a segment W_n' lying in $D_1-(P+T_1+T_2+\cdots+T_n)$ having one end point in W_n and the other end point in the boundary of D_1. For each n let L_n denote the component of $D_1-(P+T_1+T_2+\cdots+T_n)$ which contains W_n+W_n'. Then for each n, T_n separates L_n from P in D_1, and $\bar{L}_n$ contains points of both T_n and the boundary of D_1. By Axiom 5_1^* there exists in D_1 a domain U which contains P and whose boundary is a subset of the sum of a finite number of continua lying in D_1-U. There is a continuum H of this set which, for infinitely many values j of n, contains a point of L_j while U contains T_j. But if k is the smallest value of j, $P+H+\sum L_j$ is a connected point set in D_1-T_k containing P and L_k. This is a contradiction.

DEFINITIONS. If P is a point and there exists a sequence of simple domains closing down on P, then P is said to be a *simple point*. A non-simple point is said to be an *edge point*.

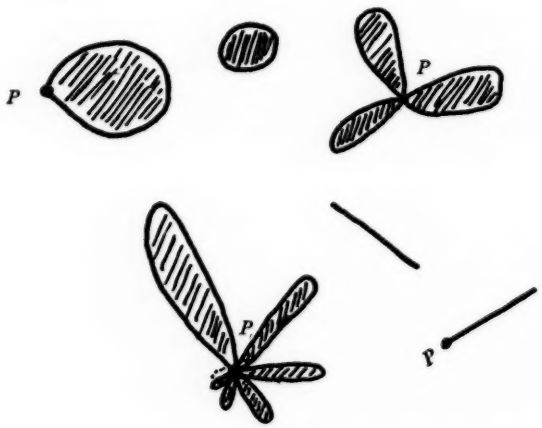

FIGURE 2

If from the plane of Figure 2 the shaded domains and their boundaries except for the points P are removed, the result is a space which satisfies Axioms 0–4 and 5_1^*. The points P are *edge points* while all other points are *simple points*.

THEOREM 23. *The set of simple points is everywhere dense.*

Proof. Suppose that R is a region. With the help of repeated applications

of Theorem 22 it may be shown that there exists a sequence $D_1, D_2, D_3, \cdots$ of simple domains lying in R such that, for each n, (1) D_n contains $\overline{D}_{n+1}$, and (2) $\overline{D}_n$ lies in some region of the collection G_n of Axiom 1. By (4) of Axiom 1, there is a point P common to $\overline{D}_1, \overline{D}_2, \overline{D}_3, \cdots$ and hence, to $D_1, D_2, D_3, \cdots$; and by (3) of Axiom 1, P is the only common point. The point P is simple and lies in R.

THEOREM 24. *No completely separable point set contains uncountably many edge points.*

Proof. Let H denote the collection of all edge points belonging to a completely separable point set M. If X is a point of H, there exists no sequence of simple domains closing down on X. Hence for each point X of H, there exists an integer n_x such that no region of G_{n_x} of Axiom 1 contains a simple domain containing X. If H is uncountable, there exist an integer k and an uncountable subset K of H, such that if X is a point of K, $n_x = k$. Since M is completely separable, K has a point P of condensation. Let Q denote a connected domain containing P and lying in a region of G_k of Axiom 1. By Theorem 22 there exists in Q a region R containing P such that every point X of $R-P$ is contained in a simple domain lying in Q. This is clearly a contradiction.

THEOREM 25. *If M is a countable set of simple points, $S-M$ is arc-wise connected.*

This theorem may be proved without using Axioms 4 and 5_1^* by a modification of R. L. Moore's proof† of Theorem 1 of Chapter II of *Foundations*.‡ The kernel of this proof is the construction of a sequence $C_1, C_2, C_3, \cdots$ of simple chains of connected domains from a point A to a point B such that for each n, C_n^* contains $\overline{C}_{n+1}^*$ and the common part of the point sets $C_1^*, C_2^*, C_3^*, \cdots$ is an arc from A to B. Now for each n, C_n^* is a connected domain, and it is easy to see that no finite number of simple points disconnects a connected domain. Hence if (1) A and B are two distinct points of $S-M$, (2) $M = P_1 + P_2 + \cdots + P_n + \cdots$, and (3) for each n the chain C_n is constructed so that C_n^* contains no point of $P_1 + P_2 + \cdots + P_n$, then the arc from A to B of Moore's construction will not contain any point of M.

THEOREM 26. *If M is a countable set of simple points, $S-M$ is cyclicly connected.*

Proof. If A and B are any two points of $S-M$, then by Theorem 25 there exists an arc AXB in $S-M$. With the help of Theorems 5, 7 and 17, and

† Cf. J. R. Kline, *Concerning the complement of a countable infinity of point sets of a certain type*, Bulletin of the American Mathematical Society, vol. 23 (1917), pp. 290–292.

‡ See footnote p. 54.

Theorem 9 of Chapter II of *Foundations*, one may show that $S-\widehat{AXB}$ is a space satisfying Axioms 0, 1, 2, and 3. Hence by Theorem 25 there exists an arc AYB lying in $S-(M+\widehat{AXB})$. The sum of the two arcs, AXB and AYB, is a simple closed curve lying in $S-M$ and containing $A+B$.

THEOREM 27. *No compact set of edge points separates two simple points from each other.*

Proof. Suppose that A and B are two simple points and that M is a compact set of edge points. Suppose that n is any fixed integer. Then there exists from A to B a simple chain of simple domains such that each element of the chain is a subset of some region of G_n of Axiom 1. For suppose the contrary. Let H_A denote the set of all points X such that there exists from A to X a chain of simple domains whose elements are each a subset of some region of G_n. Since A is a simple point, H_A is a connected domain. This domain does not contain B, and its boundary β is a subset of M. Hence β is a closed and compact point set which separates A from B. By Theorem 8 and Axiom 3, β contains a non-degenerate continuum separating A from B. Hence M is uncountable, which contradicts Theorem 24.

Now with the help of Theorem 9 of Chapter II of *Foundations* and Theorem 5 one may show that a simple domain is a space satisfying Axioms 0–4 and 5_1^*, and that the complementary domain of a point is also such a space. Let $M=P_1+P_2+P_3+\cdots$. Let C_1 denote a simple chain of simple domains from A to B such that (1) each element of C_1 is a subset of some region of G_1, and (2) C_1^* does not contain P_1. Now the boundary of the domain C_1^* is a closed and compact point set not separating A from B; so there exists a simple closed curve separating it from $A+B$ (Theorem 3 in modification). Hence there exists in C_1^* a simple domain D_1 containing $A+B$. Let C_2 denote a simple chain of simple domains from A to B such that if c is an element of D_2, (1) $\bar{c}$ is a subset of some element of C_1, (2) c is a subset of some region of G_2, and (3) c does not contain P_2. This process may be continued, and by Theorem 80 of Chapter I of *Foundations*, $C_1^*\cdot C_2^*\cdot C_3^*\cdot\cdots$ is a continuum. Hence M does not separate A from B.

As a matter of fact the above process may be carried out along the lines of the proof of Theorem 1 of Chapter II of *Foundations*, so as to show that $S-M$ is arc-wise, and even cyclicly, connected.

Nevertheless, a countable but non-compact set of edge points may separate two simple points from each other as is shown by the following example. Let space consist of all points of the number plane whose ordinates are not zero together with those whose ordinates are zero but whose abscissas are rational. If (X, Y) is a point whose ordinate Y is not zero, then for each integer

such that $1/n < |Y|$, let $V_n(X, Y)$ denote the point set consisting of all points which lie at a distance less than $1/n$ from (X, Y). If (X, Y) is a point whose ordinate Y is zero and whose abscissa X is rational then for each integer n, let $V_n(X, Y)$ denote the point set consisting of (X, Y) together with all points which are interior to the isosceles triangles bounded by the lines whose equations are $y=1/n$, $y=-1/n$, $y=n(x-X)$, and $y=-n(x-X)$. These point sets V_n shall be called regions, and for each integer n, G_n of Axiom 1 shall denote the collection of all of the point sets V_{n+q}, $q=0, 1, 2, \cdots$. It can be shown that Axioms 0–4 and 5_1^* hold true in this space. Furthermore, the space is completely separable and locally compact at all but a countable nowhere-dense set of points. Let M denote the set of all points whose abscissas are rational and whose ordinates are zero. Each point of M is an edge point. If A and B are points with positive and negative ordinates respectively, M separates A from B, and the point set H_A of the above argument consists of all points whose ordinates are positive.

THEOREM 28. *No compact set of edge points separates space.*

Theorem 28 is a consequence of Theorems 23 and 27.

DEFINITIONS. If D_1 and D_2 are domains such that D_1 contains $\overline{D}_2$, and $\hat{T}$ is a segment lying in $D_1-\overline{D}_2$ having one end point on the boundary of D_1 and the other on the boundary of D_2, then $\hat{T}$ is said to *cross* $D_1-\overline{D}_2$. Furthermore, if C is the component of $D_1-\overline{D}_2$ which contains $\hat{T}$, then $\hat{T}$ is said to *cross* C. Also the arc T is said to *cross* $D_1-\overline{D}_2$ and to *cross* C.

DEFINITION. If D_1 and D_2 are domains such that D_1 contains $\overline{D}_2$, C is a component of $D_1-\overline{D}_2$, and $\hat{T}_1$, $\hat{T}_2$, and $\hat{T}_3$ are segments crossing C, then if $\hat{T}_2$ separates $\hat{T}_1$ from $\hat{T}_3$ in C, $\hat{T}_2$ is said to be *between* $\hat{T}_1$ and $\hat{T}_3$ in C, and T_2 is said to be *between* T_1 and T_3 in C.

THEOREM 29. *Suppose that D_1 and D_2 are simply connected domains such that D_1 contains $\overline{D}_2$ but no simple domain lying in D_1 contains D_2, C is a component of $D_1-\overline{D}_2$, and $\hat{T}_1$, $\hat{T}_2$, and $\hat{T}_3$ are three mutually exclusive segments crossing C. Then one and only one of these segments is between the other two in C.*

Proof. That not more than one of these segments lies between the other two in C is an immediate consequence of a well known theorem.

Suppose that no one of them is between the other two in C. Then either (1) $C-(\hat{T}_1+\hat{T}_2+\hat{T}_3)$ contains a component W which has limit points in each of the three segments, or (2) $C-(\hat{T}_1+\hat{T}_2+\hat{T}_3)$ contains three mutually exclusive components W_{12}, W_{13}, and W_{23} having limit points in the segments indicated by their subscripts.

Case I. Suppose (1) holds. Then there exists in W three mutually exclusive segments $\widehat{AA}_1$, $\widehat{AA}_2$, and $\widehat{AA}_3$ having a common end point A in W such that A_1, A_2, and A_3 belong to $\hat{T}_1$, $\hat{T}_2$, and $\hat{T}_3$ respectively. For each i, $i=1, 2, 3$, let O_i and P_i denote the end points of T_i which belong to the boundaries of D_1 and D_2 respectively. Since D_2 is connected, there exist in $D_1-(\hat{T}_1+\hat{T}_2+\hat{T}_3+AA_1+AA_2+AA_3)$ three mutually exclusive segments $\widehat{BB}_1$, $\widehat{BB}_2$, and $\widehat{BB}_3$ having a common end point B in D_2 and having their other end points B_1, B_2, and B_3 in the intervals A_1P_1 of T_1, A_2P_2 of T_2, and A_3P_3 of T_3 respectively. If AB_nB denotes $AA_n+A_nB(\text{of } A_nP_n)+BB_n$, $n=1, 2, 3$, then the segments AB_1B, AB_2B, and AB_3B are mutually exclusive. By Theorem 5 of Chapter III of *Foundations*, the sum of two of these arcs, say AB_1B and AB_3B, forms a simple closed curve J whose interior I with respect to O_1 contains the segment from A to B of the other one. Since D_1 is simply connected, I is a subset of D_1. But T_2 contains B_2 and no point of J and is, therefore, a subset of I. Hence I contains O_2. This is a contradiction.

Case II. Suppose (2) holds. There exist segments $\widehat{A_1B_3}$, $\widehat{B_2A_3}$, and $\widehat{B_1A_2}$ lying in W_{13}, W_{23}, and W_{12} respectively and having end points on the arcs T_1, T_2, and T_3 as indicated by the subscripts in the notation. The point set $A_1B_3+A_1B_1(\text{of } T_1)+B_1A_2(\text{of } T_2)+B_2A_3+A_3B_3(\text{of } T_3)$ is a simple closed curve J lying in $D_1-\bar{D}_2$. Since D_1 is simply connected, one of the complementary domains I of J is a subset of D_1. The domain I either contains D_2 or is a subset of $D_1-\bar{D}_2$. By hypothesis D_1 contains no simple domain containing D_2. Hence I is a subset of $C-(\hat{T}_1+\hat{T}_2+\hat{T}_3)$ and has limit points in each of the three segments $\hat{T}_1$, $\hat{T}_2$, and $\hat{T}_3$, which is Case I again.

The reader can see that any collection of mutually exclusive segments crossing C would, in accordance with this notion of *between*, have a *linear order* in C.

THEOREM 30. *Suppose that D_1, D_2, and D_3 are simply connected domains containing the point P such that* (1) *D_3 contains $\bar{D}_2$,* (2) *D_3 is surrounded minimally with respect to D_1 and P by a finite collection Δ of continua lying in D_1-D_3, and* (3) *D_1 contains no simple domain containing P. Then if C is a component of $D_1-\bar{D}_2$, T is an arc crossing C, and J is a simple closed curve separating T from P, J contains two arcs T_a and T_b which cross C such that T is between them in C.*

Proof. Let R denote a connected domain lying in $D_1-D_1\cdot J$ and containing the end point A of T which is on the boundary of D_2. Now in $R+D_2$ there exists an arc AP from A to P, such that $J\cdot AP$ is a subset of an interval $A'P'$ of AP lying in D_2. By Theorem 10, $J+AP$ contains a simple closed curve J_1

separating A from P such that $J_1 \cdot AP$ is connected; but $J_1 \cdot AP$ is a subset of $A'P'$ and is, therefore, a subset of D_2. Since D_1 is simply connected but contains no simple domain containing P, J_1 contains a point O not belonging to D_1. The point set $J_1-(J_1 \cdot AP+O)$ is the sum of two mutually exclusive segments $\widehat{OX}$ and $\widehat{OY}$, where X and Y belong to AP (and Y belongs to the interval XP of AP) and the arcs OX and OY abut on AP from different sides. Both OX and OY contain arcs which cross $D_1-\bar{D}_2$. Let T_a denote the first interval of OX in the order from X to O that crosses $D_1-\bar{D}_2$, and let T_b denote the first interval of OY in the order from Y to O which crosses $D_1-\bar{D}_2$. Now C contains an element of Δ and if U denotes the component of D_1-P which contains C, U has no point in common with any other component of $D_1-\bar{D}_2$ because Δ is a minimal collection with respect to D_1 and P. Hence T_a and T_b cross C. Let X' and Y' denote the end points of T_a and T_b respectively which are on the boundary of D_2. The intervals XX' of OX and YY' of OY lie in D_1.

Suppose that T is not between T_a and T_b in C. Then there exists an arc T_1 in C irreducible from T_a to T_b not intersecting T. Let W denote the last point that OX has in common with T_1, and let Z denote the last point that OY has in common with T_1. The intervals WX of OX and ZY of OY are subsets of T_a+XX' and T_b+YY' respectively, and hence, are subsets of D_1; furthermore, WX and ZY abut on AP from different sides. Now $J_2=J_1 \cdot AP+WX+WZ(\text{of } T_1)+ZY$ is a simple closed curve lying in D_1 not containing P, or a point of T, or a point of the segment $\widehat{PY}$ of AP. Since D_1 is simply connected, one of the complementary domains I of J_2 is a subset of D_1. But since D_1 does not contain a simple domain containing P, both P and T must be subsets of the other complementary domain E of J_2. Now AX contains an arc which abuts on J_2 from the side opposite PY of AP, and since both A and $\widehat{PY}$ are in D, $\widehat{AX}$ must intersect J_2. Let A_1 and A_2 denote the first and last points respectively that $\widehat{AX}$ has in common with J_2. The segments $\widehat{AA}_1$ and $\widehat{A_2X}$ of AX lie in E and I respectively. The segment $\widehat{AX}$ does not intersect J_1; so $\widehat{AX} \cdot J_2$ is a subset of T_1. Hence $\widehat{AA}_1$ and $\widehat{A_2X}$ abut on T_1 from different sides. Let R' denote a connected domain lying in D_1 and containing A but no point of $T_1+T_a+T_b$. Then in $R'+D_2$ there exists an arc FG irreducible from AA_1 to A_2X. The simple closed curve $J_3=FG+GA_2(\text{of } A_2X)+A_1A_2(\text{of } T_1)+A_1F(\text{of } AA_1)$ lies in D_1, contains no point of T_a+T_b, but crosses the arc T_1. Hence, T_a and T_b lie in different complementary domains of J_3. This is impossible, for since D_1 is simply connected, one complementary domain of J_3 is a subset of D_1.

THEOREM 31. *In order that space be metric, it is necessary and sufficient that space be completely separable.*

Proof. That a space satisfying Axioms 0 and 1 is metric if it is completely separable has been shown by R. L. Moore.† Hence, this space is metric if it is completely separable.

It will now be shown that if space is metric, it is completely separable. Let P denote an edge point, and let R denote a region containing P but containing no simple domain containing P. Since R is not S, by Theorems 12, 14, and 15, there exist in R three simply connected domains D_1, D_2, and D_3 such that D_3 contains $\overline{D}_2$, and D_3 is surrounded minimally with respect to D_1 and P by a collection Δ of continua lying in D_1-D_3. The domain D_1 contains no simple domain containing P. Let C denote a component of $D_1-\overline{D}_2$.

Suppose that there are two mutually exclusive arcs T_1 and T_{-1} crossing C. Let α denote a well-ordered sequence (whose first element is T_1) of all arcs T which cross C such that either (1) T is T_1, or (2) T_1 lies between T_{-1} and T in C. Let α' denote a subsequence of α such that (1) the first element of α' is T_1 and (2) if an element T of α' is not T_1, then T is the first element of α which neither intersects a preceding element of α' nor lies between two preceding elements of α'. Every element of α either intersects an element of α' or lies between T_1 and some element of α', and between any element of α' and the next following element in α' there is no element of α'.

The sequence α' is countable. For suppose that α' is uncountable. For each arc T of α' let d_T denote the distance from T to the next element of α'. There exist a number ϵ and an uncountable subsequence α_ϵ' of α' such that if T belongs to α_ϵ', $d_T>\epsilon$. Let T_2 denote the first element of α' such that T is preceded in α' by infinitely many elements of α_ϵ'. Let L denote an arc from T_1 to T_2 lying in C. Infinitely many elements of α_ϵ' are between T_1 and T_2 in C and therefore intersect L. Consequently, there exists in C a connected domain D of diameter less than $\epsilon/2$ which contains points of two different elements T and T' of α_ϵ', such that T precedes T' in α'. Then, the first element of α' which follows T in α' is either T' or lies between T and T' in C and intersects D, which is a contradiction. The same may be done for the collection of all arcs T such that T_{-1} lies between T_1 and T in C. Hence there exists a countable collection G of arcs crossing C, such that any arc which crosses C either intersects an arc of G or lies between two arcs of G. This was on the assumption that there exist two mutually exclusive arcs crossing C. It is evident that G exists if this is not the case, for there exists at least one arc which crosses C.

For each pair of arcs of G let L denote an arc in C which contains points of both of them, and let M_c denote the sum of all these arcs together with the sum of the elements of G. Then if T is any arc which crosses C, T intersects

† See *Foundations*, p. 464.

M_c. Since each component C of $D_1-\overline{D}_2$ contains an element of Δ, there are only a finite number of them. Hence $M=\sum M_c$ is a closed and completely separable subset of $\overline{D}_1-P$ which separates P from the boundary of D_1. Let Q denote the component of $\overline{D}_1-M$ which contains P. Then Q contains P and lies in R, and the boundary of Q is completely separable. Thus, if P is an edge point of a region R, there exists in R a domain containing P whose boundary is separable. This is also true if P is a simple point of a region R. Therefore, space is locally peripherally separable. By a theorem of the author's such a connected, connected im kleinen metric space is completely separable.†

THEOREM 32. *If space is completely separable and P is a point, there exists a sequence of simple domains $Q_1, Q_2, Q_3, \cdots$ bounded by simple closed curves $J_1, J_2, J_3, \cdots$ respectively, such that* (1) *for each n, Q_n contains $\overline{Q}_{n+1}$, and* (2) *if M is a closed and compact point set not containing P, there exists an integer n such that $\overline{Q}_n$ does not contain a point of M.*

Proof. Let G denote the collection of all simple domains D such that $\overline{D}$ is a subset of $S-P$. By Theorem 2, G covers $S-P$, and since space is completely separable, G contains a countable subcollection $D_1, D_2, D_3, \cdots$ which covers $S-P$. For each n let C_n denote the boundary of D_n. Let J_1 denote the boundary of a simple domain containing P. For each n, $n\geqq 2$, let J_n denote a simple closed curve separating P from the closed and compact point set $C_1+C_2+\cdots+C_{n-1}+J_1+J_2+\cdots+J_{n-1}$, and let Q_n denote the complementary domain of J_n which contains P. Then $Q_1, Q_2, Q_3, \cdots$ is the required sequence of simple domains. For if M is any closed and compact point set, there exists an integer k such that $D_1+D_2+\cdots+D_k$ covers M. Hence, if $n>k$, $\overline{Q}_n$ contains no point of M.

DEFINITION. Suppose that D is a connected domain. If there exist a collection Δ of open curves, a simply connected domain Q, and a point P of D, such that (1) Q contains no simple domain containing P, and (2) Δ surrounds D minimally with respect to Q and P, then D is said to be *pseudo-simple*, and in particular, D is said to be *pseudo-simple* with respect to Q and P.

THEOREM 33. *If space is completely separable and D_1 is a simply connected domain containing an edge point P but containing no simple domain containing P, then D_1 contains a domain D which is pseudo-simple with respect to D_1 and P.*

Proof. Let D_2 and D_3 denote simply connected domains containing P such that (1) D_3 contains $\overline{D}_2$, and (2) D_3 is surrounded minimally with respect to D_1

† *A theorem concerning locally peripherally separable spaces*, Bulletin of the American Mathematical Society, vol. 41 (1935), pp. 437–439.

and P by a finite collection Δ' of continua lying in D_1-D_3. By Theorem 32 there exists a sequence of simple domains $Q_1, Q_2, Q_3, \cdots$ bounded by simple closed curves $J_1, J_2, J_3, \cdots$ respectively, such that (1) for each n, Q_n contains $\overline{Q}_{n+1}$, and (2) if M is a closed and compact subset of $S-P$, there exists an integer n such that $\overline{Q}_n$ contains no point of M. Let C denote a component of $D_1-\overline{D}_2$, and let G denote the collection of all arcs T such that T crosses C and for some n is a subset of J_n. If T and T' are arcs of G, $\widehat{T}\cdot\widehat{T}'=0$. Since D_1 is simply connected but contains no simple domain containing P, for each n, J_n contains a point of $S-D_1$. Furthermore, let AP denote an arc lying in D_2 and in the component of D_1-P which contains C; then there exists an integer n_1 such that $\overline{Q}_{n_1}$ does not contain A; so if $n>n_1$, J_n contains a point of the segment $\widehat{AP}$ and a point of $S-D_1$, and therefore contains at least two arcs of G. However for each n, J_n contains at most a finite number of arcs of G. Hence G is countably infinite. Furthermore, suppose that L is an arc lying in C from one arc of G to another arc of G. There exists an integer n_2 such that if $n>n_2$, $\overline{Q}_n$ contains no point of L. Hence, between any two arcs of G there are only a finite number of arcs of G. Each arc of G is between some two arcs of G. For, if T is an arc of G, there exists an integer n such that J_n separates T from P. By Theorem 30, J_n contains two arcs which cross C such that T is between them in C.

Let α denote a well-ordered sequence whose elements are the elements of G. Let N_1 and N_0 denote the first two elements of α. Let N_2 denote the first element of α such that N_1 is between N_0 and N_2 in C. Let N_{-1} denote the first element of α such that N_0 is between N_{-1} and N_1 in C. This process may be continued. The sequences $N_1, N_2, N_3, \cdots, N_\omega, \cdots$ and $N_0, N_{-1}, N_{-2}, \cdots, N_{-\omega}, \cdots$ may or may not be simple sequences, but since each is countable, the first contains a simple countable subsequence $T_1, T_2, T_3, \cdots$ running through it and the second contains a simple countable subsequence $T_0, T_{-1}, T_{-2}, \cdots$ running through it, such that if T is any arc crossing C, there exists an integer n such that T is between T_n and T_{-n} in C, and such that the linear order of these arcs in C is the same as the order in the sequences.

Since C contains one and only one of the continua of Δ' and by Theorem 31 space is metric, there exists in C a connected domain C' such that (1) C' contains this element of Δ', and (2) $\overline{C}'$ is a subset of C. For each pair of consecutive integers (positive or negative), a and b with $a<b$, let U_a denote an arc irreducible from T_a to T_b lying in C', and let M_a denote the interval of T_a between the end points of U_a and U_{a-1} in T_a. The point set $\sum M_a$ is closed, since each M_a is for some n a closed subset of $C\cdot J_n$, for each n J_n contains only a finite number of the arcs of G, and the limiting set of $J_1, J_2, J_3, \cdots$ is

the point P. Also $\sum U_a$ is closed; for suppose that O is a limit point of the set $\sum U_a$ not belonging to it. The point O belongs to $\overline{C}'$ and therefore, to C. There exist a connected domain V lying in C and containing O but not intersecting more than one of the simple closed curves $J_1, J_2, J_3, \cdots$, and an infinite collection H of segments $\widehat{U}_a$ such that if $\widehat{U}_a$ is a segment of H, $\widehat{U}_a$ intersects V. Hence, $V+H^*$ is a connected subset of C not intersecting more than one of the arcs $T_1, T_2, T_3, \cdots, T_0, T_{-1}, T_{-2}, \cdots$, but having limit points in infinitely many of them. This is a contradiction. So $L=\sum(M_a+U_a)$ is closed and is, because of its method of construction, an open curve.

Now suppose that T is any arc crossing C. There exists an integer n such that T lies between T_n and T_{-n} in C. Hence T intersects that interval of L which is irreducible from T_n to T_{-n}. Furthermore, if this process is carried out in each component of $D_1-\overline{D}_2$ and Δ denotes the collection of open curves so obtained, Δ^* separates P from the boundary of D_1, and since no component of D_1-P contains more than one element of Δ', no component contains more than one element of Δ. Hence, Δ is minimal with respect to D_1 and P.

Let D denote the component of $S-\Delta^*$ which contains P. The domain D is pseudo-simple with respect to D_1 and P.

THEOREM 34. *If L is an element of a collection of open curves which surrounds a pseudo-simple domain D minimally with respect to a simply connected domain Q and a point P, and C is the component of $D-P$ whose boundary contains a point of L, then no ray of L separates C from the boundary of Q in $S-P$.*

Proof. Suppose the contrary is true. Then L contains a ray $\overrightarrow{B_1B_2}$ such that B_1 belongs to $\overline{C}$ and $\overrightarrow{B_1B_2}+P$ separates C from β, the boundary of Q. By Theorem 18, $C+P+B_1$ contains an arc PB_1 from P to B_1. The ray $\overrightarrow{PB_2}=PB_1+\overrightarrow{B_1B_2}$ separates $C-C\cdot PB_1$ from β, and by Theorem 20, $S-\overrightarrow{PB_2}$ is the sum of two connected domains I and E where I contains $C-C\cdot\overrightarrow{PB_2}$ and is a subset of Q and E contains β. Let E_1F_1 and E_2F_2 denote two arcs in C which abut on PB_1 from different sides. The points E_1 and E_2 both belong to $C-C\cdot PB_1$ and, hence, to I. Therefore I contains an arc E_1E_2 from E_1 to E_2, and $E_1E_2+E_1F_1+F_1F_2(\text{of } PB_1)+E_2F_2$ contains a simple closed curve lying in Q one of whose complementary domains lies in Q and contains P. This is a contradiction.

THEOREM 35. *If L is an element of a collection Δ of open curves which surrounds a pseudo-simple domain D minimally with respect to a simply connected domain Q and a point P, then no two segments lying in D abut on L from different sides.*†

† Two segments abut on a ray or open curve from different sides provided the ray or open curve contains an arc on which the segments abut from different sides.

Proof. Suppose, on the contrary, that D contains two mutually exclusive segments which abut on L from different sides. It follows, using Theorem 11(b), that there exist an arc A_1A_2 and two segments, $\widehat{A_1B_1}$ and $\widehat{A_2B_2}$, all lying in the same component C of $D-P$, such that the segments $\widehat{A_1B_1}$ and $\widehat{A_2B_2}$ abut on L from different sides and the point set $J=A_1B_1+B_1B_2(\text{of } L)+A_2B_2+A_1$ is a simple closed curve lying in Q. One of the complementary domains of J is a subset of Q. Hence, L contains two rays, $\overrightarrow{B_1B_2}$ and $\overrightarrow{B_2B_1}$, whose common part is the arc B_1B, one of which has the property that it together with P separates C from the boundary of Q. This is a contradiction of the preceding theorem.

THEOREM 36. *If D is a pseudo-simple domain with respect to a simply connected domain Q and a point P, β is the boundary of D, and J is a simple closed curve lying in $\overline{D}$ such that $J\cdot\beta$ is connected, then one of the complementary domains of J is a subset of D.*

Proof. Let Δ denote a collection of open curves which surrounds D minimally with respect to Q and P, let L denote the element of Δ which contains $J\cdot\beta$, and let C denote the component of $D-P$ which contains $J\cdot(D-P)$. By Theorems 7 and 9, $J\cdot\beta+P$ does not separate space, and there exists an arc T from a point of $L-J\cdot L$ to a point on the boundary of Q which contains no point of J. One of the complementary domains I of J is a subset of Q. Therefore, I contains no point of T and, with the help of Theorem 35, it may be shown that I contains no point of L. But I does not contain P.† Hence, since I contains points of C, I is a subset of C and, consequently, of D.

THEOREM 37. *If D is a pseudo-simple domain, L is an element of a collection Δ of open curves which surrounds D minimally with respect to a simply connected domain Q and a point P, and OX and $O'X'$ are arcs lying in $S-\overline{D}$ except for their end points X and X' which lie on L, then the arcs OX and $O'X'$ abut on L from the same side.*

Proof. With the help of Theorems 11(b) and 34 it can be shown that there exists an arc FF' lying in D except for the end points F and F', which lie on L, such that the interval FF' of L contains X and X' as interior points. Then by Theorem 36, $J=FF'+FF'(\text{of } L)$ is a simple closed curved one of whose complementary domains is a subset of D. The other complementary domain of J contains the segments $\widehat{OX}$ and $\widehat{O'X'}$. Hence, the arcs approach FF' of L and, therefore, L from the same side.

THEOREM 38. *If D is a pseudo-simple domain with respect to a simply connected domain Q and a point P, and OX and $O'X'$ are arcs which have their*

† From the definition of a pseudo-simple domain, Q contains no simple domain containing P.

end points X and X' in an element L of a collection Δ of open curves which surrounds D, but have no other points in common with the boundary of D, and approach L from different sides, then one of these arcs lies in D and the other lies in $S-\bar{D}$.

Theorem 38 is a consequence of Theorems 35 and 37.

THEOREM 39. *If Δ is a collection of open curves which surrounds a pseudo-simple domain minimally with respect to a simply connected domain Q and a point P, then the boundary of D is Δ^*, and every point of Δ^* is accessible from D.*

Proof. Let X denote a point of an open curve L of Δ, and let AXB denote an interval of Δ containing X as an interior point. By Theorem 19 there exist two segments, $\widehat{EX}$ and $\widehat{E'X}$, which abut on AXB at X from different sides. The segments, $\widehat{EX}$ and $\widehat{E'X}$, contain segments, $\widehat{FX}$ and $\widehat{F'X}$, respectively, which abut on L from different sides and do not contain P or any point of the boundary of D except X. By Theorem 38 one of the latter two segments is a subset of D.

THEOREM 40. *If X is a point of the component L of the boundary β of a domain D which is pseudo-simple with respect to a simply connected domain Q and a point P, and M is a closed and compact point set not containing X, there exists a simple closed curve lying in $S-(D+M)$ whose common part with β is an interval of L containing X as an interior point, and whose interior with respect to P contains no point of M.*

Proof. Let XP denote an arc from X to P lying in $D+X$, and let J denote a simple closed curve separating X from P such that $J \cdot XP$ is a connected subset of D. The curve J contains a point of $S-D$. Let O denote the first point that XP has in common with J, and let AOB denote the interval of J containing O and lying in D except for its end points A and B which lie on L. The interval AB of L contains X because one of the complementary domains of the simple closed curve $AOB+AB$ is a subset of D and since it can not contain P, it must contain the segment $\widehat{OX}$ of XP. Let AZB denote the arc of J such that $J=AXB+AZB$. In the order A, Z, B, let A_0 denote the last point that AZB has in common with the ray $\overrightarrow{XA}$ of L, and let B_1 denote the next point that AZB has in common with β if it is not in L. Let L_1 denote the component of β containing B_1, and let A_1 denote the last point that AZB has in common with L_1. Let B_2 denote the next point that AZB has in common with β if it is not in L, let L_2 denote the open curve of β containing B_2, and let A_2 denote the last point that AZB has in common with L_2. This process can be continued (for a finite number of steps) until for some integer n the next point that AZB has in common with β following A_n is a point B_0 of

L. Since A_0 precedes B_0 in AZB, B_0 belongs to the ray $\overrightarrow{AB}$ of L. Let J_1 denote the simple closed curve A_0XB_0(of L)$+A_0B_1$(of AZB)$+A_1B_1$(of L_1)$+A_1B_2$(of AZB)$+\cdots+A_{n-1}B_n$(of AZB)$+A_nB_n$(of L_n)$+A_nB_0$(of AZB). (See Figure 3.) It is clear that if B_1 exists, J_1 lies in $S-D$, because by Theorem 11(b) any

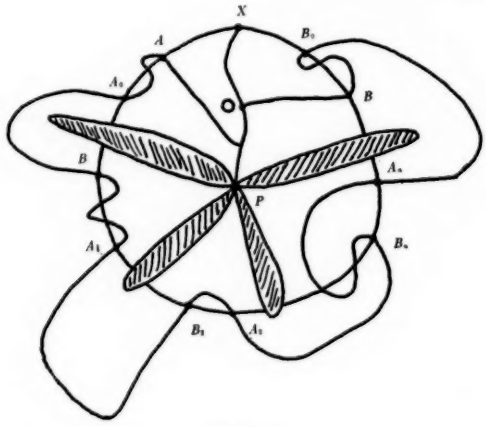

FIGURE 3

segment lying in D having its end points in different components of β contains P, and J_1 does not contain P. Now in case $J_1=A_0XB_0$(of L)$+A_0B_0$(of AZB), suppose that $\widehat{A_0B_0}$ lies in D. Then J_1 lies in $\overline{D}$ and $\beta\cdot J_1$ is connected; so by Theorem 36 one of the complementary domains I of J_1 is a subset of D. Let $P'X$ denote an arc lying in I except for X which is on the boundary of I. Now I does not contain P and J_1 does not intersect $\widehat{PX}+P$; hence, I does not contain $\widehat{PX}$, and therefore PX and $P'X$ abut on AXB from different sides. But since both $\widehat{PX}$ and $\widehat{P'X}$ lie in D, this contradicts Theorem 35.

Thus in either case J_1 is a subset of $S-D$. By Theorem 16 one of the complementary domains I_1 of J_1 is a subset of $S-\overline{D}$ and is the interior of J_1 with respect to P. Let X' denote a point of $J_1-J_1\cdot L$, and let J_2 denote a simple closed curve separating X from $P+M+X'$. Let I_2 denote the interior of J_2 with respect to P. By Theorem 11 of Chapter III of *Foundations*, J_1+J_2 contains a simple closed curve J_3 such that (1) I_3, the interior of J_3 with respect to P, is a subset both of I_1 and of I_2, and (2) $J_3\cdot L$ is an arc containing X as an interior point. Let $A'X'B'$ denote an arc lying in I_3 except for points A' and B' which belong to $J_3\cdot\widehat{A_0X}$ and $J_3\cdot\widehat{B_0X}$ respectively. Then $J_4=A'XB'$(of L)$+A'X'B'$ is a simple closed curve lying in $S-(D+M)$ whose interior with respect to P is a subset of I_3 and hence, contains no point of M. Furthermore,

the common part of J_4 with β is the arc $A'XB'$ which is an interval of L containing X as an interior point.

THEOREM 41. *If D is a pseudo-simple domain, then $S-\overline{D}$ is connected.*

Proof. The boundary β of D is the sum of a finite number of mutually exclusive open curves $L_1, L_2, \cdots, L_j$. By Theorem 40 there exists, for each point X of β, a simple closed curve J_X such that J_X is a subset of $S-D$ and $J_X \cdot \beta$ is for some $n \leqq j$ an interval T_X of L_n containing X as an interior point. Let I_X denote the complementary domain of J_X which does not contain D. By Theorem 16, I_X is a subset of $S-\overline{D}$. Now for each n, $n \leqq j$, let $D_n = \sum I_X$ for X on L_n. All the segments $J_X - T_X$ for X on L_n lie in $S-\overline{D}$ and abut on L_n from the same side. Hence the domains I_X overlap, so that D_n is a connected subset of $S-\overline{D}$. Furthermore, if T is any arc in $S-D$ which abuts on L_n at a point X, it abuts on L_n from the same side as $J_X - T_X$, and hence, contains points of I_X and D_n. So for each $n \leqq j$, only one component of $S-\overline{D}$ has boundary points on L_n.

Now suppose that $S-\overline{D}$ is not connected. Then there exist two mutually exclusive domains H and K such that $S-\overline{D} = H+K$. Since space is connected, both H and K have boundary points in β, but for no n, $n \leqq j$, does L_n contain boundary points of both H and K. Hence $\beta \cdot \overline{H}$ and $\beta \cdot \overline{K}$ are mutually exclusive closed point sets. Let AB denote an arc in $S-P$ irreducible from $\beta \cdot \overline{H}$ to $\beta \cdot \overline{K}$. Now since each D_n, $n \leqq j$, belongs either to H or to K, $\beta = \beta \cdot \overline{H} + \beta \cdot \overline{K}$; so AB contains only two points A and B of β, and these points belong to different open curves of the collection $L_1, L_2, \cdots, L_j$. Therefore, since AB does not contain P, the segment AB lies in $S-\overline{D}$, and hence, contains a point of I_A and a point of I_B. But, since I_A and I_B belong to H and K respectively, this is a contradiction.

THEOREM 42. *Every pseudo-simple domain is strongly simply connected.*

This theorem is a consequence of Theorems 16 and 41.

DEFINITION. A domain D is said to be *internally simple* provided that for each point X of D there exists a simple domain containing X and lying together with its boundary in D.

THEOREM 43. *If D is a pseudo-simple domain with respect to a simply connected domain Q and a point P, then each component of $D-P$ is internally simple.*

Proof. Let U denote a component of $D-P$, and let X denote a point of U. Let AP denote an arc from A to P lying except for P in U. By Theorems 2 and 10 there exists a simple closed curve J crossing XP. If J intersects the boundary of D, then J contains an arc AYB crossing XP and lying except

for the points A and B of L in D. Let J_1 denote the simple closed curve $AB(\text{of } L)+AYB$. One of the complementary domains of J_1 is a subset of D and contains X. Let J_X denote a simple closed curve separating J_1+P from X. If J does not intersect the boundary of D, let J_X denote J. Then, one of the complementary domains I_X of J_X contains X, and I_X+J_X is a subset of U. Hence, U is internally simple.

THEOREM 44. *If D is a pseudo-simple domain, $S-\overline{D}$ is internally simple.*

Proof. The domain D is pseudo-simple with respect to a domain Q and a point P. Let Δ denote the collection of mutually exclusive open curves whose sum is the boundary of D, and let L denote an element of Δ. Let X denote a point of $S-\overline{D}$, and let AX denote an arc irreducible from L to X and containing no point of Δ^*-L. Suppose that $S-(\overline{D}+AX)$ is the sum of two mutually exclusive domains H and K. Because no arc separates space, both H and K have boundary points in Δ^*. Furthermore, because of Theorem 40 no element of $\Delta-L$ contains points of both $\Delta^*\cdot\overline{H}$ and $\Delta^*\cdot\overline{K}$, nor does either of the rays of L from A contain points of both $\Delta^*\cdot\overline{H}$ and $\Delta^*\cdot\overline{K}$. Let AP denote an arc in $D+A$ from A to P. Since $AX+AP$ does not separate space, there exists an arc T irreducible from $\Delta^*\cdot\overline{H}$ to $\Delta^*\cdot\overline{K}$ in $S-(AX+AP)$. Now the segment $\hat{T}$ is a subset of $S-\overline{D}$, for if $\hat{T}$ were a subset of D, it would have one end on one ray of L from A and the other end on the other ray from A without intersecting AP, which by Theorems 35 and 36 is impossible. But it is also impossible for $\hat{T}$ to lie in $S-\overline{D}$, for in this case H and K would have boundary points in the same open curve of Δ. Hence, $S-(\overline{D}+AX)$ is connected.

Let EYF denote an arc lying except for Y in $S-(\overline{D}+AX)$ and crossing AX at the point Y. Since $S-(\overline{D}+AX)$ is connected, there exists an arc EZF lying in it from E to F. The sum of the arcs, EYF and EZF, contains a simple closed curve J crossing AX at the point Y whose interior with respect to P is a simple domain I containing X such that $I+J$ is a subset of $S-\overline{D}$.

THEOREM 45. *If D_1 and D_2 are pseudo-simple domains with respect to domains Q_1 and Q_2 and a point P respectively, Q_2 is a subset of D_1, and C is a component of $D_1-\overline{D}_2$, then C is an internally simple, strongly simply connected domain.*

Proof. The component C is strongly simply connected; for if a simple closed curve J is a subset of $\overline{C}$, then by Theorems 16 and 41 the interior I of J with respect to P is a subset of both D_1 and $S-\overline{D}_2$.

If X is a point of C, then by Theorems 43 and 44 there exist two simple domains, I_1 and I_2, each containing X such that $\overline{I}_1$ and $\overline{I}_2$ are subsets of D_1-P and $S-\overline{D}_2$ respectively. By Theorem 12 of Chapter III of *Foundations*, $I_1\cdot I_2$

contains a simple domain I_X which contains X and whose boundary is a subset of the sum of the boundaries of I_1 and I_2.† Therefore, $\overline{I}_X$ is a subset of C, and C is internally simple.

THEOREM 46. *If D is an internally simple, simply connected domain and AB is an arc lying in D except for its end points which belong to the boundary of D, then $D-D\cdot AB$ is the sum of two mutually exclusive, internally simple, simply connected domains, each having AB on its boundary.*

Proof. For each point X of $\widehat{AB}$ let I_X denote a simple domain containing X such that $\overline{I}_X$ is a subset of D. Let J_X denote the boundary of I_X and let C denote a simple closed curve separating X from $A+B+J_X$. By Theorem 10, $C+\widehat{AB}$ contains a simple closed curve C' separating X from $A+B+J_X$, whose common part with AB is the sum of two continua. Let $A'XB'$ denote the interval of AB containing X such that $A'XB'\cdot C'=A'+B'$. One of the complementary domains I' of C' is a subset of D and contains the segment $A'XB'$. The point set $I'-\widehat{A'XB'}$ is the sum of two simple domains I_{X1} and I_{X2}, one on one side of AB and the other on the other side of AB. Now if the subscripts 1 and 2 are so used so that all the domains I_{X1} for X on AB are on the same side of AB and all the domains I_{X2} are on the other side of AB, $D_1=\sum I_{X1}$ and $D_2=\sum I_{X2}$ are two connected domains which are subsets of D. Now each point O of $D-\widehat{AB}$ may be joined to a point of $\widehat{AB}$ by an arc OE lying in $D-\widehat{AB}$ except for its one end point E, and $\widehat{OE}$ must intersect either D_1 or D_2. If for each i, $i=1, 2$, H_i denotes the set of all points O such that the segment $\widehat{OE}$ intersects D_i, then H_1 and H_2 are connected domains and $H_1+H_2=D-\widehat{AB}$. But if D is simply connected, no arc segment lying in D can abut on AB from different sides since neither A nor B belongs to D. Hence H_1 and H_2 are mutually exclusive, and AB belongs to the boundary of each of them.

Now suppose that J is a simple closed curve lying in H_1. One of the complementary domains I of J is a subset of D. Therefore, I contains no point of AB. But I has limit points in H_1, namely, the points of J, and hence contains points of H_1. Consequently I is a subset of H_1, and therefore H_1 is simply connected. Likewise, H_2 is simply connected.

Let X denote a point of H_1. Since D is internally simple, there exists a simple domain I_X' containing X and lying together with its boundary J_X' in D. By Theorem 3 there exists a simple closed curve J_X separating X from $J_X'+AB$. The curve J_X is a subset of H_1, and hence one of its complementary domains I_X is a subset of H_1 and contains X. Consequently, H_1 is internally simple and likewise H_2 is internally simple.

† The notation here does not correspond to that of Theorem 12 of Chapter III of *Foundations*.

THEOREM 47. *If D is an internally simple, simply connected domain, β is the boundary of D, J is a simple closed curve lying in D, and AB is an arc irreducible from J to the boundary of D, then $D-(J+\widehat{AB})$ is the sum of two mutually exclusive, internally simple, simply connected domains, one having J for its boundary, and the other having for its boundary $J+AB+\beta$.*

Theorem 47 may be established by methods similar to those used in proving Theorem 46.

THEOREM 48. *If D is a pseudo-simple domain and AB is an arc lying in $S-\overline{D}$ except for the points A and B which lie on the boundary of D, then $S-(\overline{D}+AB)$ is the sum of two connected domains whose boundaries have as their common part the arc AB.*

Proof. From Theorems 16, 41, 44, and 46, $S-(\overline{D}+AB)$ is the sum of two connected domains whose boundaries contain AB in their common part. And if β denotes the boundary of D, it is clear from Theorems 38 and 40 that this common part cannot contain a point of $\beta-(A+B)$.

THEOREM 49. *If under the hypothesis of Theorem* 45, *AXB is an arc lying in C except for the points A and B which lie on the boundary of C, then $C-\widehat{AXB}$ is the sum of two connected domains whose boundaries have as their common part the arc AXB.*

Theorem 49 may be proved by an argument similar to that of the preceding theorem if we use Theorems 45 and 46 and a theorem similar to Theorem 40 and established in a similar manner.

DEFINITION. Suppose that X_1, X_2, X_3, and X_4 are points of β, the boundary of a pseudo-simple domain. Then X_1+X_3 is said to *ordinally separate* X_2 from X_4 on β if there exists a simple closed curve J whose common part with β is X_1+X_3 and which separates X_2 from X_4.

THEOREM 50. *Suppose that D is a pseudo-simple domain, β is the boundary of D, and X_1, X_2, X_3, and X_4 are four distinct points of β. Then one pair of them ordinally separates the other two on β; and if X_1+X_3 ordinally separates X_2 from X_4 on β, then* (1) *X_2+X_4 ordinally separates X_1 from X_3 on β,* (2) *every simple closed curve J containing a point of D and $S-D$ such that $J\cdot\beta=X_1+X_3$ separates X_2 from X_4, but* (3) *X_1+X_2 does not ordinally separate X_3 from X_4 on β.*

Proof. Let X_1AX_2 and X_1BX_2 denote arcs lying except for X_1 and X_2 in D and $S-\overline{D}$ respectively. Suppose that $J'=X_1AX_2+X_1BX_2$ does not separate X_3 from X_4. Now let AX_3 and BX_3 denote arcs lying except for X_3 in D and $S-\overline{D}$ respectively, such that $AX_3\cdot J'=A$ and $BX_3\cdot J'=B$. With respect to X_4, the interior of the simple closed curves formed by one pair of the three arcs AX_1B, AX_2B, and AX_3B contains the other arc segment. Hence, one

pair of the points X_1, X_2, X_3, and X_4 ordinally separates the other two on β.

Suppose that X_1+X_3 ordinally separates X_2 from X_4 on β. Then there exists by a construction similar to that of the above paragraph two arcs X_2AX_4 and X_2BX_4 lying except for X_2 and X_4 in D and $S-\overline{D}$ respectively whose sum is a simple closed curve separating X_1 from X_3. If J is a simple closed curve containing a point of D and a point of $S-\overline{D}$ whose common part with β is X_1+X_3, then J is the sum of two arcs $X_1A'X_3$ and $X_1B'X_3$ lying except for X_1 and X_3 in D and $S-\overline{D}$ respectively. Let C denote the simple closed curve $X_2AX_4+X_2BX_4$. By a double application of Theorem 10, $C+\widehat{X_1A'X_3}+\widehat{X_1B'X_3}$ contains a simple closed curve C' whose common part with $\widehat{X_1A'X_3}$ is connected and whose common part with $\widehat{X_1B'X_3}$ is also connected and which separates X_2 from X_4. It is clear now that J separates X_2 from X_4.

Now suppose that X_1+X_2 ordinally separates X_3 from X_4 on β. Then there would exist a simple closed curve, having the properties of J in the preceding paragraph, which does not separate X_2 from X_4. This is impossible if X_1+X_3 ordinally separates X_2 from X_4.

Arguments of a similar nature show that if order on β is interpreted by the notion of ordinal separation, then we have

THEOREM 51. *If β is the boundary of a pseudo-simple domain, the points of β have a cyclic order which preserves the ordinary order on any open curve component of β.*

THEOREM 52. *Suppose that D_1 and D_2 are pseudo-simple domains with respect to domains Q_1 and Q_2 and a point P respectively, Q_2 is a subset of D_1, X_1, X_2, X_3, and X_4, and Y_1, Y_2, Y_3, and Y_4 are the end points of four mutually exclusive arcs T_1, T_2, T_3, and T_4 which cross $D_1-\overline{D}_2$. Then if X_1+X_3 ordinally separates X_2 from X_4 on β_1, the boundary of D_1, then Y_1+Y_2 ordinally separates Y_2 from Y_4 on β_2, the boundary of D_2, and conversely.*

Proof. Let X_1AX_2 denote an arc lying except for the points X_1 and X_3 in $S-\overline{D}_1$, and let Y_1BY_3 denote an arc lying except for the points Y_1 and Y_3 in D_2. Let J denote the simple closed curve $T_1+X_1AX_3+T_3+Y_1BY_3$. Since X_1+X_3 ordinally separates X_2 from X_4 on β_1, J separates X_2 from X_4. But J does not intersect T_2 or T_4 and hence, J separates Y_2 from Y_4. Therefore, Y_1+Y_2 ordinally separates Y_2 from Y_4 on β_2. Likewise, the converse is true.

2. CONSEQUENCES OF AXIOMS 0−4, 5_1^* AND 7

AXIOM 7. *The set of all points is completely separable.*†

† This axiom is numbered 7 because it is so numbered in *Foundations*. See *Foundations*, p. 412.

The purpose of this section is to outline an argument showing that a space satisfying Axioms 0–4, 5_1^*, and 7 is a subset of a space satisfying Axioms 0–4, 5_1^*, and 7 which contains no edge points. Considerable detail will be left to the reader, and notation introduced in a definition, when used subsequently without explanation, will be understood to have the significance given it in the definition.

DEFINITION. Suppose that P is an edge point. Suppose, furthermore, that there exist for each n, a pseudo-simple domain D_n containing $\bar{D}_{n+1}$ whose boundary β_n is the sum of a finite number of open curves, $L_{n1}, L_{n2}, \cdots, L_{nq_n}$, and for each n and q, $q \leqq q_n$, two sequences of mutually exclusive arcs, $T_{1,nq}, T_{2,nq}, \cdots$ and $T_{-1,nq}, T_{-2,nq}, \cdots$, crossing $D_n - \bar{D}_{n+1}$ such that, denoting $T_{m,nq} \cdot \beta_n$ and $T_{m,nq} \cdot \beta_{n+1}$ by $X_{m,nq}$ and $Y_{m,nq}$ respectively,†

(0) for each n, D_n is pseudo-simple with respect to P and a domain Q_n which is a subset of a region of G_n of Axiom 1, which is in turn a subset of D_{n-1}, $D_0 = S$;

(1) for each m, L_{nq} contains $X_{m,nq}$;

(2) for each n, the points $X_{1,n1}, X_{2,n1}, \cdots, \cdots, X_{-2,n2}, X_{-1,n2}, X_{1,n2}, X_{2,n2}, \cdots, \cdots, \cdots, X_{-2,nq_n}, X_{-1,nq_n}, X_{1,nq_n}, X_{2,nq_n}, \cdots, \cdots, X_{-2,n1}, X_{-1,n1}$ have on β_n the cyclic order indicated;‡

(3) for each $Y_{m,nq}$, there exists an integer r, $r \geqq q$, such that $X_{m,(n+1)r} = Y_{m,nq}$;

(4) for each n and q, $q \leqq q_n$, neither of the sequences, $T_{1,nq}, T_{2,nq}, \cdots$ and $T_{-1,nq}, T_{-2,nq}, \cdots$ has a sequential limit set. Let Ws denote the point set consisting of P together with all the sets L_{nq} and $T_{m,nq}$. Such a set, Ws, is said to be a *radial web skeleton* about P.

THEOREM 53. *If R is a region containing an edge point P, R contains a radial web skeleton about P.*

Theorem 53 may be proved with the help of Theorems 15, 18, 32, 33, 45, and 52 and a method similar to that of Theorem 33.

DEFINITION AND NOTATION. Suppose that Ws is a radial web skeleton about an edge point P. For each n and q, $q \leqq q_n$, L_{nq} of Ws is the sum of two rays L_{nq}^+ and L_{nq}^- having only one point in common and containing $X_{1,nq}, X_{2,nq}, \cdots$ and $X_{-1,nq}, X_{-2,nq}, \cdots$ respectively. Such rays are said to be *rays on* β_n and to have the cyclic order of the points $X_{m,nq}$. The point set Ws is said to connect L_{nq}^+ to $L_{(n+1)r}^+$ if $T_{1,nq}$ contains points of both of them. Likewise, Ws is said to connect L_{nq}^- to $L_{(n+1)r}^-$ if $T_{-1,nq}$ contains points of both of them.

† When an integral subscript letter is followed by a comma, its range is both the positive and negative integers.

‡ This order is selected arbitrarily on β_1: then it is transferred from β_1 to β_2, from β_2, to β_3, $\cdots$ by the arcs $T_{m,nq}$ using Theorem 52.

THEOREM 54. *Suppose that Ws is a radial web skeleton about an edge point P, and Ws connects* L^+_{nq} *to* $L^+_{(n+1)r}$ *and* $L^-_{nq'}$ *to* $L^-_{(n+1)r'}$. *Then if* $q'=q+1$, $r'=r+1$, *and conversely.*

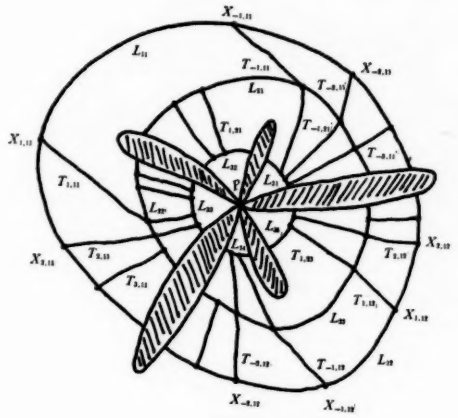

FIGURE 4

Proof. Suppose on the contrary, for instance, that $T_{1,11}$ contains a point of L^+_{11} and of L^+_{23} while $T_{-1,12}$ contains a point of L^-_{12} and of L^-_{25}. Since $T_{1,11}+T_{-1,12}+P$ does not separate space, there exists an arc T from L_{24} to β_1 of Ws lying except for its end points in $D_1-\overline{D}_2$. Let $T\cdot\beta_1=X$ and $T\cdot\beta_2=Y$. Since Y is between $Y_{1,11}$ and $Y_{-1,12}$ on β_2, X is between $X_{1,11}$ and $X_{-1,12}$ on β_1 by Theorem 52. Hence X is either on the subray of L^+_{11} from $X_{1,11}$ or on the subray of L^-_{12} from $X_{-1,12}$. Suppose the former, and a similar argument will apply to the other case. Since the sequence of arcs $T_{1,11}$, $T_{2,11}$, $T_{3,11}$, $\cdots$ has no sequential limit set, there is some one of them, $T_{m,11}$, which has no point in common with T such that $X_{m,11}$ is between X and $X_{-1,12}$ on β_1. But $Y_{m,11}$ belongs to L^+_{23} and is not between Y and $Y_{-1,12}$ on β_2, which contradicts Theorem 52.

A similar argument will show the converse to be true; that is, if $r'=r+1$, then $q'=q+1$.

DEFINITION. Suppose that Ws is a radial web skeleton. Then L^+_{nq} and $L^-_{n(q+1)}$ of Ws are said to be *adjacent rays* on β_n.†

THEOREM 55. *Suppose that Ws is a radial web skeleton. Then Ws connects each pair of adjacent rays on* β_n *to a pair of adjacent rays on* β_{n+1}, *and if Ws*

† When $q=q_n$, $q+1$ is to be interpreted as 1.

connects either of a pair of adjacent rays on β_n to a ray on β_{n-1}, then Ws connects this pair to a pair of adjacent rays on β_{n-1}.

Notation. If *Ws* is a radial web skeleton, let C_{nq} denote the component of $D_n - \overline{D}_{n+1}$ whose boundary β_{nq} contains L_{nq} of *Ws*.

THEOREM 56. *If Ws is a radial web skeleton, then $\beta_{nq} \cdot \beta_{n+1}$ is the sum of a finite collection H_{nq} of consecutive rays on β_{n+1} consisting of two extreme rays, $L^-_{(n+1)q}$ and $L^+_{(n+1)q'}$, $q \leqq q'$, and all other rays on β_{n+1} between $L^-_{(n+1)q}$ and $L^+_{(n+1)q'}$; and $L^-_{(n+1)q}$ and $L^+_{(n+1)q'}$ are the only rays of H_{nq} which are connected by Ws to rays of β_n.*

THEOREM 57. *If Ws is a radial web skeleton and n and q are integers such that neither L^+_{nq} nor $L^-_{n(q+1)}$ of Ws is connected to a ray on β_{n-1} by Ws, then there exists a sequence of mutually exclusive arcs $U_{1[nq]}$, $U_{2[nq]}$, $U_{3[nq]}$, $\cdots$ † such that* (1) *for each m, $m>0$, $U_{m[nq]}$ is an arc from $X_{m,nq}$ to $X_{-m,n(q+1)}$ lying except for these two points in $D_{n-1} - \overline{D}_n$, $D_0 = S$, and* (2) *the sequence has no sequential limit set.*

Proof. Let C denote the component of $D_{n-1} - \overline{D}_n$ whose boundary contains $L^+_{nq} + L^-_{n(q+1)}$, and let L denote $\overline{C} \cdot \beta_{n-1}$.‡ Let G denote the collection of all simple domains which lie together with their boundaries in C. By Theorems 44 and 45, G covers C. Since space is completely separable, G contains a countable subcollection $I_1, I_2, I_3, \cdots$ covering C. For each integer m, $m>0$, let M_m denote an arc irreducible from J_m, the boundary of I_m, to L lying except for one end point in C. By Theorem 47, $C - C \cdot (J_m + M_m)$ is the sum of two connected domains, I_m and E_m, where the boundary of E_m consists of $J_m + M_m$ together with the boundary of C. For each m, let $U'_{m[nq]}$ denote an arc from $X_{m,nq}$ to $X_{-m,n(q+1)}$ lying except for these end points in E_m. By Theorem 49, $U'_{m[nq]}$ separates C into two internally simple, simply connected domains. One of these domains V'_m has for its boundary $U'_{m[nq]}$ plus the subray of L^+_{nq} from $X_{m,nq}$ and the subray of $L^-_{n(q+1)}$ from $X_{-m,n(q+1)}$, and contains no point of I_m. If the arcs $U'_{m[nq]}$ are mutually exclusive, then they form the sequence $U_{1[nq]}$, $U_{2[nq]}$, $U_{3[nq]}$, $\cdots$. If this is not the case, let $U_{1[nq]} = U'_{1[nq]}$ and $V_1 = V'_1$, and suppose that $U_{1[nq]} \cdot U'_{2[nq]}$ is not vacuous. Let A_1B_1 and A_2B_2 denote intervals of $U_{1[nq]}$ and $U'_{2[nq]}$ respectively which lie in C and contain $U_{1[nq]} \cdot U'_{2[nq]}$. By Theorems 10 and 45 and Theorem 13 of Chapter III of *Foundations*, there exists a simple closed curve J lying in C and separating $A_1B_1 + A_2B_2$ from $X_{2,nq} + X_{-2,n(q+1)}$ such that the common part of J with each of the intervals $A_2X_{2,nq}$ and $B_2X_{-2,n(q+1)}$ of $U'_{2[nq]}$ is connected. Now since

† This notation, $[nq]$, is used to restrict n and q to values for which the hypothesis of Theorem 57 is satisfied.

‡ When $n=1$, $\overline{C} \cdot \beta_{n-1}$ is to be interpreted as a point of L^-_{nq}.

$\widehat{A_2X_{2,nq}}$ and $\widehat{B_2X_{-2,n(q+1)}}$ are subsets of V_1 and contain no points of V_2', J contains an arc AB lying in $V_1 \cdot V_2'$ except for A and B irreducible from $A_2X_{2,nq}$ to $B_2X_{-2,n(q+1)}$. Let $U_{2[nq]} = AB + AX_{2,nq}$ of $A_2X_{2,nq} + BX_{-2,n(q+1)}$ of $B_2X_{-2,n(q+1)}$, and let V_2 denote the component of $V_1 - \widehat{U}_{2[nq]}$ which has the subray of L_{nq}^{+} from $X_{2,nq}$ and the subray of $L_{n(q+1)}^{-}$ from $X_{-2,n(q+1)}$ on its boundary. The domain V_2 is a subset of $V_1 \cdot V_2'$. This process may be continued and the required sequence of arcs constructed, for it is clear that since $V_1 \supset V_2 \supset V_3 \supset \cdots$ and $\overline{V}_1 \cdot \overline{V}_2 \cdot \overline{V}_3 \cdot \cdots = 0$, the sequence of arcs has no sequential limit set.

Notation and construction. Now on a radial web skeleton it is possible to construct what will be called a *radial web*. Let Ws denote a radial web skeleton about an edge point P. For each integer triplet, m (positive or negative), $n>0$, and $0<q\leqq q_n$, there exists a continuous reversible transformation of the point set $T_{m,nq} - Y_{m,nq}$ of Ws into the number set $1/n \geqq t > 1/(n+1)$.

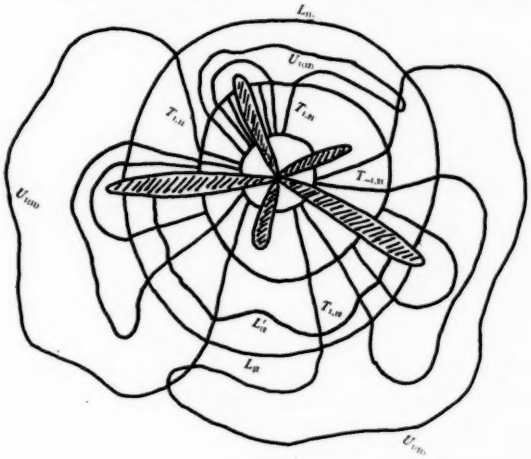

FIGURE 5

Let $X_{m,nq}^{t}$ denote the point of $T_{m,nq}$ whose transform is the number t. Then, of course, $X_{m,nq} = X_{m,nq}^{1/n}$. Let ρ denote a countable set of numbers everywhere dense on $0<t<1$ and containing all the reciprocals of positive integers except 1. For each pair of positive integers, n and q, $q\leqq q_n$, and for each t of ρ such that $1/n \geqq t > 1/(n+1)$, let L_{nq}^{t} denote an open curve lying in C_{nq} such that (1) $L_{nq}^{t} = L_{nq}$ if $t = 1/n$, (2) L_{nq}^{t} intersects $T_{m,nq}$ in the point $X_{m,nq}^{t}$, and (3) if t and t' are different numbers of ρ, $L_{nq}^{t} \cdot L_{nq}^{t'} = 0$. Since ρ is countable, these open curves may be constructed one at a time by the use of Theorem 18 and repeated applications of Theorem 46.

For each pair of integers, n and q, $n>0$ and $q\leqq q_n$, such that L_{nq}^{+} is not

connected by Ws to a ray on β_{n-1}, there exists by Theorem 57 a sequence $U_{1[nq]}, U_{2[nq]}, U_{3[nq]}, \cdots$ of mutually exclusive arcs such that (1) for each m, $m>0$, $U_{m[nq]}$ is an arc from $X_{m,nq}$ to $X_{-m,n(q+1)}$ lying except for these points in $D_{n-1}-\bar{D}_n$, $D_0=S$, and (2) the sequence has no sequential limit set. For each t of ρ, let $\beta^t=\sum_{q=1}^{q_n} L^t_{nq}$. The point set W consisting of Ws together with all the point sets β^t and $U_{m[nq]}$ is called a *radial web* about P, and Ws is said to be the skeleton of W.

Notation and definition. If $X_{m,nq}$ is a point of a ray on β_n of the skeleton Ws of the radial web W which is not connected to a ray on β_{n-1} by Ws, let $N_{m,[nq]}$ denote $P+\sum_{r=n}^{\infty} T_{m,rp_r}$, where $p_n=q$ and for $r>n$, p_r is the integer such that $X_{m,rp_r}=Y_{m,(r-1)p_{r-1}}$. For each $[nq]$ and m, $m>0$, let $J_{m[nq]}$ denote the simple closed curve $N_{m,[nq]}+U_{m[nq]}+N_{-m,[n(q+1)]}$, and let $I_{m[nq]}$ denote the complementary domain of $J_{m[nq]}$ which contains $J_{m+1[nq]}-P$.† For each t of ρ, let $U^t_{m[nq]}$ denote the interval of $J_{m[nq]}-P$ which is irreducible from one component of β^t to another component of β^t with $t\leqq 1/n$.

For each pair of values, t' and t'' of ρ, $t'>t''$, let $\beta^{t't''}_{m,[nq]}$ denote the point set consisting of the interval of $N_{m,[nq]}$ between $\beta^{t'}$ and $\beta^{t''}$, when both t' and t'' are less than $1/n$, together with the rays of $\beta^{t'}$ and $\beta^{t''}$ from the end points of this interval which intersect $N_{m+1,[nq]}$. It is clear that $\beta^{t't''}_{m,[nq]}$ separates space into two mutually exclusive connected domains, one of which, $I^{t't''}_{m,[nq']}$, is a subset of $I_{m[nq]}$, is the sum of an infinite number of simple domains, and has $\beta^{t't''}_{m,[nq]}$ for its complete boundary. For each m, $m>0$, let $R^{t't''}_{m[nq]}$ denote $I^{t't''}_{m,[nq]}+I^{t't''}_{-m,[n(q+1)]}$. The boundary, $\beta^{t't''}_{m[nq]}$, of $R^{t't''}_{m[nq]}$ is $\beta^{t't''}_{m,[nq]}+\beta^{t't''}_{-m,[n(q+1)]}$. Let $P^t_{[nq]}$ denote the collection of all domains $R^{t't''}_{m[nq]}$, where $t''<t<t'$. The collection $P^t_{[nq]}$ will be hereinafter referred to as an *ideal point*.‡ The set of ideal points, $\sum_t P^t_{[nq]}$, will be called the *spur* $N_{[nq]}$ and $\sum N_{[nq]}$ will be called the *cluster* at P or about P.

THEOREM 58. *For each* $[nq]$, *the sequence* $J_{1[nq]}, J_{2[nq]}, J_{3[nq]}, \cdots$ *has* P *for its sequential limit set and* $I_{1[nq]}\cdot I_{2[nq]}\cdot I_{3[nq]}, \cdots = 0$.

THEOREM 59. *For each* m, $W-W\cdot\sum_{[nq]} I_{m[nq]}$ *is a closed and compact point set.*

THEOREM 60. *If* $P^t_{[nq]}$ *is an ideal point and* M *is a compact point set, there exists a number* δ *and an integer* $\bar{m}$, *such that if* $0<t'-t''<\delta$, $m>\bar{m}$, *and* $R^{t't''}_{m[nq]}$ *belongs to* $P^t_{[nq]}$, *then* $\bar{R}^{t't''}_{m[nq]}$ *contains no point of* M.

THEOREM 61. *If* t *is a number of* ρ *and* M *is a compact point set, there exists a number* $\bar{m}$ *such that if* $m>\bar{m}$, *then for no* $[nq]$ *does* $U^t_{m[nq]}$ *contain a point of* M.

† The point sets $U_{m[nq]}$ may intersect the sets β^t in more than one point.

‡ This is analogous to a method used by R. L. Moore to define point in terms of *piece*. See R. L. Moore, *A set of axioms for plane analysis situs*, Fundamenta Mathematicae, vol. 25 (1935), pp. 13–28.

Notation and construction. By Theorem 24 the set of edge points may be put into a simple countable sequence $P_1, P_2, P_3, \cdots$. Let $W_1, W_2, W_3, \cdots$ and, for each integer j, $R_{1j}, R_{2j}, R_{3j}, \cdots$ and $D_{1j}, D_{2j}, D_{3j}, \cdots$ denote sequences of point sets such that (1) for each n and j, R_{nj} is a region of G_{n+j} of Axiom 1, (2) for each n and j, D_{nj} is a pseudo-simple domain with respect to P_j and a simply connected domain Q_{nj} lying with its boundary in R_{nj}, and D_{nj} contains $R_{(n+1)j}$, (3) for each j, W_j is a radial web about P_j whose skeleton is constructed on the domains, $D_{1j}, D_{2j}, D_{3j}, \cdots$, and lies in R_{1j}, (4) if $j'>j$ and $R_{1j'}$ contains a point of D_{1j}, then $\overline{R}_{1j'}$ is a subset of R_{1j}, and (5) for each integer j', there exists an integer m such that if $j<j'$, then $\overline{R}_{1j'}$ contains no point of $\overline{\sum_{[nq]} I_{m[nq]}}$ of W_j. Let G denote a countable collection of simple domains such that if X is a simple point of S and R is a region containing X, some domain of G contains X and lies together with its boundary in R. For each integer i, let H_i denote the collection of all the domains D such that either (1′) D is a simple domain of G which is a subset of a region of G_i of Axiom 1, or (2′) for some pair of integers n and j, $n+j \geqq i$, D is a pseudo-simple domain D_{nj}, or (3′) for some triplet of integers, m, n, and q, and some pair of numbers, t' and t'', of ρ, with $m \geqq i$ and $0<t'-t'' \leqq 1/i$, $D=R^{t''t'}_{m[nq]}$ of W_j for some j.

DEFINITION. If for some i and j, D is a domain of H_i and $P^i_{[nq]}$ is an ideal point defined from W_j, then D is said to *inclose* $P^i_{[nq]}$ provided that some domain of $P^i_{[nq]}$ together with its boundary is a subset of D.

DEFINITION. Let S' be a space in which *point* is to be interpreted as meaning either a point of S or one of the ideal points defined by W_j for some j, and in which *region* is to be interpreted as meaning a point set g' consisting of the points of some domain D of H_i, for some i, together with the ideal points which D incloses. A point of S' which is also a point of S will be called an "ordinary" point and a point of S' which is not a point of S will be called an "ideal" point. Regions in S' will be referred to as *G'-regions*.

From here on the space to be considered is S'.

THEOREM 63. *In S, point and limit point are unchanged by the above definition of point and region in S'.*

THEOREM 64. *Every point of S' is a limit point of S.*

THEOREM 65. *If g' is a G'-region, then $\bar{g}'=\overline{g' \cdot S}$.*

THEOREM 66. *If J is a simple closed curve of "ordinary" points, then $S'-J$ is the sum of two mutually exclusive connected domains each having J for its complete boundary.*

Proof. The point set $S-J$ is the sum of two mutually separate connected sets H and K. Let $H'=\overline{H}-\overline{H}\cdot J$ and $K'=\overline{K}-\overline{K}\cdot J$. Suppose that H' has a point A in common with K'. The point A is an "ideal" point $P^t_{[nq]}$. By Theorem 60 there exists a number $\bar{m}_1$ such that if $m>\bar{m}_1$ and $R^{t't''}_{m[nq]}$ belongs to $P^t_{[nq]}$, then $\overline{R}^{t't''}_{m[nq]}\cdot S$ contains no point of J. But $R^{t't''}_{m[nq]}$ contains points of both H and K. Hence, $I^{t't''}_{m,[nq]}+\beta^{t't''}_{m,[nq]}$ is a subset of H, say, and $I^{t't''}_{-m,[n(q+1)]}+\beta^{t't''}_{-m,[n(q+1)]}$ is a subset of K. By Theorem 61 there exists a number $\bar{m}_2$ such that if $m>\bar{m}_2$, then $U^t_{m[nq]}$ does not intersect J. But for each m, $U^t_{m[nq]}$ intersects both $\beta^{t't''}_{m,[nq]}$ and $\beta^{t't''}_{-m,[n(q+1)]}$, $t''<t<t'$, which leads to a contradiction when m is greater than both $\bar{m}_1$ and $\bar{m}_2$. Hence, H' and K' have no point in common. Since H and K are connected and have J on their respective boundaries, H' and K' are connected and have J for their respective boundaries. By Theorem 64, $S'=J+H'+K'$.

THEOREM 67. *If $P^t_{[nq]}$ belongs to the cluster of "ideal" points about the point P_j, and g' is a G'-region containing the point $P^t_{[nq]}$ such that $g'\cdot S$ is not a domain of $P^t_{[nq]}$, then $\bar{g}'\cdot S$ contains P_j.*

Proof. Suppose that $g'\cdot S$ is not a domain of $P^t_{[nq]}$ but contains a domain R of $P^t_{[nq]}$ and $\bar{g}'\cdot S$ does not contain P_j.

Case I. If $g'\cdot S$ is of type (1′),† its boundary J separates R from P_j. Therefore, infinitely many of the simple closed curves $J_{1[nq]}, J_{2[nq]}, J_{3[nq]}, \cdots$ of W_j would intersect J. But since J is a compact subset of S, the sequential limit set of $J_{1[nq]}, J_{2[nq]}, J_{3[nq]}, \cdots$ of W_j would contain points of S other than P_j, contrary to Theorem 58.

Case II. If $g'\cdot S$ is of type (2′) and its boundary was used in the construction of the web $W_{j'}$, then j' is not j and one may see, from Theorem 58 and a method similar to that of Theorem 40, that the boundary of $g'\cdot S$ and a finite number of the arcs of the type $U^t_{m[nq]}$ of $W_{j'}$ contains a simple closed curve J lying in S and separating $g'\cdot S$, and hence R, from P_j. This leads to a contradiction as in Case I.

Case III. If $g'\cdot S$ is of type (3′), there exists an integer j', such that j' is not j and $g'\cdot S$ was defined using the web $W_{j'}$. Hence, $P_{j'}$ is not P_j and the boundary of $g'\cdot S$ together with one of the simple closed curves of the type $J_{m[nq]}$ of $W_{j'}$ contains a simple closed curve J lying in S and separating $g'\cdot S$, and hence R, from P_j. This leads to a contradiction as in Case I.

DEFINITION. For each i, let G'_i denote the collection of all G'-regions g' of S' such that $D=g'\cdot S$ is an element of H_i. The region g' is said to be *defined by* or *obtained from* D.

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† See **Notation and construction**, p. 87.

THEOREM 68. *If $P^t_{[nq]}$ is an "ideal" point, there exists an integer k such that no region of G_k' contains $P^t_{[nq]}$ except those regions of G_k' defined from domains belonging to $P^t_{[nq]}$.*

Proof. Let $P_{\bar{j}}$ denote the edge point P such that the cluster about P contains $P^t_{[nq]}$. Let $\bar{n}$ denote an integer such that $t>1/\bar{n}$. There exists an integer $\bar{m}$ such that if i is an integer greater than $\bar{m}$ and R is a region in S of G_i of Axiom 1 which contains $P_{\bar{j}}$, then R is a subset of $D_{\bar{n}\bar{j}}$ in S. Suppose that g' is a region of G_i', $i>\bar{m}$, of the type (1′) or (2′) which contains $P^t_{[nq]}$.† Then by Theorem 67, $\bar{g}'\cdot S$ contains $P_{\bar{j}}$. But $\bar{g}'\cdot S$ is a subset of a region R of G_1 of Axiom 1. Hence, R contains $P_{\bar{j}}$ and points not belonging to $D_{\bar{n}\bar{j}}$, which is a contradiction. Likewise, if $j\geqq\bar{m}$ and $j\neq\bar{j}$, no region of type (3′) obtained from a domain of an "ideal" point of the cluster about P_j contains $P^t_{[nq]}$ because each such domain together with its boundary in S is a subset of a region of G_i of Axiom 1 with $i>\bar{m}$. Now by Theorem 58, for each integer $j, j<\bar{m}$ and $j\neq\bar{j}$, there exists an integer $\bar{m}_j$ such that for $m>\bar{m}_j$, $\overline{\sum_{[nq]} I_{m[nq]}}$ defined from W_j does not contain $P_{\bar{j}}$. Then if g' is a region of type (3′) of G_i', $i>\bar{m}_j$, defined by W_j, $\bar{g}'\cdot S$ does not contain $P_{\bar{j}}$. Hence, by Theorem 67, R does not contain $P^t_{[nq]}$. Let k denote an integer greater than $\bar{m}$ and for each $j, j<\bar{m}$ and $j\neq\bar{j}$, greater than $\bar{m}_j$. Thus, if $i>k$ and g' is a region of G_i' which contains $P^t_{[nq]}$, then g' was obtained from a domain of $P^t_{[nq]}$, for no two regions of type (3′) defined by $W_{\bar{j}}$ containing points of different spurs of the cluster about $P_{\bar{j}}$ have a point in common.

THEOREM 69. *The space S' satisfies Axiom* 1.

Proof. It can be shown with the help of Theorems 67 and 68 that S' satisfies parts (1), (2), and (3) of Axiom 1.‡ Considerable argument is required but is left to the reader.

It will now be shown that part (4) of Axiom 1 is satisfied. Suppose that $M_1, M_2, M_3, \cdots$ is a sequence of closed subsets of S' such that for each i, M_i contains M_{i+1} and is a subset of a region g_i' of G_i'.

Case I. Suppose that for infinitely many values of i, g_i' is of type (1′) or (2′). Suppose for simplicity that this is true for all integers i. Then for each i, $K_i=(\bar{g}_1'\cdot S)\cdot(\bar{g}_2'\cdot S)\cdots(\bar{g}_i'\cdot S)$ is a non-vacuous point set closed with respect to S such that K_i contains K_{i+1}. But for each i, $\bar{g}_i'\cdot S$, and hence K_i, is a subset of a region R_i of G_i of Axiom 1. By Axiom 1 for S, there exists a point P of S which for each i belongs to K_i. Hence in S', P belongs to the

† See **Notation and construction**, p. 87.

‡ See footnote p. 54. The role of the collections $G_1, G_2, G_3, \cdots$ is now to be taken by the collections $G_1', G_2', G_3', \cdots$.

sequential limit set of $M_1, M_2, M_3, \cdots$ and therefore for each i, P belongs to M_i.

Case II. If Case I does not occur, then there exists an integer $\bar{j}$ such that for infinitely many values of i, g_i' is a G'-region of type (3') defined by the web $W_{\bar{j}}$; for otherwise infinitely many of these regions of type (3') could be replaced by regions of type (2') satisfying the above conditions. Suppose for simplicity that for each i, g_i' is of type (3') and defined by $W_{\bar{j}}$. Since no two regions of type (3') defined by $W_{\bar{j}}$ which have a point in common contain points of different spurs of the cluster about $P_{\bar{j}}$, there exist for each i, integers m, n, and q, and numbers t' and t'', $0<t''<t'<1$, such that (1) $R^{t't''}_{m[nq]}$ defined from $W_{\bar{j}}$ is $g_i' \cdot S$, (2) $[nq]$ does not vary with i, and (3) $m\to\infty$ and $(t'-t'')\to 0$, as $i\to\infty$. Since for each i, g_i' contains points of g_1', there exists a number t, $0<t<1$, such that as $i\to\infty$, $t'\to t$ and $t''\to t$. Hence $P^t_{[nq]}$ is common to $M_1, M_2, M_3, \cdots$.

THEOREM 70. *Every G'-region is a simple domain.*

THEOREM 71. *The space S' satisfies Axioms 2, 3, 5_1^*, and 7 and contains no edge points.*

Theorem 71 is a consequence of Theorems 64, 69, and 70 and the fact that the collection of all G'-regions is countable.

THEOREM 72. *If J is a simple closed curve of "ordinary" points, I is one of its complementary domains, the points A and B separate the points C and D on J, and AXB is an arc of "ordinary" points such that $\widehat{AXB}$ is a subset of I, then $I-\widehat{AXB}$ is the sum of two simple domains I_1 and I_2 such that the boundary of I_1 is the simple closed curve ACB(of J)$+AXB$ and the boundary of I_2 is the simple closed curve ADB(of J)$+AXB$.*

Proof. Since the theorem holds for S, it can be shown to hold for S' with the help of Theorem 66.

THEOREM 73. *Theorem 72 remains true when the segment AXB contains just one "ideal" point.*

Proof. Suppose for simplicity of notation that X is the "ideal" point of the segment AXB. Let W_j denote the web used in the definition of the domains of X, and let $N_{[nq]}$ denote the spur which contains X. A number of cases occur depending upon where the edge point P_j is. Only one of these cases will be treated. The others may be dealt with in a somewhat similar but more complicated manner.

Suppose that P_j does not belong to $J+AXB$. Let $J_{1[nq]}, J_{2[nq]}, J_{3[nq]}, \cdots$ denote the sequence of simple closed curves in W_j which has as its sequential limit set $P_j+N_{[nq]}$. It can be shown by using Theorem 10 that there exists a

sequence $C_1, C_2, C_3, \cdots$ of simple closed curves whose sequential limit set is $P_j+N_{[nq]}$ such that for each k, (1) there is an integer m, $m \geqq k$, so that C_k is a subset of $J_{m[nq]}+AXB$, (2) $C_k \cdot \widehat{AX}$ is an arc $A_k'A_k''$ and $C_k \cdot \widehat{BX}$ is an arc $B_k'B_k''$, (3) the components of $C_k - C_k \cdot AXB$ are two segments, $A_k'X_kB_k'$ and $A_k''Y_kB_k''$, which approach the rays $\overrightarrow{AX}$ and $\overrightarrow{BX}$ in X from different sides but $A_k'X_kB_k'$ and $A_{k+1}'X_{k+1}B_{k+1}'$ approach these rays from the same side, and (4) the common part of C_k and C_{k+1} is at most P_j, C_k separating $C_{k+1}-C_{k+1}\cdot C_k$ from J. For each k, let I_{1k} and I_{2k} denote respectively the complementary domains of AA_k' (of AX)$+A_k'X_kB_k'+BB_k'$ (of BX)$+ACB$ or AA_k'' (of AX)$+A_k''Y_kB_k''+BB_k''$ (of BX)$+ACB$ and AA_k' (of AX) $+A_k'X_kB_k'+BB_k'$ (of BX)$+ADB$ or $AA_k''+A_k''Y_kB_k''+BB_k''$ (of BX) $+ADB$ which are subsets of I. Let I_1 denote $I_{11}+I_{12}+I_{13}+\cdots$ together with the limit points of this set which do not belong to $J+AXB$, and let I_2 denote $I_{21}+I_{22}+I_{23}+\cdots$ together with the limit points of this set which do not belong to $J+AXB$. These point sets, I_1 and I_2, can be shown to be domains satisfying the conclusions of Theorem 72.

THEOREM 74. *Theorem 72 remains true if the arc AXB contains only a finite number of "ideal" points.*

Proof. The method of proof will be that of mathematical induction. Theorem 73 shows that Theorem 74 holds true when the number of "ideal" points is one. We shall suppose that the theorem holds for $j-1$ or less points and show that it holds for j points. Suppose that $AXB\cdot(S'-S)=Z_1+Z_2+\cdots +Z_j$. Let $N_{[nq]}$ denote a spur containing one of these "ideal" points, and let Z_r denote that one of them on $N_{[nq]}$ which is farthest in $N_{[nq]}$ from its end point. Now the argument of Theorem 73 may be applied, Z_r playing the role of X.

THEOREM 75. *If J is a simple closed curve containing only a finite number of "ideal" points, then $S'-J$ is the sum of two mutually exclusive connected domains each having J for its boundary.*

Proof. Let O denote an ordinary simple point of J, and let R denote a simple domain whose boundary contains no "ideal" points such that $R\cdot J$ is a subset of an arc in J of "ordinary" points. Let ACB denote an arc of "ordinary" points in R which has only the two points A and B in common with J and such that an arc ADB of J is a subset of R. It is now easy to see that, with the help of Theorem 74, Theorem 75 may be established.

THEOREM 76. *Theorem 72 remains true regardless of the composition of the arc AXB.*

Proof. By slight modifications of the arguments of R. L. Moore a number

of the theorems of Chapter III of *Foundations* can be shown to hold true in S' if the simple closed curves involved contain at most a finite number of "ideal" points.† In particular Theorem 13 of Chapter III of *Foundations* holds true in S' for simple closed curves which contain at most a finite number of "ideal" points. Theorem 76 may then be established in the following manner.

Since the boundaries of G'-regions are simple closed curves containing at most a finite number of "ideal" points, it is easy to see that there exist two sequences of mutually exclusive arcs $A_1'X_1B_1'$, $A_2'X_2B_2'$, $A_3'X_3B_3'$, $\cdots$ and $A_1''Y_1B_1''$, $A_2''Y_2B_2''$, $A_3''Y_3B_3''$, $\cdots$ such that (1) for each k the segments $A_k'X_kB_k'$ and $A_k''Y_kB_k''$ are subsets of I, contain at most a finite number of "ideal" points and the points A_k', B_k', A_k'', and B_k'' belong to segments AC, BC, AD, and BD of J respectively, (2) for each k, the complementary domain of $A_k'X_kB_k'+A_k'AA_k''$ (of J)$+B_k'BB_k''$ (of$J$)$+A_k''Y_kB_k''$ which is a subset of I contains the segment of AXB and the corresponding complementary domain of $A_{k+1}'X_{k+1}B_{k+1}'+A_{k+1}'AA_{k+1}''$(of J)$+B_{k+1}'BB_{k+1}''$(of J) $+A_{k+1}''Y_{k+1}B_{k+1}''$, and (3) the sequential limit set of either of these two sequences of arcs is AXB. If for each k, I_{1k} and I_{2k} denote respectively the complementary domains of $A_k'CB_k'$(of J)$+A_k'XB_k'$ and $A_k''DB_k''$(of J) $+A_k''YB_k''$ which are subsets of I, then $I_1=\sum I_{1k}$ and $I_2=\sum I_{2k}$ satisfy the conclusions of Theorem 72.

Now as in Theorem 75 it is possible to establish

THEOREM 77. *If J is a simple closed curve containing an arc of "ordinary" points, then $S'-J$ is the sum of two mutually exclusive connected domains each having J for its boundary.*

With the help of Theorem 77 and methods similar to those in *Foundations* it can be shown that Theorem 74 remains true when the simple closed curve J is of any nature, provided it contains an arc of "ordinary" points. With this in mind, it is easy to see that Theorem 76 remains true when J is of any nature, provided it contains an arc of "ordinary" points. Then as in Theorem 74 it is possible to establish

THEOREM 78. *If J is a simple closed curve, $S'-J$ is the sum of two mutually exclusive connected domains each having J for its boundary.*

THEOREM 79. *The space S' satisfies Axiom* 4.

THEOREM 80. *The space S is homeomorphic with a subset of a space S' which satisfies Axioms* 0–4, 5_1^* *and* 7 *and contains no edge points.*

† It may be helpful for the reader to note that an arc in S' may be approximated by an arc in S' which contains at most a finite number of "ideal" points since a G'-region has at most two "ideal" points in its boundary.

THEOREM 81. *The space S is homeomorphic with a subset of a plane or a sphere.*

Proof. J. H. Roberts† has shown that S' of Theorem 80 is homeomorphic with a subset of a plane or a sphere.

Some of the preceding results will now be summarized for metric spaces

THEOREM 82. *A locally connected, complete metric space in which the Jordan Curve Theorem‡ is satisfied non-vacuously and in which Axiom 5_1^* holds true, is a cyclicly connected subset of a plane (if not compact) or a sphere (if compact).*

Proof. A locally connected complete metric space satisfies Axioms 0, 1, and 2. And such a space in which Axiom 5_1^* holds true satisfies Axioms 4 and 3 if the Jordan Curve Theorem holds true non-vacuously (Theorem B). Axiom 7 holds true in such a space by Theorem 31. Thus Theorem 82 follows from Theorem 81.

THEOREM 83. *A locally connected, locally peripherally connected, complete metric space in which the Jordan Curve Theorem is non-vacuously satisfied is a cyclicly connected subset of a plane or a sphere.*§

THEOREM 84. *A locally connected, locally peripherally compact, complete metric space in which the Jordan Curve Theorem is non-vacuously satisfied is a cyclicly connected subset of a plane or a sphere.*

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† J. H. Roberts, *Concerning compact continua in certain spaces of R. L. Moore*, Bulletin of the American Mathematical Society, vol. 39 (1933), pp. 615–621.

‡ Axiom 4.

§ See p. 53.

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THE GEOMETRY OF ISOGONAL AND EQUI-TANGENTIAL SERIES*

BY
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INTRODUCTION

We begin by considering certain simple operations on the lineal elements of the plane. A *turn* T_α converts each element into one having the same point and a direction making a fixed angle α with the original direction. By a *slide* S_k the line of the element remains the same and the point moves along the line a fixed distance k. These transformations together generate a continuous group of three parameters G_3.† We call any transformation of this group a *whirl*. The only contact transformations in this group are the dilatations.

We note that the group of whirls and the group of rigid motions are isomorphic, and that the two groups are commutative, together generating an interesting six-parameter group which will be studied elsewhere. This is a sub-group of the fifteen-parameter element group converting turbines into turbines, which I introduced in 1911, itself an extension of Lie's ten-parameter contact group of higher circular geometry.

We define ∞^1 elements to be a series of elements. By applying a turn T_α to the elements of a union, we obtain a series of elements which we call an *isogonal series*. When a slide S_k is applied to the elements of a union, the resulting series is called an *equi-tangential series*. (Finally the result of applying a whirl to the elements of a union is defined to be a *whirl series*. See a forthcoming Columbia dissertation by J. De Cicco, whom I wish to thank for verifying the calculations of the present paper.)

An element transformation is a correspondence between the elements of the plane. Any element transformation which transforms every union into a union is a contact transformation. We shall show that *any element transformation which carries every isogonal series into an isogonal series must be the product of a conformal point transformation by a turn.* Dually, any element transformation which transforms every equi-tangential series into an equi-

* Presented to the Society, September 9, 1930; received by the editors December 16, 1936.

† See Kasner, *The group of turns and slides and the geometry of turbines*, American Journal of Mathematics, vol. 33 (1911), pp. 193–202. This theory has recently been extended to spherical geometry (in an elegant synthetic treatment) by K. Strubecker, Jahresbericht der Deutschen Mathematiker Vereinigung, vol. 44 (1934), pp. 184–198, who uses the term *turbine-rotation* for whirl. The fact that G_3 can be generalized to any surface of constant curvature (but probably not to variable curvature) was noted in my paper, p. 193.

tangential series must be the product of an equi-long line transformation by a magnification by a slide.

For the analytic representation of an element, it will be found convenient to use two systems of coordinates called the cartesian and the hessian coordinate systems respectively. The cartesian coordinates of an element E are of course (x, y, p), where p is the slope of the line and (x, y) are the cartesian coordinates of the point. The hessian coordinates of the element E are (u, v, w), where v is the perpendicular distance from the origin to the line of the element E, u is the angle between the perpendicular and the initial line, and w is the distance from the foot of the perpendicular to the point of the element E.

A full discussion of the hessian system is given in Scheffer's fundamental paper (Mathematische Annalen, vol. 60 (1905)). The line (u, v) and the distance w are oriented.

If a general element transformation

$$X = \phi(x, y, p), \qquad Y = \psi(x, y, p), \qquad P = \chi(x, y, p)$$

is not a contact transformation it is known that *it will convert exactly* ∞^2 *unions into unions.** If it converts more than ∞^2, it must be a contact transformation, and thus convert all union into unions. Extensions of this theorem, involving ∞^3 and ∞^4 series of elements are found in the present paper. (See the five main theorems in italics.) In particular, it is shown that a general element transformation *converts* ∞^4 *isogonal series into isogonal series*; if more than ∞^4 arise, then all are preserved, and the resulting group of transformations is compounded of conformal (point) transformations and turns. The dual result for equi-tangential series is also valid, but demands an independent proof. There is no automatic principal of duality connecting the two theories, but only a rough analogy.

The equations of the turn T_α in cartesian coordinates are

$$X = x, \qquad Y = y, \qquad \Theta = \theta + \alpha,$$

where θ denotes the inclination of the element; and hence the equations of any isogonal series are

$$y = y(x), \qquad \arctan p = \arctan y'(x) + \arctan \alpha,$$

where α is the constant angle.

The equations of the slide S_k in hessian coordinates are

$$U = u, \qquad V = v, \qquad W = w + k,$$

* See Kasner, *The general transformation theory of differential elements*, American Journal of Mathematics, vol. 32 (1910), p. 393.

and hence the equations of any equi-tangential series are

$$v = v(u), \qquad w = v'(u) + k,$$

where k is the constant distance.

A *turbine* is a special case of both isogonal series and equi-tangential series where the base curve is a circle. All whirls convert turbines into turbines. The total group of turbines involves fifteen parameters. Lie's ten-parameter group converts circles into turbines.

PART I. ISOGONAL SERIES AND THE CONFORMAL GROUP

We now prove our theorems about element transformations.

THEOREM 1. *Under any element transformation there exists a three-parameter family of unions which are carried into isogonal series. Any element transformation which carries every union into an isogonal series must be the product of a contact transformation by a turn.*

By the element transformation

$$X = \phi(x, y, p), \qquad Y = \psi(x, y, p), \qquad P = \chi(x, y, p),$$

the elements of a union*

$$y = y(x), \qquad p = y'(x),$$

become the elements

$$X = \phi(x, y, y'), \qquad Y = \psi(x, y, y'), \qquad P = \chi(x, y, y').$$

If this series is an isogonal series, then we must have

$$\arctan \chi = \arctan \frac{d\psi}{d\phi} + \text{constant}. \tag{1}$$

Differentiating (1) with respect to x, we obtain

$$\frac{d}{dx}\left[\frac{\frac{d\psi}{dx}}{\frac{d\phi}{dx}}\right] = \frac{\frac{d\chi}{dx}}{1+\chi^2}\left[1 + \left[\frac{\frac{d\psi}{dx}}{\frac{d\phi}{dx}}\right]^2\right]. \tag{2}$$

Now let f denote the fraction, involving x, y, y', y'',

* This representation omits point-unions, but we can verify the validity of our main theorems by a separate discussion without difficulty.

$$f = \frac{\dfrac{d\psi}{dx}}{\dfrac{d\phi}{dx}} = \frac{\psi_x + y'\psi_y + y''\psi_{y'}}{\phi_x + y'\phi_y + y''\phi_{y'}}. \tag{3}$$

Then from (2) and (3), we obtain

$$\frac{df}{dx} = \frac{\chi_x + y'\chi_y + y''\chi_{y'}}{1 + \chi^2}\,[1 + f^2]. \tag{4}$$

The equation (4) is a differential equation in y and the highest order of y in this equation is the third order. For the coefficient of y''' in the expanded equation is given by

$$(\phi_x + y'\phi_y)\psi_{y'} - (\psi_x + y'\psi_y)\phi_{y'},$$

which, in general, is not zero. Thus, it follows that this differential equation has three parameters in its complete solution and therefore there exists a *three-parameter family* of unions which by the given element transformation are carried into isogonal series.

We now determine all the element transformations which carry every union into an isogonal series. Then the condition (4) where f is given by (3) must hold for every union. Thus (4) must be an identity in x, y, y', y'', y'''. The coefficient of y''' must then be zero, so that

$$(\phi_x + y'\phi_y)\psi_{y'} - (\psi_x + y'\psi_y)\phi_{y'} = 0. \tag{5}$$

Again (5) must be an identity in x, y, y'. Hence

$$\frac{\psi_x + p\psi_y}{\phi_x + p\phi_y} = \frac{\psi_p}{\phi_p}. \tag{6}$$

From (6) we obtain

$$f = \frac{\psi_x + y'\psi_y + y''\psi_{y'}}{\phi_x + y'\phi_y + y''\phi_{y'}} = \frac{\psi_x + y'\psi_y}{\phi_x + y'\phi_y} = \frac{\psi_{y'}}{\phi_{y'}} = \lambda(x, y, y'). \tag{7}$$

To determine λ, substitute it into equation (4) so that

$$\frac{\lambda_x + y'\lambda_y + y''\lambda_{y'}}{1 + \lambda^2} = \frac{\chi_x + y'\chi_y + y''\chi_{y'}}{1 + \chi^2}.$$

This is an identity. Hence

$$\frac{\lambda_{y'}}{1 + \lambda^2} = \frac{\chi_{y'}}{1 + \chi^2}, \qquad \frac{\lambda_x + y'\lambda_y}{1 + \lambda^2} = \frac{\chi_x + y'\chi_y}{1 + \chi^2}.$$

From these two equations, it follows immediately, that

$$\text{arc tan } \lambda = \text{arc tan } \chi + \alpha, \qquad (\alpha \text{ constant}). \tag{8}$$

From (6), (7), and (8) we obtain

$$\frac{\psi_x + p\psi_y}{\phi_x + p\phi_y} = \frac{\psi_p}{\phi_p} = \tan(\text{arc tan } \chi + \alpha). \tag{9}$$

From (9), we obtain our result. So Theorem 1 is proved.

We next study the transformation of isogonal series into isogonal series.

THEOREM 2. *Under any element transformation there exists a four-parameter family of isogonal series which are carried into isogonal series. If more than ∞^4 such series exist then all ∞^∞ isogonal series exist. Any element transformation which carries every isogonal series into an isogonal series must be the product of a conformal point transformation by a turn. These transformations form a group.*

The proof is rather complicated and will occupy the next three pages. Systems of partial differential equations appear, which fortunately can be integrated in finite form.

Under any element transformation, the elements of an isogonal series

$$y = y(x), \qquad \text{arc tan } p = \text{arc tan } y'(x) + \text{arc tan } k,$$

become the series

$$X = \phi(x, y, p), \qquad Y = \psi(x, y, p), \qquad P = \chi(x, y, p).$$

Now if this series is an isogonal series, we must have

$$\frac{d}{dx}\left(\frac{\frac{d\psi}{dx}}{\frac{d\phi}{dx}}\right) = \frac{\frac{d\chi}{dx}}{1+\chi^2}\left[1 + \left(\frac{\frac{d\psi}{dx}}{\frac{d\phi}{dx}}\right)^2\right]. \tag{10}$$

Let f again denote the fraction

$$f = \frac{\frac{d\psi}{dx}}{\frac{d\phi}{dx}} = \frac{\psi_x + y'\psi_y + p'\psi_p}{\phi_x + y'\phi_y + p'\phi_p}. \tag{11}$$

Since arc tan y' = arc tan p − arc tan k, we have from (11)

$$f = \frac{(1+kp)\psi_x + (p-k)\psi_y + (1+kp)p'\psi_p}{(1+kp)\phi_x + (p-k)\phi_y + (1+kp)p'\phi_p}. \tag{12}$$

From (10) we obtain

$$\frac{df}{dx} = \frac{\frac{d\chi}{dx}}{1+\chi^2}[1+f^2]. \tag{13}$$

Equation (13) is a differential equation in y, and the highest order of y in this equation is the third order. For the coefficient of y''' is the coefficient of p''

$$(1+kp)[\{(1+kp)\phi_x + (p-k)\phi_y\}\psi_p - \{(1+kp)\psi_x + (p-k)\psi_y\}\phi_p],$$

multiplied by the coefficient of y''' in p''. Since the coefficient of y''' in p'' in general is not zero, it follows that the coefficient of y''' in (13) in general is not zero. Therefore the complete solution of (13) will contain three constants of integration and the constant k of the original isogonal series. Thus, we see that there exists a *four-parameter family* of isogonal series which transform into isogonal series. This proves the first part of our theorem.

We now determine all the element transformations which carry every isogonal series into an isogonal series. Since a union is a special type of isogonal series, it follows that any element transformation which changes every isogonal series into an isogonal series must be the product of a contact transformation by a turn. Then from the group properties of the set of turn transformations, it follows that the contact transformation must carry every isogonal series into an isogonal series. Hence, we must find all the contact transformations which transform every isogonal series into an isogonal series.

We shall show that the contact transformation must be a point transformation. If this were not the case, then we should have

$$X = \phi(x, y, p), \qquad Y = \psi(x, y, p), \qquad P = \chi(x, y, p) = \frac{\psi_x + p\psi_y}{\phi_x + p\phi_y} = \frac{\psi_p}{\phi_p},$$

where $\phi_p \neq 0$, and $\psi_p \neq 0$. Now equation (13) must be an identity in x, y, p, p', p'' and k. Hence the coefficient of p'' must be zero so that

$$(1+kp)[\{(1+kp)\phi_x + (p-k)\phi_y\}\psi_p - \{(1+kp)\psi_x + (p-k)\psi_y\}\phi_p] = 0.$$

Since $1+kp \neq 0$, we obtain

$$[(\phi_x + p\phi_y)\psi_p - (\psi_x + p\psi_y)\phi_p] + k[(p\phi_x - \phi_y)\psi_p - (p\psi_x - \psi_y)\phi_p] = 0. \tag{14}$$

Since (14) is an identity in k, we have

$$(\phi_x + p\phi_y)\psi_p - (\psi_x + p\psi_y)\phi_p = 0, \qquad (p\phi_x - \phi_y)\psi_p - (p\psi_x - \psi_y)\phi_p = 0,$$

and therefore we obtain

$$\frac{\psi_x}{\phi_x}=\frac{\psi_y}{\phi_y}=\frac{\psi_p}{\phi_p}. \tag{15}$$

But (15) makes the jacobian of the transformation vanish. This means that the only contact transformations of the desired type are point transformations.

We now find all the point transformations

$$X=\phi(x, y), \qquad Y=\psi(x, y), \qquad P=\frac{\psi_x+p\psi_y}{\phi_x+p\phi_y},$$

which carry every isogonal series into an isogonal series. Then the equation (13) must be an identity where

$$\chi=\frac{\psi_x+p\psi_y}{\phi_x+p\phi_y}, \qquad f=\frac{(1+kp)\psi_x+(p-k)\psi_y}{(1+kp)\phi_x+(p-k)\phi_y}. \tag{16}$$

In (13) we then can set the coefficient of p' equal to zero so that

$$\frac{[(1+kp)\phi_x+(p-k)\phi_y](k\psi_x+\psi_y)-[(1+kp)\psi_x+(p-k)\psi_y](k\phi_x+\phi_y)}{[(1+kp)\phi_x+(p-k)\phi_y]^2} =\frac{\chi_p}{1+\chi^2}\left[1+\left\{\frac{(1+kp)\psi_x+(p-k)\psi_y}{(1+kp)\phi_x+(p-k)\phi_y}\right\}^2\right]. \tag{17}$$

From (16) and (17) we obtain

$$\frac{[(1+kp)\phi_x+(p-k)\phi_y][k\psi_x+\psi_y]-[(1+kp)\psi_x+(p-k)\psi_y](k\phi_x+\phi_y)}{[(1+kp)\phi_x+(p-k)\phi_y]^2+[(1+kp)\psi_x+(p-k)\psi_y]^2} =\frac{(\phi_x+p\phi_y)\psi_y-(\psi_x+p\psi_y)\phi_y}{(\phi_x+p\phi_y)^2+(\psi_x+p\psi_y)^2},$$

which, upon simplification, becomes

$$\frac{[(\phi_x+p\phi_y)+k(p\phi_x-\phi_y)](k\psi_x+\psi_y)-[(\psi_x+p\psi_y)+k(p\psi_x-\psi_y)](k\phi_x+\phi_y)}{[(\phi_x+p\phi_y)+k(p\phi_x-\phi_y)]^2+[(\psi_x+p\psi_y)+k(p\psi_x-\psi_y)]^2} =\frac{\phi_x\psi_y-\psi_x\phi_y}{(\phi_x+p\phi_y)^2+(\psi_x+p\psi_y)^2}. \tag{18}$$

Since (18) is an identity in k, we can set the coefficient of k^2 equal to zero so that

$$\frac{\psi_x(p\phi_x-\phi_y)-\phi_x(p\psi_x-\psi_y)}{(p\phi_x-\phi_y)^2+(p\psi_x-\psi_y)^2}=\frac{\phi_x\psi_y-\psi_x\phi_y}{(\phi_x+p\phi_y)^2+(\psi_x+p\psi_y)^2},$$

and thus we obtain

$$(19) \qquad \frac{-\psi_x\phi_y+\phi_x\psi_y}{(p\phi_x-\phi_y)^2+(p\psi_x-\psi_y)^2}=\frac{\phi_x\psi_y-\psi_x\phi_y}{(\phi_x+p\phi_y)^2+(\psi_x+p\psi_y)^2}.$$

Since the jacobian $\phi_x\psi_y-\psi_x\phi_y\neq 0$, we have

$$(20) \qquad (\phi_x+p\phi_y)^2+(\psi_x+p\psi_y)^2=(p\phi_x-\phi_y)^2+(p\psi_x-\psi_y)^2.$$

Now (20) is an identity in p so that we have

$$\phi_x^2+\psi_x^2=\phi_y^2+\psi_y^2, \qquad \phi_x\phi_y+\psi_x\psi_y=0.$$

These equations are then equivalent to the Cauchy-Riemann equations (together with the conjugate equations)

$$\phi_y=\pm\psi_x, \qquad \phi_x=\mp\psi_y.$$

Hence the point transformation is conformal.

The sufficiency of our result is obvious so that Theorem 2 is completely proved.

Part II. Equi-tangential series and the equi-long group

Theorem 3. *Under any element transformation there exists a three-parameter family of unions which are carried into equi-tangential series. Any element transformation which carries every union into an equi-tangential series must be the product of a contact transformation by a slide.*

By any element transformation

$$U=\phi(u,v,w), \qquad V=\psi(u,v,w), \qquad W=\chi(u,v,w),$$

the elements of a union*

$$v=v(u), \qquad w=v'(u),$$

become the elements

$$U=\phi(u,v,v'), \qquad V=\psi(u,v,v'), \qquad W=\chi(u,v,v').$$

If this series is an equi-tangential series, then

$$\chi=\frac{d\psi}{d\phi}+\text{constant},$$

and hence we obtain

* We are assuming that the union is not a straight line union. The final results are, however, true in all cases.

$$\frac{d}{du}\left(\frac{\psi_u + v'\psi_v + v''\psi_{v'}}{\phi_u + v'\phi_v + v''\phi_{v'}} - \chi\right) = 0. \tag{21}$$

Now (21) is a differential equation in v and the highest order of v is the third order. For the coefficient of v''' is

$$(\phi_u + v'\phi_v)\psi_{v'} - (\psi_u + v'\psi_v)\phi_{v'},$$

which in general is not zero. Hence the complete solution of the differential equation (21) contains three constants of integration, and therefore it is seen that there exists a three-parameter family of unions which are transformed into equi-tangential series. *If more than ∞^3 unions of this kind exist, then all ∞^∞ unions must be of this kind.*

We now determine all the element transformations which carry every union into an equi-tangential series. Then (21) must be an identity in u, v, v', v'', v'''. Thus the coefficient of v''' in (21) must be zero. That is,

$$(\phi_u + v'\phi_v)\psi_{v'} - (\psi_u + v'\psi_v)\phi_{v'} = 0.$$

This is an identity. Hence

$$\frac{\psi_u + w\psi_v}{\phi_u + w\phi_v} = \frac{\psi_w}{\phi_w}. \tag{22}$$

Then we obtain

$$\frac{\psi_u + v'\psi_v + v''\psi_{v'}}{\phi_u + v'\phi_v + v''\phi_{v'}} = \frac{\psi_u + v'\psi_v}{\phi_u + v'\phi_v} = \frac{\psi_{v'}}{\phi_{v'}} = \lambda(u, v, v'). \tag{23}$$

To determine λ, substitute it into (21) and we obtain

$$\frac{d}{du}(\lambda - \chi) = 0.$$

This immediately yields the result

$$\lambda(u, v, w) = \chi(u, v, w) + \alpha, \qquad (\alpha \text{ constant}). \tag{24}$$

From (22), (23), and (24) we obtain

$$\frac{\psi_u + w\psi_v}{\phi_u + w\phi_v} = \frac{\psi_w}{\phi_w} = \chi + \alpha. \tag{25}$$

Interpreting equations (25), we obtain Theorem 3.

THEOREM 4. *Under any element transformation there exists a four-parameter family of equi-tangential series which are carried into equi-tangential series. Any element transformation which carries every equi-tangential series into an equi-*

tangential series must be the product of an equi-long transformation (every two collinear elements are transformed into collinear elements in such a way that the distance between them is preserved), by a magnification by a slide. These transformations form a group.

We observe that this group is generated by *three* partial groups (the infinite equi-long group depending on arbitrary functions of $u+jv$, where $j^2=0$, slides, and magnifications); and is thus larger than the dual group of Theorem 2 which is generated by *two* partial groups (conformal and turns).

We proceed now to the rather long proof of Theorem 4. By any element transformation, the equi-tangential series

$$v = v(u), \qquad w = v'(u) + k,$$

becomes the series

$$U = \phi(u, v, w), \qquad V = \psi(u, v, w), \qquad W = \chi(u, v, w).$$

Now if this series is an equi-tangential series, then we must have

$$\frac{d}{du}\left(\frac{\psi_u + v'\psi_v + w'\psi_w}{\phi_u + v'\phi_v + w'\phi_w} - \chi\right) = 0,$$

and hence

$$\frac{d}{du}\left(\frac{\psi_u + (w-k)\psi_v + w'\psi_w}{\phi_u + (w-k)\phi_v + w'\phi_w} - \chi\right) = 0. \tag{26}$$

The equation (26) is a differential equation in v and the highest order of v in (26) is the third order. For the coefficient of v''' is

$$[\phi_u + (w-k)\phi_v]\psi_w - [\psi_u + (w-k)\psi_v]\phi_w,$$

which in general is not zero. Therefore the complete solution of (26) contains three constants of integration and the constant k of the original equi-tangential series. Thus, it is seen that there exists a four-parameter family of equi-tangential series which are transformed into equi-tangential series.

We now determine all the element transformations which carry every equi-tangential series into an equi-tangential series. Since a union is a special type of equi-tangential series, it follows that any element transformation which carries every equi-tangential series into an equi-tangential series must, by Theorem 3, be the product of a contact transformation by a slide. From this it follows that we must find all contact transformations by which every equi-tangential series corresponds to an equi-tangential series.

We shall show that the contact transformation is a line transformation. If this were not the case, then

$$U = \phi(u, v, w), \qquad V = \psi(u, v, w), \qquad W = \chi(u, v, w) = \frac{\psi_u + w\psi_v}{\phi_u + w\phi_v} = \frac{\psi_w}{\phi_w},$$

where $\phi_w \neq 0$ and $\psi_w \neq 0$. The equation (26) now must be an identity in u, v, w, k, w', w''. Hence the coefficient of w'' must be zero

$$[\phi_u + (w - k)\phi_v]\psi_w - [\psi_u + (w - k)\psi_v]\phi_w = 0. \tag{27}$$

Since (27) is an identity, we have

$$(\phi_u + w\phi_v)\psi_w - (\psi_u + w\psi_v)\phi_w = 0, \qquad \phi_v\psi_w - \psi_v\phi_w = 0,$$

and hence

$$\frac{\phi_u}{\psi_u} = \frac{\phi_v}{\psi_v} = \frac{\phi_w}{\psi_w}. \tag{28}$$

But (28) makes the jacobian of the transformation vanish. This means that the only possible contact transformations of the desired kind are line transformations.

Let us now suppose that our contact transformation has the properties just stated. Its equations are

$$U = \phi(u, v), \qquad V = \psi(u, v), \qquad W = \frac{\psi_u + w\psi_v}{\phi_u + w\phi_v}.$$

Then (26) becomes

$$\frac{d}{du}\left[\frac{\psi_u + (w - k)\psi_v}{\phi_u + (w - k)\phi_v} - \chi\right] = 0, \tag{29}$$

where

$$\chi = \frac{\psi_u + w\psi_v}{\phi_u + w\phi_v}. \tag{30}$$

Since (29) is an identity, the coefficient of w' is zero

$$\frac{[\phi_u + (w - k)\phi_v]\psi_v - [\psi_u + (w - k)\psi_v]\phi_v}{[\phi_u + (w - k)\phi_v]^2} = \chi_w.$$

Simplifying this last equation and taking account of (30), we have

$$\frac{\phi_u\psi_v - \psi_u\phi_v}{[\phi_u + (w - k)\phi_v]^2} = \frac{\phi_u\psi_v - \psi_u\phi_v}{(\phi_u + w\phi_v)^2}.$$

Since $\phi_u\psi_v - \psi_u\phi_v \neq 0$, we then obtain

$$[\phi_u + (w - k)\phi_v]^2 = [\phi_u + w\phi_v]^2. \tag{31}$$

Now (31) is an identity. Hence

$$\phi_v = 0,$$

so that we have

$$\phi = F(u). \tag{32}$$

Substituting (32) into (29) we obtain

$$\frac{d}{du}\left[\frac{\psi_u + (w - k)\psi_v}{F'(u)} - \frac{\psi_u + w\psi_v}{F'(u)}\right] = 0. \tag{33}$$

Now (33) is an identity. Hence the coefficient of k must be zero:

$$\frac{d}{du}\frac{\psi_v}{F'(u)} = 0,$$

and therefore

$$F'(u)[\psi_{vu} + v'\psi_{vv}] - \psi_v F''(u) = 0. \tag{34}$$

The equation (34) is an identity. Hence

$$F'(u)\psi_{vu} - \psi_v F''(u) = 0, \qquad F'(u)\psi_{vv} = 0. \tag{35}$$

From equations (32) and (35) we obtain

$$\begin{cases} U = F(u), \\ V = mvF'(u) + G(u), \\ W = mw + mv\dfrac{F''(u)}{F'(u)} + \dfrac{G'(u)}{F'(u)}, \end{cases} \tag{36}$$

where $F(u)$ and $G(u)$ are functions of u only and where m is a constant. This transformation is the product of an equi-long line transformation by a magnification.

The sufficiency of the result is obvious so that our Theorem 4 is completely proved.

From Theorems 2 and 4 we easily obtain the following result:

THEOREM 5. *Any element transformation which carries every isogonal series into an isogonal series and every equi-tangential series into an equi-tangential series must be the product of a rigid motion by a magnification. We thus obtain the similitude group.*

In conclusion we remark that the family of ∞^2 unions converted into unions by an arbitrary element transformation is of entirely general character; but that the *three-parameter and four-parameter manifolds considered in Theorems* 1–4 *are of special character*. The geometric properties of these manifolds will be considered elsewhere. See abstracts in the Bulletin of the American Mathematical Society, vol. 36 (1930), pp. 353 and 487.

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STOCHASTIC PROCESSES DEPENDING ON A CONTINUOUS PARAMETER*

BY

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Stochastic processes depending on a continuous parameter have been defined in various ways. A definition frequently given is in terms of a physical system or other entity which depends on the parameter t (time) and whose state is specified by the position of a point $Q=Q(t)$ varying in some space in accordance with a given probability law. The probability that Q will be in given point sets at given times is specified, or, if the process is a Markoff process, the probability that Q will be in a given point set at time $t+\delta$ ($\delta>0$) if it was in a given position at time t is specified. Bachelier (I, II, III)† was the first to study these processes. His work was pioneering, and without any attempt at rigor. Most investigations have studied the particular case of Markoff processes‡ or else what are called below differential processes; that is, those in which the changes of $Q(t)$ in non-overlapping intervals are independent in the usual probability sense.§

If Q represents the state of a physical system, the combination of the physical system and the probability relations is to be taken as the stochastic process. This, however, is not a mathematical definition, but a concretization of an unstated mathematical definition. Sometimes $Q(T)$ is simply described as a function varying in accordance with a given law of probability.‖ Khintchine (II) defined a stochastic process as a one-parameter family of chance variables. If Q varies on the x-axis, the probability relations of such a process are determined by specifying the probability of any set of inequalities of the form

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† Roman numerals refer to the bibliography at the end of the paper. This bibliography does not pretend to completeness. It refers only to those papers on probability which are fairly closely related to this one.

‡ Cf. Hostinsky (I), Khintchine (III, pp. 24–59), Kolmogoroff (IV). Hostinsky has an extended bibliography, including references to papers on the diffusion problem which leads to a study of Markoff processes. Many papers have also been written studying Markoff processes in which $Q=Q(t)$ can have only a finite number of positions. These have not studied the specific difficulties of definition, many of which do not arise in this special case, so specific references will not be given.

§ Bachelier (I, II, III); Khintchine (III, pp. 68–75); Kolmogoroff (I, II); Lévy (II, III, and several papers in the Paris Comptes Rendus whose results are given in II); Wiener (I, Chapters 9 and 10, and several earlier papers whose results are given in I).

‖ Finetti (I, II).

$$x(t_j) < \lambda_j, \quad j = 1, \cdots, n, \tag{0.1}$$

where $\lambda_1, \cdots, \lambda_n$ is any set of numbers and $t_1, \cdots, t_n$ any set of values of the (time) parameter.† This definition is not complete enough for the following reason. In many investigations it is necessary to examine the actual paths $Q = Q(t)$ as to their continuity, etc.‡ The probability that the paths are continuous cannot be defined, however, in terms of the elementary probabilities of the inequalities (0.1).§ It is necessary, therefore, to define a stochastic process in a way which will make it possible to study such classes of paths as continuous paths. In this paper such a definition is proposed, and proofs are given for some new theorems and for some old ones whose earlier proofs (and formulations) have presupposed such a definition. Proofs which would merely be rephrasings of earlier ones have been omitted. Only the case where Q varies on the x-axis and where the parameter t varies from $-\infty$ to $+\infty$ is considered. The method in more general cases would be the same.

The point of view is that in which probability becomes (mathematically) a study of Lebesgue and more general measures on suitable spaces. This point of view was developed fully from an abstract standpoint by Kolmogoroff (V).‖ It was applied (not in all cases in work based on Kolmogoroff's) by Doob (I, II, III), Hopf (I), Kac and Steinhaus (II, III), Khintchine (I), Lévy (IV, esp. pp. 84–88), Łomnicki and Ulam (II) to discrete stochastic processes (those in which the parameter t runs through integral values only) by choosing a suitable space of infinitely many dimensions. Essentially the same space had been used earlier by Steinhaus (II) and Wiener (I), and has been used recently by Jessen (I).

1. **Definitions.** Let Ω^* be the space of all functions $x(t)$ defined over $-\infty < t < +\infty$. If $\{t_1, \cdots, t_n\}$ is any finite set of t-values, and if $-\infty \leqq a_j < b_j \leqq +\infty$, $j = 1, \cdots, n$, the set of elements of Ω^* satisfying

$$a_j < x(t_j) < b_j, \qquad j = 1, \cdots, n, \tag{1.1}$$

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† Since a chance variable, mathematically, is merely a measurable function $\phi(\omega)$ defined on some sort of space Ω on which a measure is defined, this definition presumably means that we have given a one-parameter family of measurable functions $\phi_t(\omega)$ all defined on the same space Ω, and that the Ω-measure of the ω-set, where the inequalities

$$\phi_{t_j}(\omega) < \lambda_j, \qquad j = 1, \cdots, n$$

are satisfied, is specified. Cf. Doob (I).

‡ Kolmogoroff (II, p. 868); Lévy (II, III); Wiener (I).

§ Cf. Khintchine (III, pp. 68–69). If the space in which Q varies is the real axis, and if I is an interval on the t-axis, not even the probability that $\text{L.U.B.}_{t \in I} x(t) \leqq M$ can be defined, for any number M. Cf. Kolmogoroff (V, p. 26).

‖ Cf. also earlier papers of Cantelli (I), Łomnicki (I), Steinhaus (I).

will be called a neighborhood. If $x_0(t)$ is an element of such a neighborhood, the neighborhood will be called a neighborhood of $x_0(t)$. The topological characteristics of the space Ω^*, induced by this definition of neighborhood, have been investigated by Tychonoff.† Kolmogoroff (V, pp. 24–30) has shown that a measure function $P^*(\Lambda^*)$ which is non-negative and completely additive, and satisfies $P^*(\Omega^*) = 1$, can be defined on the sets of the Borel system of sets determined by the neighborhoods.‡ The set function is determined by its values on the neighborhoods (1.1) and the only requirement is that, for $t_1, \cdots, t_n$ fixed, $P^*(\Lambda^*)$ should become a non-negative, completely additive function defined on the sets of the Borel system over the neighborhoods (1.1), taking on the value 1 for the whole space.§ The sets on which P^* is defined will be called P^*-measurable. If M^* is any subset of Ω^*, its exterior P^*-measure, $\overline{P}^*(M^*)$, is defined as the greatest lower bound of $P^*(\Lambda^*)$ for all P^*-measurable sets Λ^* containing M^*. It is readily seen that $\overline{P}^*(M^*)$ is the greatest lower bound of $P^*(\Lambda^*)$ for all sets $\Lambda^* \supset M^*$ which are finite or denumerable sums of neighborhoods.

THEOREM 1.1. *Let Ω be a subset of Ω^*. A necessary and sufficient condition that $\Lambda_1^*\Omega = \Lambda_2^*\Omega$ imply that $P^*(\Lambda_1^*) = P^*(\Lambda_2^*)$ for every pair of P^*-measurable sets Λ_1^*, Λ_2^* of Ω^* is that $\overline{P}^*(\Omega) = 1$.*

The condition is sufficient. For if $\overline{P}^*(\Omega) = 1$, and if $\Lambda_1^*\Omega = \Lambda_2^*\Omega$,

$$(\Lambda_1^* - \Lambda_1^* \cdot \Lambda_2^*)\Omega = 0. \tag{1.2}$$

Then

$$\Omega^* \supset (\Lambda_1^* - \Lambda_1^* \cdot \Lambda_2^*) + \Omega \tag{1.3}$$

implies

$$1 \geqq P^*(\Lambda_1^* - \Lambda_1^* \cdot \Lambda_2^*) + \overline{P}^*(\Omega) = P^*(\Lambda_1^*) - P^*(\Lambda_1^* \cdot \Lambda_2^*) + 1, \tag{1.4}$$

or

$$P^*(\Lambda_1^*) = P^*(\Lambda_1^* \cdot \Lambda_2^*). \tag{1.5}$$

Interchanging Λ_1^*, Λ_2^*,

$$P^*(\Lambda_2^*) = P^*(\Lambda_1^* \cdot \Lambda_2^*), \tag{1.5'}$$

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† Mathematische Annalen, vol. 111 (1935), pp. 762–766.

‡ The Borel system of sets determined by any collection of sets is the smallest collection of sets which includes the given collection and which contains the sets $\sum_1^\infty A_n, \prod_1^\infty A_n$ when it contains the sets $\{A_n\}$. We shall suppose also that $P^*(\Lambda^*)$ is defined for any set differing from a set Λ^* in the Borel system by a subset of a set in the system for which P^* vanishes: $P^*(\Lambda_1^*) = P^*(\Lambda^*)$.

§ This set function is merely a measure in n-dimensional Euclidean space with coordinates $x(t_1), \cdots, x(t_n)$.

so that $P^*(\Lambda_1^*) = P^*(\Lambda_2^*)$.

The condition is necessary. For if $\Lambda_1^* \cdot \Omega = \Lambda_2^* \cdot \Omega$ implies that $P^*(\Lambda_1^*) = P^*(\Lambda_2^*)$, let Λ_1^* be a P^*-measurable set containing Ω. It is sufficient to show that then $P^*(\Lambda_1^*) = 1$ necessarily. If $\Lambda_2^* = \Omega^*$,

$$\Lambda_1^* \cdot \Omega^* = \Lambda_2^* \cdot \Omega = \Omega, \tag{1.6}$$

so that $P^*(\Lambda_1^*) = P^*(\Lambda_2^*) = 1$, as was to be proved.

This theorem makes possible the following definition.

DEFINITION. *Let Ω be a subset of Ω^* of exterior P^*-measure* 1. *If Λ is a subset of Ω of the form $\Lambda = \Lambda^* \cdot \Omega$, where Λ^* is P^*-measurable, a set function $P(\Lambda)$ is defined by $P(\Lambda) = P^*(\Lambda^*)$. The topological space Ω together with the measure function thus defined, will be called a stochastic process.*†

The sets Λ on which $P(\Lambda)$ is defined will be called P-measurable. Explicit examples of stochastic processes are given below.

DEFINITION. *Ω will be called quasi-separable if for any open interval I, and for any number k, the set of elements in Ω at which*

$$\text{L.U.B.}_{t \in I}\, x(t) \leqq k \qquad (\text{G.L.B.}_{t \in I}\, x(t) \geqq k),$$

is the same as the set of elements at which

$$\text{L.U.B.}_{n \geqq 1}\, x(t_n) \leqq k \qquad (\text{G.L.B.}_{n \geqq 1}\, x(t_n) \geqq k),$$

for some sequence $\{t_n\}$ of points in I.

The conditions require that certain non-denumerable products of closed sets $\prod_{t \in I}\{x(t) \leqq k\}$‡ or $\prod_{t \in I}\{x(t) \geqq k\}$ be equivalent to denumerable partial products, and thus imply less than separability. A stochastic process will be called quasi-separable whenever the space Ω of the process is quasi-separable.

THEOREM 1.2. *Let $x^*(t)$, $x_*(t)$ be the upper and lower limiting functions of $x(t)$,*§ *respectively. Then the process is quasi-separable if and only if there is an everywhere dense sequence of points $\{r_n\}$ such that if $y^*(t)$ $(y_*(t))$ is the upper (lower) limiting function of $x(r_j)$,*‖

$$x^*(t) \equiv y^*(t), \qquad x_*(t) \equiv y_*(t). \tag{1.7}$$

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† Strictly speaking, the stochastic process should be defined as the physical system or other entity whose changing is represented by the mathematical formulation of the definition, but it seems wiser to use the term stochastic process both for the mathematical formulation and for the concretization it represents, than to introduce more terminology.

‡ The set of elements satisfying conditions C will be denoted by $\{C\}$.

§ $x^*(t) = \lim_{n \to \infty} \text{L.U.B.}_{|\tau - t| < 1/n}\, x(\tau)$, $x_*(t) = \lim_{n \to \infty} \text{G.L.B.}_{|\tau - t| < 1/n}\, x(\tau)$.

‖ $y^*(t) = \lim_{n \to \infty} \text{L.U.B.}_{|r_j - t| < 1/n}\, x(r_j)$, $y_*(t) = \lim_{n \to \infty} \text{G.L.B.}_{|r_j - t| < 1/n}\, x(r_j)$.

The condition is certainly sufficient for quasi-separability. The sequence $\{t_n\}$ of the definition can be taken as the subsequence of $\{r_n\}$ consisting of the points of $\{r_n\}$ in I. Conversely suppose that Ω is quasi-separable.

According to the quasi-separability hypothesis, to each open interval I, and to each number k correspond two sequences of points in I. Let $\{r_n\}$ be the set of all such points for all rational values of k and for all open intervals I with rational endpoints. This sequence $\{r_n\}$ is evidently everywhere dense and has the property that the least upper bound of $x(t)$ (an element of Ω) in any interval I with rational endpoints is the same as that of $x(t)$ for t running through the values of $\{r_n\}$ in I. This statement is still true even if I is not restricted to have rational endpoints, since any interval can be expressed as the sum of such intervals. The first identity (1.7) then follows at once from the definitions of the functions concerned, and the second identity is proved in the same way.

THEOREM 1.3. *If for some real number* λ, $\lambda_n \to \lambda$ *implies*

$$P\{\lim_{n\to\infty} x(\lambda_n) = x(\lambda)\} = 1, \tag{1.8}$$

then if $\{s_n\}$ *is a denumerable set with* λ *as a limit point,*

$$P\{\lim_{s_n\to\lambda} x(s_n) = x(\lambda)\} = 1. \tag{1.9}$$

It is sufficient to prove that the upper and lower limiting functions of $x(s_n)$ at λ are both $x(\lambda)$, on an Ω-set of P-measure 1. The proof will be given for the upper limiting function $x_s^*(s)$. Unless

$$x_s^*(\lambda) \leqq x(\lambda) \tag{1.10}$$

with probability 1—i.e., on an Ω-set of P-measure 1, there is a number r, a positive number ϵ, and a P-measurable set Λ of positive measure, such that

$$x(\lambda) < r < r + \epsilon < \underset{|s_j-\lambda|<1/n}{\text{L.U.B.}}\, x(s_j), \qquad n = 1, 2, \cdots, \tag{1.11}$$

on Λ. Let $u_1^{(n)}, u_2^{(n)}, \cdots$, be the points of $\{s_j\}$ in $|t-\lambda| < 1/n$, arranged in any order. Then on Λ,

$$x(\lambda) < r < r + \epsilon < \underset{j\geqq 1}{\text{L.U.B.}}\, x(u_j^{(n)}), \qquad n = 1, 2, \cdots. \tag{1.12}$$

Now

$$\underset{j\geqq 1}{\text{L.U.B.}}\, x(u_j^{(n)}) = \lim_{\nu\to\infty} \max\,[x(u_1^{(n)}), \cdots, x(u_\nu^{(n)})], \tag{1.13}$$

so there is an integer ν_n so large that

$$\max_{j\leqq\nu_n} x(u_j^{(n)}) > \text{L.U.B.}_{j\geqq 1}\, x(u_j^{(n)}) - \epsilon > r \tag{1.14}$$

on a subset Λ_n of Λ of P-measure at least $P(\Lambda)(1-2^{-n-1})$. If $u_1^{(1)}, \cdots, u_{\nu_1}^{(1)}$, $u_1^{(2)}, \cdots, u_{\nu_2}^{(2)}, \cdots$ are renamed $\lambda_1, \lambda_2, \cdots$ respectively, $\lambda_n \to \lambda$ and

$$\limsup_{n\to\infty} x(\lambda_n) \geqq r > x(\lambda) \tag{1.15}$$

on $\Lambda_0 = \prod_1^\infty \Lambda_n$, where $P(\Lambda_0) \geqq \frac{1}{2}P(\Lambda) > 0$, contradicting the hypothesis of the theorem. Then $x_*^*(\lambda) \leqq x(\lambda)$ with probability 1, i.e., on a set of P-measure 1. The inequality $x_*^*(\lambda) \geqq x(\lambda)$ with probability 1 follows at once from the hypotheses, so $x_*^*(\lambda) = x(\lambda)$ with probability 1, as was to be proved.

COROLLARY. *If for some real number* λ, $\lambda_n \to \lambda$ *implies* (1.8) *on a quasi-separable process,* $x(t)$ *is continuous at* λ, *with probability* 1.

Since the process is quasi-separable, there is an everywhere dense sequence $\{r_n\}$ such that the upper and lower limiting functions of $x(t)$ at λ are the same as those of $x(t)$ when we consider $x(t)$ defined only at the points $\{r_n\}$. By the theorem the latter limiting functions are both $x(\lambda)$, with probability 1.

2. **Measurability of stochastic processes.** In the applications of the theory of stochastic processes, P^*-measure is usually prescribed on Ω^*, and the first problem becomes that of finding the subspace Ω, $\overline{P}^*(\Omega) = 1$, whose elements are as simple as possible, e.g., measurable, monotone, etc. As an introductory example we prove the following theorem.

THEOREM 2.1. *Let* P^**-measure be so defined that there is a number* M *with the property that for each value of* t

$$P^*\{\,|\,x(t)\,|\,\leqq M\} = 1. \tag{2.1}$$

Then if Ω *is the subspace of* Ω^* *consisting of those elements of* Ω^* *for which* $|x(t)| \leqq M$, $-\infty < t < +\infty$, $\overline{P}^*(\Omega) = 1$, *and* Ω *is the space of a stochastic process.*

Let $\Gamma^* = \sum_1^\infty \Gamma_n^*$ be a sum of neighborhoods, and suppose that $\Gamma^* \supset \Omega$. If $t_1, t_2, \cdots$ are the t-values used in defining $\Gamma_1^*, \Gamma_2^*, \cdots$, and if Λ^* is the P^*-measurable set determined by $|x(t_j)| \leqq M, j \geqq 1$,

$$P^*(\Lambda^*) = 1. \tag{2.2}$$

If $x_0(t) \epsilon \Lambda^*$, there is an element $x_1(t)$ in Ω such that $x_1(t_j) = x_0(t_j)$, $j \geqq 1$. For some ν, $x_1(t) \epsilon \Gamma_\nu^*$. Since Γ_ν^* is defined by inequalities on $x(t)$ at certain of the numbers $t_1, t_2, \cdots$, $x_1(t) \epsilon \Gamma_\nu^*$ implies that $x_0(t) \epsilon \Gamma_\nu^*$. Therefore

$$\Lambda^* \subset \Gamma^*, \tag{2.3}$$

so $P^*(\Gamma^*)=1$. Thus any sum of neighborhoods which includes Ω has P^*-measure 1, implying that $\overline{P}^*(\Omega)=1$.

The space of all pairs $\{\tau, x(t)\}$, the direct product of the τ-axis with Ω, will be denoted by $T\times\Omega$. If measure is defined on the τ-axis as Lebesgue measure, measure can be defined on the product space in the usual way, being determined by the fact that the measure of the direct product of a τ-measurable set and a P-measurable set is the product of their measures.† A set in $T\times\Omega$ is measurable (i.e., its measure is defined) if and only if it is the intersection of $T\times\Omega$ with a set measurable in $T\times\Omega^*$.

DEFINITION. *A stochastic process will be called measurable if the* $T\times\Omega$ *set of elements* $\{\tau, x(t)\}$ *for which* $x(\tau)>k$ *is measurable in* $T\times\Omega$ *for every real number* k, *i.e., if* $x(\tau)$ *is a measurable function on* $T\times\Omega$.

This implies, by Fubini's theorem,‡ that the element $x(t)$ of Ω is a Lebesgue measurable function, with probability 1.

THEOREM 2.2. *The space* Ω^* *is never the space of a measurable stochastic process.*

If Ω^* were the space of some measurable stochastic process, $x(t)$ would be Lebesgue measurable, with probability 1, in terms of P^*-measure. It is therefore sufficient to show that if Λ^* is the set of elements of Ω^* which are not Lebesgue measurable, $\overline{P}^*(\Lambda^*)=1$, however P^*-measure is defined. Let $\Gamma^*=\sum_1^\infty \Gamma_n^*$ be a sum of neighborhoods covering Λ^*. Then we must show that $P^*(\Gamma^*)=1$. Let $t_1, t_2, \cdots$ be the set of t-values involved in defining Γ^*. If $x_0(t)$ is any element of Ω^*, there is a non-measurable function $x_1(t)$, an element of Λ^*, such that $x_1(t_j)=x_0(t_j), j=1, 2, \cdots$. We need only take any non-measurable function and redefine it, if necessary, at $t=t_1, t_2, \cdots$. Since $\Gamma^*\supset\Lambda^*$ contains the element $x_1(t)$, it must also contain $x_0(t)$. Therefore $\Gamma^*=\Omega^*$: $P^*(\Gamma^*)=1$, as was to be proved.

We shall need the following lemma on measurable functions.

LEMMA 2.1. *Let* $f(\tau, v)$ *be a Lebesgue measurable function defined in the strip* $0<v<1$ *of the* (τ, v)*-plane. Define* $\phi_n(t)$ by:

$$\phi_n(t) = k2^{-n} \quad \text{if} \quad (k-1)2^{-n} < t \leqq k2^{-n}. \tag{2.4}$$

Then there is a number c *and a sequence of integers* $\{a_n\}$ *such that* $f[\phi_n(t)+c, v]$ *is a measurable function in the strip* $0<v<1$ *of the* (t, v)*-plane and that*

† A treatment of measure functions in product spaces can be found in S. Saks, *Théorie de l'Intégrale*, Warsaw, 1933, pp. 257–263.

‡ The importance of Fubini's theorem in work of this kind was stressed by P. Lévy (V). Cf. also the investigations of J. von Neumann on flows, Annals of Mathematics, vol. 33 (1932), pp. 587–642; esp. pp. 588–589.

$$\lim_{n\to\infty} f[\phi_{\alpha_n}(t) + c, v] = f(t + c, v) \tag{2.5}$$

almost everywhere in the strip.

(i) Consider the function $f[\phi_n(t)+s, v]$, defined in the 3-dimensional (s, t, v)-region $0<v<1$. *We show first that this function is Lebesgue measurable in the* (s, t, v)*-region.* It is sufficient to show that if $E_{u,v}$ is a measurable plane point set on the (u, v)-plane, and if

$$[\phi_n(t) + s, v]\epsilon E_{u,v}, \qquad (u = \phi_n(t) + s)$$

determines the (s, t, v)-set $E_{s,t,v}$, then $E_{s,t,v}$ is measurable. If $E_{u,v}$ is a rectangle, $E_{s,t,v}$ is certainly measurable, as the direct product of an (s, t) Borel set with a v-interval. Then if $E_{u,v}$ is any Borel set $E_{s,t,v}$ is measurable. Suppose that $E_{u,v}$ is a Borel set of measure 0. It has just been shown that $E_{s,t,v}$ is measurable, and we shall show the additional fact that it is of zero measure. The intersection of $E_{u,v}$ with a line $v=$ const. is almost always of measure 0, and it is sufficient to show that the intersection of $E_{s,t,v}$ with a line $v=$ const. is almost always of measure 0. To show this it is sufficient to show that the (s, t)-set $E_{s,t}$ determined by $\phi_n(t)+s\epsilon E_u$ is of measure 0 if it is measurable and if E_u is of measure 0. This is certainly true since the intersection of $E_{s,t}$ with a line $t=$ const. is congruent to E_u and so has the same measure. Thus if $E_{u,v}$ is a Borel set, $E_{s,t,v}$ is measurable, and has measure 0 if $E_{u,v}$ has measure 0. If $E_{u,v}$ is any measurable set of zero measure, it can be enclosed in a Borel set of zero measure, so in this case $E_{s,t,v}$ is still measurable. Since any measurable set $E_{u,v}$ differs from some Borel set by at most a set of measure 0, if $E_{u,v}$ is measurable, $E_{s,t,v}$ is also measurable, as was to be proved.

(ii) For (t, v) fixed, $f[\phi_n(t)+s, v]$ is a measurable function of s (except possibly for a (t, v)-set of measure 0) so by a theorem of Auerbach† if $\{\alpha_n\}$ is any sequence of integers, there is a subsequence $\{\beta_n\}$ with the property that

$$\lim_{n\to\infty} f[\phi_{\beta_n}(t) + s, v] = f(t + s, v) \tag{2.6}$$

for almost all s. Let I_s be any finite s-interval, ϵ any positive number, and let $\xi_n(t, v)$ be the s-measure of the subset of I_s where

† Fundamenta Mathematicae, vol. XI (1928), pp. 196–197. Auerbach's theorem states that if $F(x)$ is a Lebesgue measurable function of x and if $\{\eta_n\}$ is a sequence of numbers converging to 0, there is a subsequence $\{\eta_{\alpha_n}\}$ with the property that $F(x+\eta_{\alpha_n})\to F(x)$ almost everywhere on the x-axis. This can be restated in terms of convergence in measure: $F(x+h)$ converges in measure to $F(x)$ on any finite interval as $h\to 0$. (Convergence in measure was defined and discussed by F. Riesz, Paris Comptes Rendus, vol. 148 (1909), pp. 1303–1309.)

$$|f[\phi_n(t)+s, v]-f(t+s, v)| \geqq \epsilon. \tag{2.7}$$

Then $\xi_n(t, v)$ is defined almost everywhere in the (t, v)-strip $0<v<1$, is uniformly bounded by the length of I_s, and converges to 0 with $1/n$.† If $R_{t,v}$ is the (t, v)-rectangle $[t_0<t<t_1, 0<v<1]$, the integral

$$\int\int_{R_{t,v}} \xi_n(t, v)dtdv$$

which represents the (s, t, v)-measure of the (s, t, v)-set where (2.7) is true, restricting (s, t) to $[s\epsilon I_s, t_0<t<t_1]$, converges to 0 with $1/n$. This implies that $f[\phi_n(t)+s, v]$ converges in measure to $f(t+s, v)$ on the (s, t, v)-set $I_s\times R_{t,v}$ determined by $s\epsilon I_s$, $(t, v)\epsilon R_{t,v}$. There is then a sequence of integers $\{\alpha_n\}$ such that $f[\phi_{\alpha_n}(t)+s, v]$ converges almost everywhere on $I_s\times R_{t,v}$ to $f(t+s, v)$. By a familiar use of the diagonal process, there is a subsequence $\{a_n\}$ of $\{\alpha_n\}$ such that

$$\lim_{n\to\infty} f[\phi_{a_n}(t)+s, v] = f(t+s, v)$$

almost everywhere on the (s, t, v)-set determined by $0<v<1$. Then except for an s-set of zero measure $f[\phi_{a_n}(t)+s, v]$ converges (s fixed) to $f(t+s, v)$ almost everywhere on the (t, v)-strip $0<v<1$. The number c of the theorem can be taken as any number not on this exceptional s-set.

THEOREM 2.3. *If P^*-measure is so defined that a constant c and a sequence of integers $\{a_n\}$ exist for which*

$$P^*\{\lim_{n\to\infty} x[c+\phi_{a_n}(t)] = x(c+t)\} = 1‡ \tag{2.8}$$

for almost all t, there is a space $\Omega\subset\Omega^*$ which is the space of a measurable stochastic process.

Conversely, if there is a measurable stochastic process with space $\Omega\subset\Omega^*$, the corresponding P^*-measure must satisfy the above condition.

Suppose that the given P^*-measure satisfies the condition of the theorem. It will simplify the notation without restricting generality, if we assume that $c=0$. If $x(t)\epsilon\Omega^*$, we define $x^{(i)}(t)$, $x^{(s)}(t)$ by

$$\begin{aligned} x^{(i)}(t) &= \liminf_{n\to\infty} x[\phi_{a_n}(t)], \\ x^{(s)}(t) &= \limsup_{n\to\infty} x[\phi_{a_n}(t)]. \end{aligned} \tag{2.9}$$

† Cf. the preceding note.

‡ The function $\phi_n(t)$ is defined by (2.4).

The functions $x(t), x[\phi_n(t)], x^{(i)}(t), x^{(s)}(t)$ determine functions $x(\tau), x[\phi_n(\tau)]$, $x^{(i)}(\tau), x^{(s)}(\tau)$ on $T\times\Omega^*$. The function $x[\phi_{a_n}(\tau)]$ is readily seen to be measurable on $T\times\Omega^*$, $n=1, 2, \cdots$, so the functions $x^{(i)}(\tau), x^{(s)}(\tau)$ are also measurable on $T\times\Omega^*$. If t is not in an exceptional set S of measure 0,

$$P^*\{x^{(i)}(t) = x(t) = x^{(s)}(t)\} = 1. \tag{2.10}$$

Let Ω be the set of all elements $x(t)\epsilon\Omega^*$ satisfying

$$x^{(i)}(t) \leqq x(t) \leqq x^{(s)}(t) \tag{2.11}$$

for all values of t not in S for which $x^{(i)}(t) < +\infty$, $x^{(s)}(t) > -\infty$, and otherwise unrestricted. We shall show that $\overline{P}^*(\Omega) = 1$ by showing that if $\Gamma^* = \sum_1^\infty \Gamma_n^*$ is a sum of neighborhoods covering Ω, $P^*(\Gamma^*) = 1$ necessarily. Let $t_1, t_2, \cdots$ be the values of t used in defining Γ^*, of which $t_{a_1}, t_{a_2}, \cdots$ are not in S,† and let Λ^*be the Ω^*-set determined by

$$x^{(i)}(t_{a_j}) = x(t_{a_j}) = x^{(s)}(t_{a_j}), \qquad j = 1, 2, \cdots. \tag{2.12}$$

By (2.10), $P^*(\Lambda^*) = 1$. If $x_0(t)$ is an element of Λ^*, let $x_1(t)$ be an element of Ω^* equal to $x_0(t)$ at $t_1, t_2, \cdots$ and at $t = k2^{-n}$, for $k = 0, \pm 1, \cdots, n = 1, 2, \cdots$, and otherwise restricted only by

$$x_1^{(i)}(t) = x_0^{(i)}(t) \leqq x_1(t) \leqq x_0^{(s)}(t) = x_1^{(s)}(t) \tag{2.13}$$

at the values of $t\epsilon S$ where $x_0^{(i)}(t) < +\infty$, $x_0^{(s)}(t) > -\infty$. Then $x_1(t)\epsilon\Omega$: for $t = k2^{-n}$, $x_1^{(i)}(t) = x_1^{(s)}(t) = x_1(t)$ by the definition of the functions $x^{(i)}(t), x^{(s)}(t)$ so that (2.11) is satisfied; at $t = t_{a_1}, t_{a_2}, \cdots$, (2.12) implies (2.11); and for all other values of t, (2.13) implies (2.11) (the qualifying conditions on the two are identical). Since $x_1(t)\epsilon\Gamma^*$ there is an integer ν such that $x_1(t)\epsilon\Gamma_\nu^*$ and since $x_1(t_j) = x_0(t_j)$, $j = 1, 2, \cdots$, this means that $x_0(t)\epsilon\Gamma_\nu^* : \Lambda^* \subset \Gamma^*$, $P^*(\Gamma^*) = 1$, as was to be proved.

The space Ω is thus of exterior P^*-measure 1 and so the space of a stochastic process, which we shall now show is measurable. The functions $x^{(i)}(\tau)$, $x^{(s)}(\tau)$ are measurable on $T\times\Omega^*$ and therefore on $T\times\Omega$. For fixed $\tau = t$ they are equal, with probability 1, by (2.10), except for a t-set of measure 0, so they are equal to each other almost everywhere on $T\times\Omega$. Since for fixed $\tau = t\epsilon S$ they are equal to $x(t)$ with probability 1, they are finite valued almost everywhere on $T\times\Omega$. The function $x(\tau)$ on $T\times\Omega$ then satisfies

$$x^{(i)}(\tau) \leqq x(\tau) \leqq x^{(s)}(\tau) \tag{2.11'}$$

almost everywhere on $T\times\Omega$ and is therefore equal to $x^{(i)}(\tau) = x^{(s)}(\tau)$ almost everywhere on $T\times\Omega$, proving the measurability of the process.

† The set $\{t_{a_n}\}$ may be empty.

Conversely, if P^*-measure is so defined that there is a measurable stochastic process with space $\Omega \subset \Omega^*$, the function $x(\tau)$ is a measurable function on $T \times \Omega$. We apply Lemma 2.1 to this function $x(\tau)$ defined on $\{\tau, x(t)\}$ space $T \times \Omega$, finding that there is a number c and a sequence of integers $\{a_n\}$ such that

$$P\{\lim_{n\to\infty} x[c + \phi_{a_n}(t)] = x(c + t)\} = 1,$$

equivalent to (2.8) because of the definition of P-measure.†

The condition of the theorem is in unwieldy form, but has the advantage of being both necessary and sufficient for the existence of a measurable stochastic process. The following theorem gives a necessary condition and a sufficient condition in more usable form.

THEOREM 2.4. (i) *Let P^*-measure be so defined that there is a space $\Omega \subset \Omega^*$ which is the space of a measurable stochastic process. Then if $\epsilon > 0$*

$$P^*\{|x(t + h) - x(t)| > \epsilon\} \tag{2.14}$$

is a measurable function of t (h fixed) which converges in measure to 0 on every finite t-interval, as $h \to 0$.

(ii) *If P^*-measure is so defined that for every positive ϵ (2.14) converges to 0 for almost all values of t, as $h \to 0$, there is a subspace $\Omega \subset \Omega^*$ which is the space of a measurable stochastic process.*

If P^*-measure is so defined that the P^*-measure of a neighborhood (and so of any P^*-measurable set) is independent of translations of the t-axis, the conditions (i) and (ii) become identical and the resulting condition is then both necessary and sufficient.

We first prove (i). By hypothesis $x(\tau)$ is a measurable function of $\{\tau, x(t)\}$. Then $|x(\tau+h) - x(\tau)|$ is measurable, so by Fubini's theorem

$$P\{|x(t + h) - x(t)| > \epsilon\} \tag{2.14'}$$

is a measurable function of t. The function of (2.14) is equal to this function for all t, h, and is therefore also measurable in t. The element $x(t)$ of Ω is measurable, with probability 1. Applying the theorem of Auerbach as in the proof of Lemma 2.1, if I is any finite t-interval, $x(t+h)$ converges in measure on I to $x(t)$ as $h \to 0$; i.e., if $\xi[x(t), h]$ is the Lebesgue measure of the t-subset of I where $|x(t+h) - x(t)| > \epsilon$, ξ is bounded by the length of I and converges to 0 with h. The measure of the $T \times \Omega$-set (restricting τ to I), where $|x(\tau+h) - x(\tau)| > \epsilon$, is

† The lemma was stated, for convenience, for a function $f(\tau, v)$ defined on (τ, v)-space, but the proof holds for this case also, with no essential change.

$$\int \xi[x(t), h]dP, \dagger \tag{2.15}$$

which converges to 0 with h. The measure of this set can also be expressed as

$$\int_I P\{\,|\,x(t+h) - x(t)\,| > \epsilon\}dt, \tag{2.16}$$

and the convergence of this integral to 0 with h, remembering the identity of (2.14) and (2.14′), is equivalent to the statement of the theorem.

Next we prove (ii). We shall show that P^*-measure satisfies the condition of Theorem 2.3 with $c=0$ if (ii) of the present theorem is satisfied.

Consider the function $x[\phi_n(t)]$, inducing the function $x[\phi_n(\tau)]$ on $T\times\Omega^*$. The latter function is measurable on $T\times\Omega^*$. The hypotheses of (ii) imply that $x[\phi_n(t)]$ (t fixed) converges (to $x(t)$) in measure almost everywhere on the t-axis, and by the device used in the proof of the first part of this theorem, this means that $x[\phi_n(\tau)]$ converges in measure on $T\times\Omega^*$ for τ restricted to any finite interval I. There is then a sequence of integers $\{\alpha_n\}$ such that $x[\phi_{\alpha_n}(\tau)]$ converges almost everywhere on $I\times\Omega^*$, and by the diagonal procedure, a subsequence $\{a_n\}$ of $\{\alpha_n\}$ can be obtained for which $x[\phi_{a_n}(\tau)]$ converges almost everywhere on $T\times\Omega^*$. By Fubini's theorem, $x[\phi_{a_n}(t)]$ (t fixed) converges with probability 1, necessarily to $x(t)$, for almost all values of t, as was to be proved.

THEOREM 2.5. *Let P^*-measure be so defined that $t_n\to t$ implies*

$$P^*\{\lim_{n\to\infty} x(t_n) = x(t)\} = 1, \tag{2.17}$$

for all values of t except possibly those in a set S of measure 0. Then

(i) *there is a measurable stochastic process with space Ω, whose elements $x(t)$ are continuous almost everywhere on the t-axis, with probability 1, and $x(t)$ is continuous at $t=t_0$ with probability 1, for almost all values of t_0;*

(ii) *every quasi-separable stochastic process with space $\Omega\subset\Omega^*$ is measurable, and its elements fulfill the regularity conditions of* (i);

(iii) *if the exceptional t-set S is empty, and if there is an everywhere dense denumerable set $\{r_n\}$ with the property that*

$$\lim_{k\to\infty} P^*\{\text{L.U.B.}_{r_j\in I}\,|\,x(r_j)\,| \leqq k\} = 1, \tag{2.18}$$

for every finite interval I, a quasi-separable process with space $\Omega\subset\Omega^*$ exists (to which (ii) *can then be applied).*‡

† The integral of a P-measurable function $f[x(t)]$ over a P-measurable set Λ will be denoted by $\int_\Lambda f[x(t)]dP$.

‡ Cf. Lévy (II, p. 345).

First we prove (i). By Theorem 2.4, (ii) there is a measurable process with space Ω, whose elements $x(t)$ satisfy (2.11′) almost everywhere on $T\times\Omega$. Let $\bar{x}_n(t)$, $(\underline{x}_n(t))$ be defined as L.U.B. $x(r_j)$ (G.L.B. $x(r_j)$) for all numbers r_j of the form $\mu 2^{-\nu}$ $(\mu=0, \pm 1, \cdots, \nu=1, 2, \cdots)$ in the interval

$$\frac{k-1}{2^n} \leqq r \leqq \frac{k+1}{2^n}, \tag{2.19}$$

where k is determined by

$$\frac{k}{2^n} \leqq t < \frac{k+1}{2^n}. \tag{2.20}$$

Then $\bar{x}_n(\tau)$, $\underline{x}_n(\tau)$ are measurable on $T\times\Omega$. Define $\bar{x}(t)$, $\underline{x}(t)$ by

$$\bar{x}(t) = \lim_{n\to\infty} \bar{x}_n(t), \qquad \underline{x}(t) = \lim_{n\to\infty} \underline{x}_n(t).$$

The functions $\bar{x}(t)$, $\underline{x}(t)$ are respectively the upper limiting function of $x^{(s)}(t)$ and the lower limiting function of $x^{(i)}(t)$. The functions $\bar{x}(\tau)$, $\underline{x}(\tau)$ are measurable on $T\times\Omega$ and satisfy

$$\underline{x}(\tau) \leqq x^{(i)}(\tau) \leqq x(\tau) \leqq x^{(s)}(\tau) \leqq \bar{x}(\tau) \tag{2.11″}$$

almost everywhere on $T\times\Omega$. If $x^*(t)$, $x_*(t)$ are respectively the upper and lower limiting functions of $x(t)$, (2.11″) implies

$$\underline{x}(\tau) \leqq x_*(\tau) \leqq x(\tau) \leqq x^*(\tau) \leqq \bar{x}(\tau) \tag{2.11‴}$$

almost everywhere on $T\times\Omega$. Applying Theorem 1.3, if $t\notin S$, there is equality of the extremes in (2.11‴) for $\tau=t$, with probability 1. Then there is equality almost everywhere on $T\times\Omega$. The equality of $x^*(t)$, $x_*(t)$ at t_0 is the condition of continuity at t_0. Applying Fubini's theorem, we have the statement (i) of Theorem 2.5.

We now prove (ii). If P^*-measure has the property described, and if Ω is the space of a quasi-separable process, let $\{r_n\}$ be a sequence of numbers with the properties described in Theorem 1.2. Let $\bar{x}(t)$, $\underline{x}(t)$ be defined as in (i), in terms of the values of $x(t)$ at the points $\{r_n\}$, which no longer have the special form prescribed there. Then if $x(t)\in\Omega$, $\underline{x}(t)$, $\bar{x}(t)$ are its lower and upper limiting functions respectively, whose equality is the condition of continuity. We have on Ω

$$\underline{x}(t) \leqq x(t) \leqq \bar{x}(t) \tag{2.21}$$

for all values of t. Now for fixed τ, $\underline{x}(\tau)$, $\bar{x}(\tau)$ are equal with probability 1 (except for $\tau\in S$), applying Theorem 1.3 to the present stochastic process.

Then by Fubini's theorem, $\underline{x}(\tau) = \bar{x}(\tau)$ almost everywhere on $T \times \Omega$, so $x(\tau)$, equal to a measurable function almost everywhere on $T \times \Omega$, is itself measurable on that space. The process with space Ω is thus measurable, and the continuity properties of the elements $x(t)$ are deduced by applying Fubini's theorem, as in the proof of (i).

Finally we consider (iii). Suppose that the conditions of (iii) are satisfied. Form the functions $\underline{x}(t)$, $\bar{x}(t)$ as in (i), using the sequence $\{r_n\}$ with the properties described in the statement of (iii). Let Ω be the space of all elements $x(t)$ satisfying (2.21) for every value of t. We shall show that $\overline{P}^*(\Omega) = 1$ by showing that if $\Gamma^* = \sum_1^\infty \Gamma_n^*$ is a sum of neighborhoods covering Ω, $P^*(\Gamma^*) = 1$ necessarily. Let $t_1, t_2, \cdots$ be the values of t used in defining Γ^*, and let $\Lambda_{k,n}^*$ be the Ω^*-set determined by

$$\underset{|r_j| < n}{\text{L.U.B.}} \; | \, x(r_j) \, | \leqq k.$$

The set

$$\Lambda_0^* = \prod_{n=1}^{\infty} \sum_{k=1}^{\infty} \Lambda_{k,n}^*$$

then has P^*-measure 1. It consists of the elements $x(t)$ which are bounded functions (considered defined only at the points $\{t_j\}$) on every finite interval. Let Λ^* be the subset of Λ_0^* for which

$$\left.\begin{aligned} \underline{x}(t_j) &= x(t_j) = \bar{x}(t_j) \\ \underline{x}(r_j) &= x(r_j) = \bar{x}(r_j) \end{aligned}\right\} \qquad j = 1, 2, \cdots.$$

The set Λ^* also has P^*-measure 1. If $x_0(t) \epsilon \Lambda^*$, let $x_1(t)$ be an element of Ω^* equal to $x_0(t)$ at the points $\{t_j\}$ and $\{r_j\}$ and otherwise restricted only by

$$\underline{x}_1(t) = \underline{x}_0(t) \leqq x_1(t) \leqq \bar{x}_0(t) = \bar{x}_1(t).$$

Then $x_1(t) \epsilon \Omega \subset \Gamma^*$. Since $x_1(t) \epsilon \Gamma^*$ there is an integer ν such that $x_1(t) \epsilon \Gamma_\nu^*$, and since $x_1(t_j) = x_0(t_j)$, $j \geqq 1$, this means that $x_0(t_j) \epsilon \Gamma_\nu^* : \Lambda^* \subset \Gamma^* : P^*(\Gamma^*) = 1$, as was to be proved. The sequence $\{r_n\}$ can be used as the sequence of t-values required by the condition of quasi-separability (cf. Theorem 1.2).

THEOREM 2.6. *Suppose that P^*-measure is so defined that for every number t and positive number δ,*

$$P^*\{x(t+\delta) - x(t) \geqq 0\} = 1. \tag{2.22}$$

There is then a finite or denumerable sequence of points $d_1, d_2, \cdots$ such that if Ω is the set of all elements $x(t)$ of Ω^* which are monotone non-decreasing functions and continuous on the right, except possibly at $d_1, d_2, \cdots$, $\overline{P}^*(\Omega) = 1$: Ω is

the space of a quasi-separable measurable stochastic process. If $t_0 \neq d_1, d_2, \cdots$, $x(t)$ *in* Ω *is continuous at* t_0 *with probability* 1.†

(i) Let $t_1, t_2, \cdots$ be any sequence of numbers. If $t_j > t_k$,

$$P^*\{x(t_j) \geqq x(t_k)\} = 1. \tag{2.23}$$

Since the set of all pairs (t_j, t_k) is denumerable, $x(t)$, considered as defined only at $t_1, t_2, \cdots$ is monotone non-decreasing, with probability 1 (in terms of P^*-measure).

(ii) Evidently if $\epsilon > 0$ and if $a < b$,

$$P^*\{x(b) - x(a) \geqq \epsilon\} \tag{2.24}$$

is monotone non-decreasing in b for fixed a and in $-a$ for fixed b. Then if $a < t < b$ and if $a \to t$, $b \to t$ the expression (2.24) converges to a limit which is unaltered if a, b are restricted to run through rational values. Let $S(\epsilon, \eta)$ be the t-set for which this limit is at least $\eta > 0$. Let Λ_r^* be the subset of Ω^* consisting of the elements $x(t)$ which are monotone non-decreasing when considered only for rational t-values. By (i), $P^*(\Lambda_r^*) = 1$. If $t_0 \epsilon S(\epsilon, \eta)$, $x(r)$ (r rational) has a jump of magnitude ϵ at t_0, if $x(t)$ is in some set $M_{t_0}^*$ of P^*-measure at least η. The set $S(\epsilon, \eta)$ has only a finite number of points in common with any finite interval I: otherwise, if $t_j \epsilon I \cdot S(\epsilon, \eta)$, $j = 1, 2, \cdots$, there is a P^*-measurable set M^* of P^*-measure at least η each element of which is in infinitely many of the sets $M_{t_j}^*$,‡ and if $x(t) \epsilon M^*$, $x(r)$ has infinitely many jumps in I of magnitude ϵ, an impossibility. The set $S(\epsilon, \eta)$ is therefore at most denumerably infinite, and the set $\{d_j\}$ of the theorem is the sum of the sets $S(1/m, 1/n)$, $m, n = 1, 2, \cdots$.

(iii) We show that $\overline{P}^*(\Omega) = 1$ by showing that if $\Gamma^* = \sum_1^\infty \Gamma_n^*$ is any sum of neighborhoods covering Ω, $P^*(\Gamma^*) = 1$. Let $t_1, t_2, \cdots$ be any everywhere dense sequence which includes the sequence $\{d_n\}$ and those values of t used in defining the neighborhoods of Γ^*, and let Λ^* be the subset of Ω^* consisting of those elements $x(t)$ which, when considered defined only at $t_1, t_2, \cdots$, are monotone non-decreasing. By (i), $P^*(\Lambda^*) = 1$. If $x(t) \epsilon \Lambda^*$, $x(t)$ (considered defined only at $t_1, t_2, \cdots$) is either continuous on the right at t_j or has a jump there. If $t_j \neq d_\nu$, $\nu = 1, 2, \cdots$, the first case is true, with probability 1. Then if Λ_0^* is the subset of Ω^* consisting of those elements $x(t)$ which are continuous on the right (considered defined only at $t_1, t_2, \cdots$) at each point $t_1, t_2, \cdots$ not in the set $\{d_n\}$, $P^*(\Lambda_0^*) = P^*(\Lambda^*) = 1$. If $x_0(t) \epsilon \Lambda_0^*$ define $x_1(t)$ by

† This does not mean that $x(t)$ in Ω is continuous except at $d_1, d_2, \cdots$. The probability that $x(t)$ is continuous at any given point t_0 may be 1 for every value of t_0, although at the same time the probability that $x(t)$ is continuous in an interval is less than 1. An example of this is given below. The last statement of Theorem 2.6 implies that the condition on P^*-measure of Theorem 2.5 is satisfied.

‡ The set M^* can be defined as $\prod_{N=1}^\infty \sum_N^\infty M_{t_j}^*$.

$$
\begin{aligned}
x_1(t) &= \lim_{t_n \downarrow t} x_0(t_n)\dagger \text{ if } t \neq t_j, \qquad && j = 1, 2, \cdots, \\
x_1(t_j) &= x_0(t_j), && j = 1, 2, \cdots.
\end{aligned} \tag{2.25}
$$

Then $x_1(t)$ is monotone non-decreasing and continuous on the right, except perhaps at $t=d_j, j=1, 2, \cdots$, so $x_1(t)\epsilon\Omega$. For some ν, $x_1(t)\epsilon\Gamma_\nu^*$. Since Γ_ν^* is defined by inequalities on $x(t)$ at certain of the numbers $t_1, t_2, \cdots$, $x_1(t)\epsilon\Gamma_\nu^*$ implies that $x_0(t)\epsilon\Gamma_\nu^*$. Therefore $\Lambda_0 \subset \Gamma^*: P^*(\Gamma^*)=1$, $\overline{P}^*(\Omega)=1$.

(iv) The numbers $\{t_n\}$ required in the definition of quasi-separability (cf. §1) can be taken as the rational numbers in the interval. The process is therefore quasi-separable. Let t_0 be any value of t not in the sequence $\{d_n\}$. Then

$$
\lim_{t_n \downarrow t_0} x(t_n) = x(t_0 + 0),
$$

if $x(t)\epsilon\Omega$, and at the same time $x(t_n)$ converges in measure on Ω to $x(t_0)$, so $x(t_n)$ converges to $x(t_0)=x(t_0+0)$ with probability 1. Similarly

$$
\lim_{t_n \uparrow t_0} x(t_n) = x(t_0 - 0) = x(t_0)
$$

with probability 1, and we can say in general that if $t_n \to t_0$,

$$
P^*\{\lim_{n\to\infty} x(t_n) = x(t_0)\} = P\{\lim_{n\to\infty} x(t_n) = x(t_0)\} = 1. \tag{2.26}
$$

The P^*-measure thus satisfies the condition (ii) of Theorem 2.5, and the process is therefore measurable. We have proved that

$$
x(t_0 + 0) = x(t_0 - 0) = x(t_0) \tag{2.27}
$$

on Ω with probability 1, if $t_0 \neq d_1, d_2, \cdots$, completing the proof of the theorem.

The following example‡ of a stochastic process of the type just discussed has no point d_n, and so $x(t)$ is continuous, with probability 1 at every value of t. If $t_1 < t_1+\delta_1 \leqq t_2 < \cdots \leqq t_n < t_n+\delta_n$, and if $\nu_1, \cdots, \nu_n$ are any positive integers or 0, we define P^*-measure by

$$
\begin{aligned}
&P^*\{x(0) = 0\} = 1, \\
&P^*\{x(t_j + \delta_j) - x(t_j) = \nu_j, \qquad j = 1, \cdots, n\} = \prod_1^n e^{-\delta_j} \frac{\delta_j^{\nu_j}}{\nu_j!}.
\end{aligned} \tag{2.28}
$$

The elements of Ω are monotone non-decreasing and everywhere continuous on the right. It can be shown that the subspace Ω_0 of Ω whose elements $x(t)$ all vanish at $t=0$, and are constant except for jumps of magnitude 1, is also

† The notation $t_n \uparrow t$ ($t_n \downarrow t$) will be used to denote approach from below (above).

‡ Cf. Khintchine (III, pp. 23–24).

of exterior P^*-measure 1. This space Ω_0 is the natural space of the time series with Poisson distribution.

Let M_τ^* be an Ω^*-set in the Borel system over the neighborhoods (1.1) with $t_j \leqq \tau, j = 1, \cdots, n$, but otherwise unrestricted. If Ω is the space of a stochastic process, and if Λ is P-measurable, $P(\Lambda \cdot M_\tau)$ is a completely additive non-negative function of the sets $M_\tau = M_\tau^* \cdot \Omega$ (τ fixed), and $P(M_\tau) = 0$ implies that $P(\Lambda \cdot M_\tau) = 0$. There is therefore a P-measurable function $P_{\tau,\Lambda}[x(t)]$ such that

$$\int_{M_\tau} P_{\tau,\Lambda}[x(t)]\,dP = P(\Lambda \cdot M_\tau), \dagger \tag{2.29}$$

for all M_τ. The function $P_{\tau,\Lambda}$ is uniquely determined up to a set of P-measure 0 and $P_{\tau,\Lambda} > k$ defines an M_τ-set. This function $P_{\tau,\Lambda}$ is called the conditional probability function of Λ if $x(t)$ is known for $t \leqq \tau$.

THEOREM 2.7. *If $P_{\tau,\Lambda}$ is the conditional probability function of Λ when $x(t)$ is known for $t \leqq \tau$, and if $\phi[x(t)]$ is the characteristic function of the set Λ,‡ $\tau_n \to \infty$ implies*

$$\lim_{n\to\infty} P_{\tau_n,\Lambda}[x(t)] = \phi[x(t)] \tag{2.30}$$

with probability 1.

The proof of this is the same as that for the corresponding theorem on discrete stochastic processes, proved by Lévy (IV, pp. 88–89), and will therefore be omitted.

Let T_s^* be the transformation of Ω^* taking the element $x(t)$ into the element $x(t+s)$. This point transformation takes P^*-measurable sets into P^*-measurable sets. If P^*-measure has the property that the transformations $\{T_s^*\}$ are measure preserving, the P^*-measure, and any process with space $\Omega \subset \Omega^*$ with the corresponding P-measure, will be called temporally homogeneous. In the study of temporally homogeneous processes, a natural tool is the ergodic theorem. The following lemma will make its application possible.

LEMMA 2.2. *Let $f[x(t)]$ be a P-measurable function defined on the space Ω of a temporally homogeneous measurable stochastic process, and suppose that $T_s^*\Omega = \Omega$ for all values of s. Then*

$$f[x(t+\tau)] = f[T_\tau^* x(t)]$$

is a measurable function in $\{\tau, x(t)\}$ space $T \times \Omega$.

† The existence of such a density function was proved by Nikodym, Fundamenta Mathematicae, vol. 15 (1930), pp. 168–179. The discussion here is a specialization of one by Kolmogoroff, (V, pp. 41–43).

‡ The characteristic function of a point set is defined as 1 on the set and 0 on its complement.

The statement that the process is measurable gives the lemma for $f[x(t)]=x(0)$. The proof in the general case consists of showing that the lemma is true for $f[x(t)]$ the characteristic function of a P-measurable set (going from neighborhoods to more general sets), and then approximating f in the general case by linear combinations of such functions.

If $f[x(t)]$ is as described in the lemma, $f[T_\tau^* x(t)]$ is Lebesgue measurable in τ (for $x(t)$ a fixed element of Ω), with probability 1. If we suppose that the integral

$$\int_\Omega f[x(t)]dP$$

exists, the ergodic theorem states that

$$\lim_{Z\to\infty}\frac{1}{Z}\int_0^Z f[x(t+\tau)]d\tau$$

exists, with probability 1.†

3. **Differential stochastic processes.** If P^*-measure is defined in such a way that for $t_1<t_2<\cdots<t_n$

$$x(t_2)-x(t_1),\cdots,x(t_n)-x(t_{n-1})$$

form an independent set of chance variables,‡ the process will be called a differential process.§ The time series discussed above (p. 122) is an example of a differential process.

An Ω-set determined by

$$(3.1)\quad a_j<x(t_j+\delta_j)-x(t_j)<b_j,\quad j=1,\cdots,n,\quad(-\infty\leqq a_j<b_j\leqq+\infty)$$

will be called a differential neighborhood. An Ω-set in the Borel system of sets over the differential neighborhoods, or differing from such a set by a subset of such a set which is of P-measure 0, will be called a differential set. A function $f[x(t)]$ will be called measurable on the differential sets if the set defined by $f>k$ is a differential set for every number k.

† Cf. Doob (I); Khintchine (II, pp. 613–615); Wiener (I, Chapter X); Khintchine's treatment is open to the objection that he does not justify integrating his chance variables with respect to t.

Doob (II, pp. 763–765), Hopf (I, p. 95), and Khintchine (I) applied the ergodic theorem, in the discrete case, to derive the law of large numbers.

A simple presentation of the ergodic theorem, first proved by Birkhoff, was given by Khintchine (Mathematische Annalen, vol. 107 (1933), pp. 485–488).

‡ This means that for any real numbers $a_1,\cdots,a_n$

$$P^*\{x(t_j)-x(t_{j-1})>a_j,\ j=2,\cdots,n\}=\prod_{j=2}^{n}P^*\{x(t_j)-x(t_{j-1})>a_j\}.$$

§ Differential processes have been discussed by Bachelier (I, II, III), Khintchine (III, pp. 68–75), Kolmogoroff (I, II), Lévy (II, III), and Wiener (I, Chapters IX and X).

The definition of temporal homogeneity given in the preceding section is not the most convenient one, when differential processes are under investigation. Throughout this section, we shall understand by the temporal homogeneity of a differential process merely the invariance of the P^*-measure of *differential* sets under the transformations $\{T_s^*\}$.

It will be convenient to define a transformation $T_s\Lambda$ on the P-measurable sets of the space Ω of a stochastic process by

$$T_s\Lambda = \Omega \cdot T_s^*\Lambda^*,$$

where Λ^* is a P^*-measurable set satisfying $\Lambda^* \cdot \Omega = \Lambda$. If T_s^* takes sets of P^*-measure 0 into sets of P^*-measure 0, $T_s\Lambda$ is uniquely determined up to a set of P-measure 0. If $T_s^*\Omega = \Omega$,

$$T_s\Lambda = \Omega \cdot T_s^*\Lambda^* = T_s^*(\Omega \cdot \Lambda^*) = T_s^*\Lambda,$$

so in this case T_s becomes T_s^* on the P-measurable sets. If the process is a temporally homogeneous differential one, $T_s\Lambda$ can be defined as above for Λ any differential set, and the definition will be unique, up to differential sets of P-measure 0.

THEOREM 3.1. *Let B_τ be the Borel system of sets over the Ω-sets determined by the differential neighborhoods* (3.1) *with t_i, $t_i + \delta_i > \tau$, $j = 1, \cdots, n$, where Ω is the space of a stochastic differential process. Then if Λ is a P-measurable set which is in B_τ for all values of τ, $P(\Lambda) = 1$ or $P(\Lambda) = 0$.*†

Using the notation of Theorem 2.7, if M_τ is P-measurable,

$$P(\Lambda \cdot M_\tau) = P(\Lambda) \cdot P(M_\tau). \tag{3.2}$$

Then it follows that if $P(\Lambda) > 0$, $P_{\tau,\Lambda} = P(\Lambda)$ almost everywhere on Ω, so that, by Theorem 2.7, $P(\Lambda) = 1$.

Using this theorem, if $t_n \to \infty$, the probability of convergence of $x(t_n)$ is either 0 or 1. Or by a simple transformation of the t-axis, if t_n approaches t from one side, the probability of convergence is either 0 or 1.

THEOREM 3.2. *Any differential set, invariant (up to a set of P-measure 0) under the transformations $\{T_s\}$ on the differential sets of Ω, the space of a temporally homogeneous differential stochastic process, has P-measure 0 or 1.*‡

---

† Cf. Jessen (I, pp. 270–271), Kolmogoroff (III, pp. 60–61), and Lévy (IV, pp. 88–89), for the analogous theorem on discrete stochastic processes.

‡ Another way of stating this theorem is that the transformations $\{T_s\}$ are metrically transitive over the collection of differential sets. The corresponding theorem was proved for discrete processes by Doob (II, pp. 761–763) and Hopf (I, p. 95). The method used here can also be used in that case.

Let Λ be a differential set of P-measure ρ invariant under the transformations $\{T_s\}$. If $0<\rho<1$, there is a number $\epsilon>0$ so small that

$$(\rho+\epsilon)^2 < \rho - 2\epsilon. \tag{3.3}$$

Let Γ be a finite sum of differential neighborhoods satisfying

$$P(\Gamma) < \rho+\epsilon, \qquad P\{\Lambda\cdot c\Gamma\} < \epsilon.\dagger \tag{3.4}$$

Suppose that the t-values involved in defining Γ are all in the interval $a<t<b$, and let $\Gamma' = T_{b-a}\Gamma$. Then

$$P(\Gamma') = P(\Gamma) < \rho+\epsilon, \quad P(\Gamma\cdot\Gamma') < (\rho+\epsilon)^2, \quad P(\Lambda\cdot c\Gamma') < \epsilon, \tag{3.5}$$

so

$$\begin{aligned}(\rho+\epsilon)^2 > P(\Gamma\cdot\Gamma') &\geqq P(\Lambda\cdot\Gamma\cdot\Gamma') \\ &\geqq P(\Lambda) - P(\Lambda\cdot c\Gamma) - P(\Lambda\cdot c\Gamma') > \rho - 2\epsilon,\end{aligned} \tag{3.6}$$

contradicting (3.3).

LEMMA 3.1. *Let $\Delta_1, \Delta_2, \cdots$ be a sequence of independent chance variables. If there is a chance variable Δ such that*

$$\Delta - \sum_1^n \Delta_j$$

is independent of the set $\{\Delta_1, \cdots, \Delta_n\}$ for $n=1, 2, \cdots$ there is a sequence of constants $\{m_n\}$ such that

$$\lim_{n\to\infty}\left[\sum_1^n \Delta_j - m_n\right] \tag{3.7}$$

exists and is finite with probability 1.

The existence of constants $\{m_n\}$ with the property described was proved by Lévy (I, p. 132).

THEOREM 3.3. *Choose $m(t)$ to satisfy*

$$\int_\Omega \arctan\,[x(t) - m(t)]\,dP = 0\ddagger \tag{3.8}$$

for each value of t, and set $y(t)=x(t)-m(t)$. Then if the process is a differential one, and if t is any real number, there is a chance variable $y(t-0)$ $(y(t+0))$ such that $t_n \uparrow t$ $(t_n \downarrow t)$ implies

† The notation $c\Gamma$ will be used to denote $\Omega-\Gamma$. The existence of Γ can be proved, for example, by transfinite induction.

‡ For t fixed the integral is a monotone decreasing function of m, varying from $\pi/2$ to $-\pi/2$ as m varies from $-\infty$ to $+\infty$, so there is a unique value of m satisfying (3.8).

$$\lim_{n\to\infty} y(t_n) = y(t-0), \qquad \lim_{n\to\infty} y(t_n) = y(t+0) \tag{3.9}$$

with probability 1. *Except at most at a denumerable t-set,*

$$y(t-0) = y(t) = y(t+0)$$

with probability 1.†

Suppose that $t_n \uparrow t$. We can assume that $t_1 < t_2 < \cdots$. Then

$$y(t_n) - y(t_1) = \sum_1^n \Delta_j, \qquad \Delta_j = y(t_{j+1}) - y(t_j). \tag{3.10}$$

Taking the variable Δ of Lemma 3.1 as $y(t) - y(t_1)$, it follows that a sequence of constants $\{m_n\}$ exists such that

$$\lim_{n\to\infty} [y(t_n) - m_n] = y \tag{3.11}$$

exists and is finite with probability 1. If $\limsup_{n\to\infty} |m_n| = \infty$, some subsequence of $\{\arctan y(t_n)\}$ must approach $-\pi/2$ or $\pi/2$, contradicting

$$\int_\Omega \arctan\, y(t)\,dP = 0.‡ \tag{3.8'}$$

Unless the sequence $\{m_n\}$ is convergent, there are two subsequences $\{m_{a_n}\}$, $\{m_{b_n}\}$ converging to different finite limits m', m'' respectively. Then

$$\lim_{n\to\infty} y(t_{a_n}) = y + m', \qquad \lim_{n\to\infty} y(t_{b_n}) = y + m'', \tag{3.12}$$

contradicting (3.8) again. Thus $\lim_{n\to\infty} m_n$ exists and is finite, implying the same (with probability 1) for $\lim_{n\to\infty} y(t_n)$. The latter limit must be independent of the particular sequence $\{t_n\}$ chosen since any pair of such sequences can be arranged in a single sequence. The limit y is thus the $y(t-0)$ of (3.9). The other half of (3.9) is proved in the same way. We shall need the fact that

$$\int_\Omega \arctan\, y(t-0)\,dP = \int_\Omega \arctan\, y(t+0)\,dP = 0. \tag{3.8''}$$

To prove the last part of the theorem, define $\alpha_t(v)$ by

---

† This theorem is due to Lévy (II, pp. 342–343). Lévy centers his distributions by a different function of t.

‡ We shall use repeatedly in this discussion the theorem of Lebesgue that a uniformly bounded convergent sequence of measurable functions defined on a measurable set of finite measure can be integrated term by term.

$$\alpha_t(v) = \left| \int e^{iv[y(t)-y(0)]} dP \right|, \qquad t > 0. \tag{3.13}$$

Then $\alpha_t(v)$ is non-negative and continuous in v. Since $\alpha_{t+h}(v)$, $h>0, t>0$, is the product of $\alpha_t(v)$ (which is less than or equal to 1) and a function obtained similarly from the distribution of $y(t+h)-y(t)$,† $\alpha_t(v)$ is monotone non-increasing in t. It can readily be shown that at any t-point where $y(t-0)-y(t)$, $y(t+0)-y(t)$ are not identically constant, with probability 1, $\alpha_t(v)$ must have a jump for some rational value of v. Then the set of such points is at most denumerable. Moreover if either of these two differences is identically constant with probability 1, the constant must be 0, because of (3.8″), so the theorem is completely proved, for $t>0$. A similar discussion can be given for $t<1$, and the combination of the two results gives the theorem in full.

COROLLARY. *If the process of Theorem* 3.3 *is measurable, $m(t)$ is a Lebesgue measurable function, and the process whose elements are $\{y(t)\}$ is a measurable differential process.*

The inequality $m(t)<k$ is equivalent to

$$\int_\Omega \arctan\,[x(t) - k]\,dP < 0. \tag{3.14}$$

Now $\arctan\,[x(\tau)-k]$ is a measurable bounded function in $\{\tau, x(t)\}$-space $T\times\Omega$, so the integral of (3.14), a Lebesgue measurable function of t (Fubini's theorem), is negative on a Lebesgue measurable t-set. The function $m(t)$ is thus Lebesgue measurable. Let Ω' be the space whose elements are the functions $y(t)$. We define P'-measure by

$$P'\{y(t_j) < \mu_j, j = 1, \cdots, n\} = P\{x(t_j) < \mu_j + m(t_j), j = 1, \cdots, n\}, \tag{3.15}$$

for any $t_1, \cdots, t_n, \mu_1, \cdots, \mu_n$. It is readily seen that in this way a measurable differential stochastic process with space Ω' is defined.

The choice of the function $m(t)$ is to a certain extent arbitrary. If $m_1(t)-m(t)$ is continuous, $m_1(t)$ would serve as well for the purposes of Theorem 3.3. If, whenever $t_n \uparrow t$ or $t_n \downarrow t$,

$$\lim_{t_n \uparrow t} x(t_n), \qquad \lim_{t_n \downarrow t} x(t_n)$$

† This follows at once from the fact that if f and g are bounded P-measurable functions, which are independent in the usual sense of probability,

$$\int f\cdot g\,dP = \int f\,dP \cdot \int g\,dP,$$

a fact which in turn is simply the fact that an iterated integral, if the limits of integration are constants, is the product of its component integrals.

exist and are finite with probability 1, and if the differences $x(t-0)-x(t)$, $x(t+0)-x(t)$ can never be identically constant with probability 1 unless the constant is 0, the process will be said to be centered If a process is centered, the proof of Theorem 3.3 shows that at most a denumerably infinite set of t-values are points of "fixed discontinuities" (where $P\{x(t-0)=x(t)\}<1$ or $P\{x(t+0)=x(t)\}<1$). Even these cannot occur if the process is temporally homogeneous. The P^*-measure of a centered differential process thus satisfies the condition of Theorem 2.5 on P^*-measure, so a measurable process always exists with the same measure relations. This result will be considerably strengthened by Theorem 3.8. The above corollary can be restated as follows. *If the process of Theorem 3.3 is measurable, $m(t)$ is a Lebesgue measurable function, and the process whose elements are $\{y(t)\}$ is a centered measurable process.*

THEOREM 3.4. *If $x(t)$ is the general element of a quasi-separable differential stochastic process with the property that the integrals*

$$\int_\Omega [x(t) - x(0)]dP, \qquad \int_\Omega [x(t) - x(0)]^2 dP \tag{3.16}$$

exist for all t, and if the first vanishes identically, σ_t^2 is monotone non-decreasing in t for $t>0$, and if $T>0$, $\mu>0$,

$$P\{\text{L.U.B.}_{0<t<T} \mid x(t) - x(0) \mid > \mu\} < \sigma_T^2/\mu^2. \tag{3.17}$$

Since if $\delta>0$

$$\sigma_{t+\delta}^2 = \sigma_t^2 + \int_\Omega [x(t+\delta) - x(t)]^2 dP, \tag{3.18}$$

$\sigma_{t+\delta} \geqq \sigma_t^2$. The Ω-set Λ where

$$\text{L.U.B.}_{0<t<T} \mid x(t) - x(0) \mid > \mu \tag{3.19}$$

is the set where

$$\text{L.U.B.}_{j\geqq 1} \mid x(\lambda_j) - x(0) \mid > \mu \tag{3.20}$$

for some sequence $\{\lambda_n\}$ in $0<t<T$, since the process is quasi-separable. The set Λ is P-measurable and

$$P(\Lambda) = \lim_{n\to\infty} P\{\max_{j\leqq n} \mid x(\lambda_j) - x(0) \mid > \mu\}. \tag{3.21}$$

By Kolmogoroff's generalization of the Tchebycheff inequality (III, p. 310,

Theorem 1), the P-measure of the nth set on the right is at most $\sigma(n)^2/\mu^2$, where if $\overline{\lambda}_n = \max_{j \leqq n} \lambda_j$, $\sigma(n)^2 = \sigma^2_{\overline{\lambda}_n}$. Then

$$P(\Lambda) \leqq \limsup_{n \to \infty} \sigma(n)^2/\mu^2 \leqq \sigma_T^2/\mu^2, \tag{3.22}$$

as was to be proved.

This theorem could be used to derive the analogue of Theorem 3.3 in the special case being considered. Under the hypotheses, the process is already centered.

THEOREM 3.5. *Let $\overline{B}_\tau$ be the Borel system of sets over the Ω-sets determined by inequalities of the form*

$$a_j < x(t_j - \delta_j) - x(t_j) < b_j, \quad \delta_j > 0, \quad t_j \leqq \tau, \quad j = 1, \cdots, n, \tag{3.23}$$

where Ω is the space of a centered temporally homogeneous differential stochastic process. Let $s[x(t)]$ be a P-measurable function with the property that the Ω-set determined by $s \geqq \tau$ is in $\overline{B}_\tau$ for every number τ. Let

$$y(t) = x(t + s[x(t)]).$$

(i) *If $s[x(t)]$ takes on at most denumerably many values,*

$$\begin{aligned} &P\{y(t_j + \delta_j) - y(t_j) < \mu_j, j = 1, \cdots, n\} \\ &\qquad = P\{x(t_j + \delta_j) - x(t_j) < \mu_j, j = 1, \cdots, n\} \end{aligned} \tag{3.24}$$

whenever $0 < t_1 < t_1 + \delta_1 \leqq t_2 < \cdots \leqq t_n < t_n + \delta_n$.

(ii) *If the process is quasi-separable, the conclusion of* (i) *holds without the restriction that s take on at most denumerably many values.*†

(i) If s assumes only the values $a_1, a_2, \cdots$ on sets $\Lambda_1, \Lambda_2, \cdots$ respectively, and if $0 < t_1 < t_1 + \delta_1 \leqq t_2 < \cdots \leqq t_n < t_n + \delta_n$,

$$\begin{aligned} &P\{y(t_j + \delta_j) - y(t_j) < \mu_j, j = 1, \cdots, n\} \\ &\quad = \sum_\nu P\{s = a_\nu, x(t_j + a_\nu + \delta_j) - x(t_j + a_\nu) < \mu_j, j = 1, \cdots, n\} \\ &\quad = \sum_\nu P\{s = a_\nu\} \cdot P\{x(t_j + a_\nu + \delta_j) - x(t_j + a_\nu) < \mu_j, j = 1, \cdots, n\} \\ &\quad = \sum_\nu P\{s = a_\nu\} \cdot P\{x(t_j + \delta_j) - x(t_j) < \mu_j, j = 1, \cdots, n\} \end{aligned} \tag{3.25}$$

which is equivalent to (3.24), since $\sum_\nu P\{s = a_\nu\} = 1$.

(ii) Define s_n by

$$s_n = (k-1)/n \quad \text{if} \quad (k-1)/n \leqq s < k/n, \qquad k = 0, 1, \cdots. \tag{3.26}$$

Then $|s_n - s| \leqq 1/n$. The hypotheses of the theorem imply that

† Doob (III) proved this theorem, in a somewhat more general form, for discrete processes.

$$\lim_{t' \to t} x(t') = x(t)$$

with probability 1 (t fixed). There is therefore a positive function $\delta(\epsilon, \eta)$, independent of t, such that if t is fixed, and if $\epsilon > 0$, $\eta > 0$,

$$(3.27) \qquad P\{\text{L.U.B.}_{|\bar{\delta}|<\delta} \mid x(t+\bar{\delta}) - x(t) \mid < \epsilon\} > 1 - \eta.$$

By (i) the distribution of

$$x(s_n + t + \bar{\delta}) - x(s_n + t), \qquad \bar{\delta} > 0,$$

is the same as that of

$$x(t + \bar{\delta}) - x(t),$$

if $t > 1/n$, and evidently the elements $x(s_n+t)$ (n fixed), $t > 1/n$, can be considered as the elements of a quasi-separable stochastic process, so

$$(3.28) \quad P\{\text{L.U.B.}_{|\bar{\delta}|<\delta} \mid x(s_n + t + \bar{\delta}) - x(s_n + t) \mid < \epsilon\} > 1 - \eta, \ (t > 1/n), \quad n = 1, 2, \cdots.$$

If $\nu = \nu(\epsilon, \eta)$ is so large that $1/\nu < \delta$,

$$(3.29) \quad \mid x(s + t) - x(s_\nu + t) \mid = \mid x(s_\nu + t + s - s_\nu) - x(s_\nu + t) \mid < \epsilon, \ t > 1/\nu$$

with probability at least $1 - \eta$: the sequence $x(s_n+t)$ converges in probability, i.e., in measure on Ω, to $x(s+t)$, if $t > 0$. Then $y(t) = x(s+t)$ is readily seen to have the same differential distribution as $x(t)$ ($t > 0$), another way of stating (3.24).

We consider now a differential process for which the integrals (3.16) exist, the first vanishing identically. We shall assume for greater simplicity, that there are no fixed t-points of discontinuity, meaning in this case that σ_t^2 is a continuous function of t. Then if $V(t)$ is defined by

$$(3.30) \qquad V(t) = \sigma_t^2, \ t > 0; \quad V(t) = -\sigma_t^2, \ t < 0; \quad V(0) = 0,$$

we shall define a chance variable (i.e., a P-measurable function)

$$(3.31) \qquad \int f(t) dx(t)$$

for any Baire function $f(t)$ for which

$$(3.32) \qquad \int f(t)^2 dV(t)$$

exists as a Lebesgue-Stieltjes integral.

Let $f(t)$ be the characteristic function of an open bounded t-set, the sum of intervals $a_j<t<b_j$, $j=1, 2, \cdots$. Then we make the definition

$$\int f(t)dx(t) = \sum_1^\infty [x(b_j) - x(a_j)], \tag{3.33}$$

where the series on the right must converge, since the terms are independent chance variables whose means are 0 and whose standard deviations squared form a convergent series:†

$$\sum_1^\infty \int_\Omega [x(b_j) - x(a_j)]^2 dP = \sum_1^\infty [V(b_j) - V(a_j)]. \tag{3.34}$$

We note that

$$\int_\Omega \left\{ \int f(t)dx(t) \right\}^2 dP = \int_O dV(t). \tag{3.35}$$

If B is any bounded Borel set, there is a monotone sequence $O_1 \supset O_2 \supset \cdots$ of bounded open sets whose product includes B and differs from it at most by a set of V-measure 0; i.e., by a set on which $\int dV(t)=0$. If $f, f_1, f_2, \cdots$ are the characteristic functions of $B, O_1, O_2, \cdots$, respectively,

$$\int_\Omega \left\{ \int f_n(t)dx(t) - \int f_m(t)dx(t) \right\}^2 dP = \int_{O_n - O_m} dV(t), \qquad m > n, \tag{3.36}$$

so that $\int f_n(t)dx(t)$ converges in the mean on Ω. It can be shown that the limit is independent of the sequence $\{O_n\}$ chosen (disregarding Ω-sets of P-measure 0). This limit is denoted by (3.31), and (3.35) remains true, substituting B for O.

If $f(t)$ is a Baire function taking on only a finite number of values, and equal to 0 outside some finite interval, $f(t)$ is a linear combination of functions of the type just treated, and (3.31) is defined as the corresponding linear combination of integrals. Equation (3.35) becomes

$$\int_\Omega \left\{ \int f(t)dx(t) \right\}^2 dP = \int_{-\infty}^\infty f(t)^2 dV(t). \tag{3.35'}$$

If $f(t)$ is any Baire function for which (3.32) exists, it can be approximated in the mean by Baire functions $\{f_n(t)\}$ each of which takes on only a finite number of values:

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} [f_n(t) - f(t)]^2 dV(t) = \lim_{n,m\to\infty} \int_{-\infty}^{+\infty} [f_n(t) - f_m(t)]^2 dV(t) = 0. \tag{3.37}$$

† Kolmogoroff (III, p. 314, Theorem 7).

Then

$$\int_\Omega \left\{ \int f_n(t)dx(t) - \int f_m(t)dx(t) \right\}^2 dP$$

$$= \int_\Omega \left\{ \int [f_n(t) - f_m(t)]dx(t) \right\}^2 dP = \int_{-\infty}^{+\infty} [f_n(t) - f_m(t)]^2 dV(t) \to 0, \tag{3.38}$$

so that the sequence $\{\int f_n(t)dx(t)\}$ converges in the mean. The chance variable (3.31) is defined as the corresponding limit, which is readily seen to be independent of the approximating sequence, neglecting Ω-sets of P-measure 0.

THEOREM 3.6. *If $f_1(t)$, $f_2(t)$ are Baire functions, and if*

$$\int_{-\infty}^{+\infty} f_j(t)^2 dV(t) < \infty, \qquad j = 1, 2, \tag{3.39}$$

then

$$\int f_j(t)dx(t), \qquad \int_\Omega \left\{ \int f_j(t)dx(t) \right\}^2 dP \tag{3.40}$$

are defined, and

$$\int_\Omega \left\{ \int f_1(t)dx(t) \cdot \int f_2(t)dx(t) \right\} dP = \int_{-\infty}^{+\infty} f_1(t) \cdot f_2(t)dV(t). \dagger \tag{3.41}$$

We have already defined $\int f_j(t)dx(t)$. Equation (3.35′) is a special case of (3.41) with $f_1 = f_2 = f$, f taking on only a finite number of values, and (3.35′) is true whenever the integral (3.32) is finite, since the corresponding equality is true for the approximating functions. Equation (3.41) is deduced from (3.35′) by writing (3.35′) for $f_1 + f_2$, $f_1 - f_2$ and finding the difference between the two equations thus obtained.

THEOREM 3.7. *Let $\phi_1(t), \phi_2(t), \cdots$ be a complete normal orthogonal set with respect to the measure function $V(t)$, for functions defined on the interval $a \leqq t \leqq b$ $(-\infty \leqq a < b \leqq +\infty)$. Then if*

$$\Phi_n[x(t)] = \int \phi_n(t)dx(t), \tag{3.42}$$

the functions $\{\Phi_n\}$ form a complete normal orthogonal set for the space of functions (3.40) *on Ω with $f(t) = 0$ if $t < a$, $t > b$:*

---

† The integral over Ω in (3.41) exists, by Schwarz' inequality, since the second pair of integrals in (3.40) exists.

$$\int_{\Omega} \Phi_n \cdot \Phi_m dP = \delta_{n,m}, \tag{3.43}$$

and

$$\begin{aligned} \int f(t)dx(t) &= \text{l.i.m.}_{N\to\infty} \sum_1^N c_n\Phi_n, \dagger \\ c_n &= \int \left\{ \Phi_n \cdot \int f(t)dx(t) \right\} dP = \int_a^b \phi_n(t)f(t)dV(t). \end{aligned} \tag{3.44}$$

In particular

$$\begin{aligned} x(\tau) - x(a) &= \text{l.i.m.} \sum_1^N c_n(\tau)\Phi_n, \\ c_n(\tau) &= \int_{\Omega} \{ \Phi_n \cdot [x(\tau) - x(a)] \} dP = \int_a^{\tau} \phi_n(t)dV(t). \end{aligned} \tag{3.45}$$

The functions $\{\Phi_n\}$ form a normal orthogonal set since by (3.41),

$$\int_{\Omega} \Phi_n \cdot \Phi_m dP = \int_a^b \phi_n \cdot \phi_m dV(t) = \delta_{n,m}, \tag{3.46}$$

and (3.44) follows by applying (3.41) to

$$f_1 = f_2 = f - \sum_1^N c_n\phi_n.$$

If $f(t)$ is taken as the characteristic function of the interval $a \leqq t \leqq \tau$, (3.44) becomes (3.45).

The elements of the process have thus been made to depend upon a denumerably infinite set of parameters $\Phi_1, \Phi_2, \cdots$. The discussion given here is a generalization of one given by Wiener, who supposes that $x(t+\delta) - x(t)$ has a Gaussian distribution with zero mean and standard deviation proportional to $(|\delta|)^{1/2}$.‡ Wiener *defines* the elements $x(t)$ by (3.45), taking advantage of the fact that in this case there is a subset Ω_0 of Ω^* of P^*-measure 1 on which the series in (3.45) converges to a continuous function of τ.§

THEOREM 3.8. *Let P^*-measure be so defined that Ω^* is the space of a centered differential stochastic process. Let Ω_0 be the space of elements $x(t)$ which are con-*

† *Limit in the mean* will be denoted by l.i.m.

‡ Cf. Wiener (I, Chapter IX).

§ Cf. our proof of Theorem 3.9.

tinuous except possibly for discontinuities of the first kind,† *and everywhere continuous on the right except perhaps at fixed points of discontinuity. Then* $\overline{P}^*(\Omega_0) = 1$: Ω_0 *is the space of a quasi-separable measurable differential stochastic process.*

If Ω *is the space of a quasi-separable centered differential stochastic process, the elements* $x(t)$ *are everywhere continuous, except possibly for discontinuities of the first kind.*‡

The first part of the theorem is really an existence theorem, and therefore the second part would have no content unless the first part, or some similar statement, were true.

It will be sufficient to prove the theorem under the assumption that there are no points of fixed discontinuity (the denumerably many jumps at such points in any finite interval can be subtracted out). Suppose then that $\Omega \subset \Omega^*$ is the space of a centered differential process, and that $t_n \to t$ implies that

$$P^*\{\lim_{n\to\infty} x(t_n) = x(t)\} = 1.$$

(i) Let R: $\{r_n\}$ be a sequence of numbers satisfying the following conditions: R includes the rational numbers; if $\rho_j \epsilon R$, $j = 1, \cdots, n$, $(\rho_1 \pm \rho_2 \pm \cdots \pm \rho_n) \epsilon R$. The set R is everywhere dense. Then by Theorem 1.3,

$$\lim_{r\to t} x(r) = x(t) \qquad (r \epsilon R) \tag{3.47}$$

with probability 1.

The oscillation of $x(r)$, $(r \epsilon R)$ at t is defined as

$$\lim_{\epsilon\to 0} \text{L.U.B.}_{|t-r|+|t-r'|<\epsilon} |x(r) - x(r')|. \tag{3.48}$$

This oscillation is 0 almost everywhere on Ω, for each value of t. Let I be any closed t-interval $a \leqq t \leqq b$. The set

$$\prod_{\nu=1}^{\infty} \sum_{\substack{|r_j - r_k|<1/\nu \\ a-1/\nu<r_j, r_k<b+1/\nu}} \{|x(r_j) - x(r_k)| > \alpha - 1/\nu\}, \qquad \alpha > 0, \tag{3.49}$$

consists of the elements $x(t)$ for which the oscillation of $x(r)$ is at least α at some point of I. This set is P-measurable, and its complement is also: *if* $S_\alpha[(t)]$ *is the* t*-set at which the oscillation of* $x(r)$ *is at least* α*, the* Ω*-set of elements for which* $I \cdot S_\alpha[x(t)] = 0$ *is* P*-measurable.*

† A function $x(t)$ is said to have a discontinuity of the first kind at a if the limits $\lim_{t\downarrow a} x(t)$, $\lim_{t\uparrow a} x(t)$ exist and are unequal.

‡ Cf. Lévy (II, pp. 359–364; III, pp. 217–218).

(ii) We shall now prove that the points of the set $S_a[x(t)]$ are points of discontinuity of the first kind, with probability 1. Unless this is true, there is an interval $I: a \leqq t \leqq b$ so large that the set of elements $x(t)$ for which there is a point t, $t \epsilon I \cdot S_a[x(t)]$, $t<b$, t a point of discontinuity of $x(r)$ of the second kind, is not of P-measure 0. The set $S_a[x(t)]$ is closed for each element $x(t)$. Let $s_1[x(t)]$ be the first point of $S_a[x(t)]$ in I, or b, if $I \cdot S_a[x(t)]=0$. According to (i), the function $s_1[x(t)]$ is P-measurable, and evidently it has the property described in Theorem 3.5. Then if $s_{1,n}$ is defined by

$$(3.50)\quad s_{1,n} = k/n \text{ if } k/n \leqq s_1 < (k+1)/n,\ k = 0, \pm 1, \cdots, n = 1, 2, \cdots,$$

$s_{1,n}+1/n$ has this same property. According to (i), if $\epsilon>0$, there is a number $\delta=\delta(\epsilon, t)$ such that

$$(3.51)\quad P\Big\{\underset{\substack{|t-r'|<\delta \\ |t-r''|<\delta}}{\text{L.U.B.}} \big| x(r') - x(r'')\big| \leqq \epsilon\Big\} \geqq 1-\epsilon, \quad r', r'' \epsilon R.$$

An immediate application of the Heine-Borel theorem shows that a function $\delta(\epsilon)$ exists, independent of t, such that (3.51) is true for $\delta=\delta(\epsilon)$ when $t \epsilon I$. Then if $\delta=\delta(\epsilon)$,

$$\begin{aligned} &P\Big\{\underset{0<r', r''<\delta}{\text{L.U.B.}} \big| x(s_{1,n}+1/n+r') - x(s_{1,n}+1/n+r'')\big| \leqq \epsilon\Big\} \\ (3.52)\quad &= \sum_k P\{s_{1,n} = (k-1)/n\} \\ &\quad \cdot P\Big\{\underset{0<r', r''<\delta}{\text{L.U.B.}} \big| x(k/n+r') - x(k/n+r'')\big| \leqq \epsilon\Big\} \geqq 1-\epsilon, \end{aligned}$$

or

$$(3.53)\quad P\{\text{L.U.B.}\big| x(r') - x(r'')\big| \leqq \epsilon\} \geqq 1-\epsilon,$$

where r', r'' are points of R satisfying

$$(3.54)\quad s_{1,n} + 1/n < r', r'' < s_{1,n} + 1/n + \delta,$$

which will surely be true (for $1/n<\delta$) if

$$(3.55)\quad s_1 + 1/n < r', r'' < s_1 + \delta.$$

Since n can be made arbitrarily large,

$$(3.56)\quad P\Big\{\underset{s_1<r', r''<s_1+\delta}{\text{L.U.B.}} \big| x(r') - x(r'')\big| \leqq \epsilon\Big\} \geqq 1-\epsilon.$$

This inequality implies that $\lim_{r \downarrow s_1} x(r)$ exists with probability 1. If $s_1<b$, s_1 is therefore an isolated point of $S_\epsilon[x(t)]$ with probability 1. It is then readily seen (applying the results just obtained to the process with elements $x(-t)$)

that $\lim_{r \downarrow s_1} x(r)$ exists with probability 1. Then $x(r)$ has at worst a discontinuity of the first kind at s_1, with probability 1. If s_2 is the second point of S_α in I (or b if there is no second point), it can be shown in the same way that $x(r)$ has at worst a discontinuity of the first kind at s_2, with probability 1. In this way we obtain a sequence $s_1, s_2, \cdots, s_1 \leqq s_2 \leqq \cdots \leqq b$: $s_n[x(t)] \epsilon S_\alpha[x(t)]$ if $s_n < b$; $x(r)$ has a discontinuity of the first kind at s_n, if $s_n < b$, with probability 1. If $\lim_{n\to\infty} s_n = s$, $s \epsilon S_\alpha[x(t)]$ if $s < b$, and the above argument shows that s is an isolated point of $S_\alpha[x(t)]$ in that case, with probability 1, an impossibility, unless

$$P\{\lim_{n\to\infty} s_n < b\} = 0. \tag{3.57}$$

This however contradicts the hypothesis that the set of elements $x(t)$ having a point $t \epsilon I \cdot S_\alpha[x(t)]$, $t < b$, t a discontinuity of $x(r)$ of the second kind, is not of zero P-measure. We have thus proved, since α is arbitrary, that *the function* $x(r)$ $(r \epsilon R)$ *has at worst discontinuities of the first kind, with probability* 1.

(iii) If in particular Ω is quasi-separable, there is a sequence R: $\{r_n\}$ of points which has the property that the upper and lower limiting functions of $x(r)$, $r \epsilon R$ are the same as those of $x(t)$. The above proof then shows that $x(t)$ is continuous except for discontinuities of the first kind, with probability 1. The number of such discontinuities for a given element $x(t)$ is at most denumerable.† This completes the proof of the second part of the theorem.

(iv) Suppose that $\Omega = \Omega^*$, and define Ω_0 as in the statement of the theorem. To show that $\overline{P}^*(\Omega_0) = 1$, it is sufficient to show that if $\Gamma^* = \sum_1^\infty \Gamma_n^*$ is a sum of neighborhoods covering Ω_0, then $P^*(\Gamma^*) = 1$. Let R: $\{r_n\}$ be the set of t-values used in defining the neighborhoods $\Gamma_1^*, \Gamma_2^*, \cdots$, enlarged, if necessary, to satisfy the requirements on R in (i). Let Λ_1^* be the Ω^*-set of elements $x(t)$ for which $x(r)$ is continuous except for discontinuities of the first kind. In accordance with (ii), $P^*(\Lambda_1^*) = 1$. If Λ^* is the subset of Λ_1^* for which $x(r)$ is continuous at r_j, $j = 1, 2, \cdots$, $P^*(\Lambda^*) = 1$, by (3.47). Let $x_0(t)$ be an element of Λ^*. Define $x_1(t)$ by

$$x_1(t) = \lim_{r \downarrow t} x_0(r), \qquad r \epsilon R. \tag{3.58}$$

Then $x_1(r_j) = x_0(r_j)$, $j = 1, 2, \cdots$, and $x_1(t) \epsilon \Omega_0 \subset \Gamma^*$. There is therefore an integer ν such that $x_1(t) \epsilon \Gamma_\nu^*$. Since Γ_ν^* is defined by inequalities on $x(t)$ at certain of the numbers $r_1, r_2, \cdots$, $x_1(t) \epsilon \Gamma_\nu^*$ implies that $x_0(t) \epsilon \Gamma_\nu^*$. Therefore $\Lambda^* \subset \Gamma^*$, so

$$P^*(\Gamma^*) \geqq P^*(\Lambda^*) = 1 : \overline{P}^*(\Omega_0) = 1,$$

as was to be proved.

† Hobson, *The Theory of Functions of a Real Variable*, vol. 1, 3d edition, 1927, p. 304.

Differential processes for which the distribution of $x(t+\delta)-x(t)$ is Gaussian have been discussed by many authors,† who have shown completely or partially that this is the only case in which the elements $x(t)$ can all be supposed continuous. Such processes can be made temporally homogeneous by a transformation of the t-axis.

THEOREM 3.9. *Let P^*-measure be so defined that Ω^* is the space of a temporally homogeneous differential process for which*

$$(3.59)\qquad P^*\{a < x(t+\delta) - x(t) < b\} = (2\pi\delta)^{-1/2}\int_a^b e^{-x^2/(2\delta)}dx, \quad \delta > 0.$$

Let Ω_c be the space of elements $x(t)$ which are everywhere continuous. Then $\overline{P}^*(\Omega_c)=1$: Ω_c is the space of a quasi-separable measurable process.

Conversely if Ω is the space of a differential stochastic process all of whose elements $x(t)$ are everywhere continuous, $x(t+\delta)-x(t)$ has a Gaussian distribution, for all t, δ.

If P^*-measure is defined by (3.59), it is readily seen that the chance variable $\int f(t)dx(t)$ discussed above has a Gaussian distribution with mean 0 and standard deviation

$$(3.60)\qquad \left\{\int_\Omega\left[\int f(t)dx(t)\right]^2 dP\right\}^{1/2},$$

more generally that the set

$$(3.61)\qquad \int f_1(t)dx(t), \cdots, \int f_n(t)dx(t)$$

has an n-variate Gaussian distribution, the covariance between the jth and kth members being given by

$$(3.62)\qquad \int_\Omega\left\{\int f_j(t)dx(t)\cdot\int f_k(t)dx(t)\right\}dP.$$

Then the orthogonal set $\Phi_1, \Phi_2, \cdots$ becomes an independent set of functions, in the usual sense of probability, and in (3.44) there is actual convergence, almost everywhere on Ω^*: the series is a series of independent chance variables whose standard deviations squared form a convergent series.‡ In particular the series in (3.45) is convergent almost everywhere on Ω^*. Now if a, b are finite, Wiener has shown that $\phi_1, \phi_2, \cdots$ can be chosen so that

† Kolmogoroff (II, p. 868); Lévy (II, pp. 346–347); Wiener (I, Chapter IX).

‡ Kolmogoroff (III, p. 314, Theorem 7).

$$\sum_{1}^{\infty} c_n(\tau)\Phi_n$$

converges almost everywhere on Ω^* to a continuous function of τ.† Since this continuous function coincides with $x(\tau)-x(a)$ for any fixed τ with probability 1 (in terms of P^*-measure), if R: $\{r_n\}$ is an everywhere dense set of numbers, $x(r)$ $(r\epsilon R)$ is everywhere continuous, with probability 1. Then if Ω_0 is the space whose existence was proved in Theorem 3.7, the discontinuities (jumps) are absent, with probability 1. A slight variation of this discussion would show that $x(t)$ is everywhere continuous with probability 1 on any quasi-separable subspace of Ω^* which is the space of a stochastic process.

Conversely if $x(t)$ is everywhere continuous, the Laplace-Liapounoff theorem can be invoked to show that the distribution of $x(t+\delta)-x(t)$ is Gaussian.‡

Bibliography

L. Bachelier.

(I) *Calcul des Probabilités*, Paris, 1912.

(II) *Le Jeu, la Chance, et le Hasard*, Paris, 1914.

(III) *Théorie de la spéculation*, Annales de l'École Normale, series 3, vol. 17 (1900), pp. 21–86.

F. P. Cantelli.

(I) *Una teoria astratta del calcolo delle probabilità*, Giornale dell'Istituto Italiano degli Attuari, vol. 3 (1932), pp. 257–265.

J. L. Doob.

(I) *Stochastic processes and statistics*, Proceedings of the National Academy of Sciences, vol. 20 (1934), pp. 376–379.

(II) *Probability and statistics*, these Transactions, vol. 36 (1934), pp. 759–775.

(III) *Note on probability*, Annals of Mathematics, vol. 37 (1936), pp. 363–367.

B. de Finetti.

(I) *Sulle funzione a incremento aleatorio*, Rendiconti della Reale Accademia Nazionale dei Lincei, Series 6, vol. 10 (1929), pp. 163–168.

E. Hopf.

(I) *On causality, statistics, and probability*, Journal of Mathematics and Physics of the Massachusetts Institute of Technology, vol. 13 (1934), pp. 51–102.

B. Hostinsky.

(I) *Méthodes générales du calcul des probabilités*, Mémorial des Sciences Mathématiques, No. 52.

(II) *Application du calcul des probabilités à la théorie du mouvement Brownien*, Annales de l'Institut Henri Poincaré, vol. 3 (1932), pp. 1–74.

B. Jessen.

(I) *The theory of integration in a space of an infinite number of dimensions*, Acta Mathematica, vol. 63 (1934), pp. 249–323.

M. Kac.

(I) *Sur les fonctions indépendantes*, I, Studia Mathematica, vol. 6 (1936), pp. 46–58.

(II) (with H. Steinhaus) *Sur les fonctions indépendantes*, II. *La loi exponentielle; la divergence de séries*, ibid., pp. 59–66.

† Wiener (I, pp. 148–151, and earlier papers). In the reference given here the elements in the stochastic process are complex-valued.

‡ Lévy (II, pp. 346–347).

(III) (with H. Steinhaus) *Sur les fonctions indépendantes*, III. *Le mouvement Brownien; la loi de Maxwell*, ibid., pp. 88–97.

A. KHINTCHINE.

(I) *Zur mathematischen Begrundung der statistischen Mechanik*, Zeitschrift für angewandte Mathematik und Mechanik, vol. 13 (1933), pp. 101–103.

(II) *Korrelationstheorie der stationare stochastischen Prozesse*, Mathematische Annalen, vol. 109 (1934), pp. 604–615.

(III) *Asymptotische Gesetze der Wahrscheinlichkeitsrechnung*, Ergebnisse der Mathematik, vol. 2, No. 4.

A. KOLMOGOROFF.

(I) *Sulla forma generale di un processo stocastico omogeneo (Un problema di Bruno de Finetti)*, Atti della Reale Accademia Nazionale dei Lincei, Series 6, vol. 15 (1932), pp. 805–808.

(II) *Ancora sulla forma generale di un processo stocastico omogeneo*, ibid., pp. 866–870.

(III) *Über die Summen durch den Zufall bestimmter unabhangiger Grössen*, Mathematische Annalen, vol. 99 (1928), pp. 309–319.

(IV) *Über die analytische Methoden in der Wahrscheinlichkeitsrechnung*, Mathematische Annalen, vol. 104 (1931), pp. 415–458.

(V) *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Ergebnisse der Mathematik, vol. 2, No. 3.

PAUL LÉVY.

(I) *Sur les séries dont les termes sont des variables éventuelles indépendantes*, Studia Mathematica, vol. 3 (1931), pp. 119–155.

(II) *Sur les intégrales dont les éléments sont des variables aléatoires indépendantes*, Pisa Annali, Series 2, vol. 3 (1934), pp. 337–366.

(III) *Observation sur un précédent mémoire de l'auteur*, ibid., vol. 4 (1935), pp. 217–218.

(IV) *Propriétés asymptotiques des sommes de variables aléatoires enchainées*, Bulletin des Sciences Mathématiques, Series 2, vol. 59 (1935), pp. 84–96.

(V) *Sur l'application du théorème de Fubini au calcul des probabilités*, L'Enseignement Mathématique, vol. 33 (1934), pp. 265–270.

Z. ŁOMNICKI.

(I) *Nouveaux fondements du calcul des probabilités*, Fundamenta Mathematicae, vol. 4 (1923), pp. 34–70.

(II) (with S. Ulam) *Sur la théorie de la mesure dans les espaces combinatoires et son application au calcul des probabilités*, I. *Variables indépendantes*, Fundamenta Mathematicae, vol. 23 (1934), pp. 237–278.

R. E. A. C. PALEY (with N. Wiener).

(I) *Fourier Transforms in the Complex Domain*, American Mathematical Society Colloquium Publications, vol. XIX, 1934.

H. STEINHAUS.

(I) *Les probabilités dénombrables et leur rapport à la théorie de la mesure*, Fundamenta Mathematicae, vol. 4 (1923), pp. 286–310.

(II) *Sur la probabilité de la convergence de séries*, Studia Mathematica, vol. 2 (1930), pp. 21–39. Cf. M. Kac and H. Steinhaus.

S. ULAM.

Cf. Z. Łomnicki and S. Ulam.

N. WIENER.

Cf. R. E. A. C. Paley and N. Wiener, and earlier papers by Wiener, whose results are given in that reference.

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A DIRECT SUFFICIENCY PROOF FOR THE PROBLEM OF BOLZA IN THE CALCULUS OF VARIATIONS*

BY

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1. **Introduction.** The problem to be considered is that of minimizing an expression of the form

$$J = g[x^1, y(x^1), x^2, y(x^2)] + \int_{x^1}^{x^2} f(x, y, y')dx$$

in a class of arcs

$$y_i = y_i(x) \qquad (x^1 \leqq x \leqq x^2;\ i = 1, \cdots, n)$$

satisfying a set of differential equations

$$\phi_\beta(x, y, y') = 0 \qquad (\beta = 1, \cdots, m < n)$$

and having its end values $[x^1, y(x^1), x^2, y(x^2)]$ lying on a surface S in $(x^1y^1x^2y^2)$-space. The surface S may be defined by equations of the form

$$\psi_\mu[x^1, y(x^1), x^2, y(x^2)] = 0 \qquad (\mu = 1, \cdots, p \leqq 2n + 2) \tag{1.2}$$

or by parametric equations

$$x^s = x^s(\alpha_1, \cdots, \alpha_r), \qquad y_i^s = y_i^s(\alpha_1, \cdots, \alpha_r) \qquad (s = 1, 2). \tag{1.3}$$

In the latter case the function g of the end values may be considered as a function $g(\alpha)$ of the parameters $\alpha_1, \cdots, \alpha_r$. The functional J can then be taken in the form

$$J = g(\alpha) + \int_{x^1}^{x^2} f(x, y, y')dx.$$

It is immaterial which of the forms (1.2) and (1.3) is used. The theory for the one can readily be obtained from that of the other.

A sufficiency proof for the problem of Bolza independent of assumptions of normality has recently been given by the author (IV)† and in a somewhat modified form by Bliss (VII). The proof given is indirect in the sense that

* Presented to the Society, November 28, 1936; received by the editors November 9, 1936. The results here given were obtained at the University of Chicago during the Summer Quarter, 1936, and came as a result of a discussion of the problem with Professor Bliss.

† Roman numerals in parentheses refer to the references at the end of the paper.

it is applicable only after the problem has been transformed into an equivalent problem of somewhat different form. This proof, although adequate, is not entirely satisfactory. For a satisfactory theory one should have a sufficiency proof which can be applied directly to the problem at hand. The purpose of this paper is to give such a proof. The proof here given is essentially a generalization of the earlier one and makes use of the notion of families of Mayer fields which is the basis of Hahn's theorem as presented by Bliss (II). The only other sufficiency proof for the problem of Bolza known to the author which is independent of assumptions of normality is the one recently announced by Reid (IX). This proof has the same limitations as the earlier proof of the author.*

2. **Preliminary remarks.** The hypotheses upon which the analysis of the present paper is based and the terminology used will be that of the author (IV) in case the end conditions are of the form (1.3) and that of Bliss (VII) in case the end conditions are of the form (1.2). The definition of admissible arcs here used will be that of Bliss; that is, an arc (1.1) will be said to be *admissible* if it is continuous and is composed of a finite number of subarcs having a continuously turning tangent, satisfying the equations $\phi_\beta = 0$, and having its elements (x, y, y') all admissible. As is customary we center our attention on a particular admissible arc E_0 with end values $(x_0^1, y_0^1, x_0^2, y_0^2)$ on S. We shall assume that the parameters (α) in equations (1.3) have been chosen so that the end values of E_0 are given by the values $(\alpha) = (0)$. We shall assume further that the matrix of the derivatives of the second members of equations (1.3) has rank r at $(\alpha) = (0)$.

The arc E_0 will be said to satisfy the *condition* I with a set of multipliers $\lambda_0, \lambda_\beta(x)$ if the Euler-Lagrange equations

$$F_{y_i'} = \int_{x^1}^{x} F_{y_i} dx + c_i, \qquad \phi_\beta = 0 \qquad (i = 1, \cdots, n) \tag{2.1}$$

hold along E_0, where the c's are constants and

$$F(x, y, y', \lambda) = \lambda_0 f + \lambda_\beta(x)\phi_\beta \qquad (\beta = 1, \cdots, m)$$

and if further the transversality condition

$$\left[(F - y_i' F_{y_i'})dx + F_{y_i'} dy_i\right]_1^2 + \lambda_0 dg = 0 \tag{2.2}$$

holds on E_0 for all directions $dx^1, dy_i^1, dx^2, dy_i^2$ tangent to S at the point on S determined by the end values of E_0. Here and elsewhere it will be understood

* Added in proof. Reid has recently removed these limitations in a paper entitled *A direct expansion proof of sufficient conditions for the non-parametric problem of Bolza* to appear in an early issue of these Transactions.

that the multiplier λ_0 is a constant and that the multipliers $\lambda_\beta(x)$ have continuity properties like those of the functions $y_i'(x)$ belonging to E_0. An admissible arc (1.1) and a set of multipliers λ_0, $\lambda_\beta(x)$ having continuous derivatives y_i', y_i'', λ_β' will be called an extremal if it satisfies equations (2.1) with a set of constants c_i.

The arc E_0 will be said to satisfy the *Weierstrass condition* $\text{II}'_{\mathfrak{N}}$ with a set of multipliers λ_0, $\lambda_\beta(x)$ if at each element (x, y, y', λ) in a neighborhood $\mathfrak{N}$ of those on E_0 and having (x, y, y') admissible, the inequality

$$E(x, y, y', \lambda, Y') > 0$$

holds for every admissible set $(x, y, Y') \neq (x, y, y')$, where

$$E = F(x, y, Y', \lambda) - F(x, y, y', \lambda) - (Y_i' - y_i')F_{y_i'}(x, y, y', \lambda).$$

The arc E_0 will be said to satisfy the *Clebsch condition* III' if at each element (x, y, y', λ) on E_0 the inequality

$$F_{y_i'y_k'}\pi_i\pi_k > 0 \qquad (i, k = 1, \cdots, n)$$

holds for every set of constants $(\pi) \neq (0)$ satisfying the conditions $\phi_{\beta y_i'}\pi_i = 0$.

The arc E_0 will be said to satisfy the *condition* IV' with a set of multipliers $\lambda_0 = 1$, $\lambda_\beta(x)$ if the second variation J_2 of J formed with these multipliers is positive definite along E_0. In order to define this condition more precisely, recall that when the end conditions are taken in the form (1.3) the second variation J_2 of J along E_0 is expressible in the form (II, pp. 520–521).

$$J_2(\eta, w) = b_{hl}w_hw_l + \int_{x^1}^{x^2} 2\omega(x, \eta, \eta')dx,$$

where $h, l = 1, \cdots, r$; $i, k = 1, \cdots, n$; s not summed,

$$\begin{aligned} 2\omega &= F_{y_iy_k}\eta_i\eta_k + F_{y_iy_k'}\eta_i\eta_k' + F_{y_i'y_k'}\eta_i'\eta_k', \\ b_{hl} &= g_{hl} + \left[(F - y_i'F_{y_i'})x^s_{hl} + F_{y_i'}y^s_{ihl}\right]_{s=1}^{s=2} \\ &\quad + \left[(F_x - y_i'F_{y_i})x^s_hx^s_l + F_{y_i}(x^s_hy^s_{il} + x^s_ly^s_{ih})\right]_{s=1}^{s=2}. \end{aligned} \tag{2.3}$$

In the last expression the subscripts h, l denote the derivatives of the functions $g(\alpha)$, $x^s(\alpha)$, $y_i^s(\alpha)$ with respect to α_h, α_l at $(\alpha) = (0)$. The variations $\eta_i(x)$ are assumed to have derivatives like those of the functions $y_i(x)$ and the w's are constants. The second variation $J_2(\eta, w)$ is said to be *positive definite* along E_0 if the inequality $J_2(\eta, w) > 0$ holds for every set of variations $(\eta, w) \neq (0, 0)$ satisfying along E_0 the differential equations

$$\Phi_\beta(x, \eta, \eta') = \phi_{\beta y_i}\eta_i + \phi_{\beta y_i'}\eta_i' = 0 \qquad (\beta = 1, \cdots, m)$$

and end conditions (s not summed; $s = 1, 2$)

$$\eta_i(x_0^s) = c_{ih}^s w_h, \qquad c_{ih}^s = y_{ih}^s - y_i'(x_0^s)x_h^s, \tag{2.4}$$

the functions $y_i'(x)$ being the slope functions of E_0.

The purpose of the present paper is to give a direct proof of the following theorem:

THEOREM 2.1. *If the arc E_0 has no corners and satisfies the conditions* I, $\mathrm{II}_{\mathfrak{N}}'$, III′, IV′, *there exist neighborhoods $\mathfrak{F}$ of E_0 in xy-space and N of the end values of E_0 in $(x^1y^1x^2y^2)$-space such that the inequality $J(C) > J(E_0)$ holds for every admissible arc C in $\mathfrak{F}$ with end values on S in N and not identical with E_0.*

The condition IV′ can be replaced by other conditions, commonly called conditions of Mayer, which together with the condition III′ imply the condition IV′ here defined. A discussion of these results has been given by Bliss (VII, pp. 95–140) and the author (IV, pp. 810–813; VIII). One should also consult the papers of Morse (V) and Reid (VI).

The differential equations associated with the second variation are given by the equations

$$\Omega_{\eta_i} - (d/dx)\Omega_{\eta_i'} = 0, \qquad \Phi_\beta = 0, \tag{2.5}$$

where $\Omega = \omega + \mu_\beta(x)\Phi_\beta$. When the condition III′ is satisfied, a solution η_i, μ_β of equations (2.5) is completely determined by the functions η_i, $\zeta_i = \Omega_{\eta_i'}$. A system η_{ik}, ζ_{ik} $(i, k = 1, \cdots, n)$ of n linearly dependent solutions of equations (2.5) will be called a conjugate system if the equations

$$\zeta_{ij}\eta_{ik} - \zeta_{ik}\eta_{ij} = 0 \qquad (i, j, k = 1, \cdots, n) \tag{2.6}$$

hold identically on x^1x^2.

3. **Fundamental sufficiency theorems.** In this section we shall assume that the arc E_0 forms an extremal arc with a set of multipliers $\lambda_0 = 1$, $\lambda_\beta(x)$. The sufficiency theorems here given depend upon the notion of families of Mayer fields. The definition of Mayer fields here used is that of Bliss (I, p. 730). Let $\mathfrak{F}$ and N be respectively neighborhoods of E_0 in xy-space and of the end values of E_0 in (x^1, y^1, x^2, y^2)-space such that the points (x^1, y^1), (x^2, y^2) determined by points in N are in $\mathfrak{F}$. Suppose that for each fixed point (x^1, y^1, x^2, y^2) in N the functions

$$p_i(x, y, x^1, y^1, x^2, y^2), \qquad l_0 = 1, \qquad l_\beta(x, y, x^1, y^1, x^2, y^2) \tag{3.1}$$

are the slope functions and multipliers of a Mayer field over the neighborhood $\mathfrak{F}$ of E_0 and that E_0 is an extremal of the field determined by its end values $(x_0^1, y_0^1, x_0^2, y_0^2)$. Let $W(x^1, y^1, x^2, y^2)$ be the function defined over N by the equation

$$W(x^1, y^1, x^2, y^2) = g(x^1, y^1, x^2, y^2) + \int_{(x^1,y^1)}^{(x^2,y^2)} \{Fdx + (dy_i - p_i dx)F_{y_i'}\},$$

where the integral in this expression is the Hilbert integral formed with the functions (3.1) and evaluated along an arc in $\mathfrak{F}$ joining the points (x^1, y^1), (x^2, y^2). Since E_0 is an extremal of the field determined by its end values, it follows that

$$J(E_0) = W(x_0^1, y_0^1, x_0^2, y_0^2). \tag{3.2}$$

A first sufficiency theorem is the following (cf. VII, p. 107).

THEOREM 3.1. *If at each point (x, y) in $\mathfrak{F}$ and each element (x^1, y^1, x^2, y^2) on S in N the Weierstrass E-function formed with the functions* (3.1) *satisfies the condition*

$$E(x, y, p, l, y') > 0 \tag{3.3}$$

for every admissible set $(x, y, y') \neq (x, y, p)$ and if the end values of E_0 afford a proper minimum for the function $W(x^1, y^1, x^2, y^2)$ on S in N, then the inequality $J(C) > J(E_0)$ holds for every admissible arc C in $\mathfrak{F}$ with end values on S in N and not identical with E_0.

For from the definitions of the function W and the Weierstrass E-function it follows readily that

$$J(C) = W(x^1, y^1, x^2, y^2) + \int_C E(x, y, p, l, y')dx,$$

where (x^1, y^1, x^2, y^2) are the end values of C and the functions p_i, l_0, l_β appearing in the E-function are the functions (3.1) determined by the end values of C. Since the end values of the arc E_0 afford a proper minimum to the function W on S in N it follows from the relations (3.2), (3.3), and the last equation that

$$J(C) \geqq W(x^1, y^1, x^2, y^2) \geqq W(x_0^1, y_0^1, x_0^2, y_0^2) = J(E_0), \tag{3.4}$$

the equality holding in both cases only in case the end values of C and E_0 coincide and the equations $y_i' = p_i$ hold along C. But in this case the arc C would be an extremal of the field determined by the common end values of C and E_0 and hence would coincide with E_0 since there is but one extremal of the field through each point of $\mathfrak{F}$. This proves the theorem.

When the surface S is defined by the parametric equations (1.3), the parameter values $(\alpha) = (0)$ giving the end values of E_0, the sufficiency theorem described above can be stated in somewhat different form. Let $\mathfrak{F}$ be a neighborhood of E_0 in xy-space and A be a neighborhood of $(\alpha) = (0)$ in $(\alpha_1, \cdots, \alpha_r)$-

space such that the points (x^1, y^1), (x^2, y^2) determined by points on S with parameter values (α) in A are in $\mathfrak{F}$. Suppose that for each element (α) in A the functions

$$p_i(x, y, \alpha), \qquad l_0 = 1, \qquad l_\beta(x, y, \alpha) \tag{3.5}$$

are the slope functions and multipliers of a Mayer field defined over $\mathfrak{F}$ and that E_0 is an extremal of the field determined by $(\alpha) = (0)$. Let $W(\alpha_1, \cdots, \alpha_r)$ be the function defined by the equation

$$W(\alpha) = g(x^1, y^1, x^2, y^2) + \int_{(x^1,y^1)}^{(x^2,y^2)} \{F dx + (dy_i - p_i dx) F_{y_i'}\}, \tag{3.6}$$

where (x^1, y^1, x^2, y^2) is the point (1.3) on S determined by $(\alpha_1, \cdots, \alpha_r)$ and the integral is the Hilbert integral formed with the functions (3.5) and evaluated along an arc in $\mathfrak{F}$ joining the points (x^1, y^1), (x^2, y^2). One readily verifies that $J(E_0) = W(0)$. The following sufficiency theorem can now be established by an argument like that given above.

THEOREM 3.2. *If at each point (x, y) in $\mathfrak{F}$ and each element (α) in A the inequality*

$$E[x, y, p(x, y, \alpha), l(x, y, \alpha), y'] > 0 \tag{3.7}$$

holds for every admissible element $(x, y, y') \neq (x, y, p)$ and the element $(\alpha) = (0)$ affords a proper relative minimum to the function $W(\alpha)$, then the inequality $J(C) > J(E_0)$ is true for every admissible arc C in $\mathfrak{F}$ not identical with E_0 and having its end values on S in a sufficiently small neighborhood N of the end values of E_0.

One readily verifies that Theorems 3.1 and 3.2 are equivalent whenever, as we have assumed, the surface S is non-singular. The earlier sufficiency theorem given by Hestenes (IV, p. 805) and in somewhat different form by Bliss (VII, p. 107) is the special case in which the functions (3.1) and (3.5) depend only on the values (x, y). When this earlier theorem is applied to the problem of Bolza with separated end conditions obtained from the original problem of Bolza here described by a suitably chosen transformation, and the result is reinterpreted in terms of the original problem, one obtains a theorem of the type here described.

4. **Two lemmas.** The proof of Theorem 2.1 to be given in the next section will be based on Theorem 3.2 and two lemmas, the first of which is the following:

LEMMA 4.1. *Let E_0 be a non-singular extremal arc with multipliers $\lambda_0 = 1$, $\lambda_\beta(x)$ for which there exists a conjugate system η_{ik}, ζ_{ik} of solutions of the accessory*

equations (2.5) *whose determinant* $|\eta_{ik}(x)|$ *is different from zero along* E_0. *Let* u_{ih}, v_{ih} $(h=1, \cdots, r)$ *be a set of* r *further solutions of equations* (2.5). *There exists an* $(n+r)$*-parameter family of extremals*

$$\begin{aligned} y_i &= y_i(x, a_1, \cdots, a_n, \alpha_1, \cdots, \alpha_r), \\ \lambda_0 &= 1, \quad \lambda_\beta = \lambda_\beta(x, a_1, \cdots, a_n, \alpha_1, \cdots, \alpha_r) \end{aligned} \tag{4.1}$$

containing E_0 *for values* $x_0^1 \leqq x \leqq x_0^2$, $a_i=0$, $\alpha_h=0$ *and such that the functions* y_i, y_{ix}, $z_i=F_{y_i'}$, z_{ix} *have continuous first and second partial derivatives in a neighborhood of the values* (x, a, α) *belonging to* E_0 *and have as its variations along* E_0

$$y_{ia_k} = \eta_{ik}, \qquad z_{ia_k} = \zeta_{ik}, \qquad y_{i\alpha_h} = u_{ih}, \qquad z_{i\alpha_h} = v_{ih}. \tag{4.2}$$

Moreover there is a neighborhood $\mathfrak{F}$ *of* E_0 *in* xy*-space and* A *of* $(\alpha)=(0)$ *such that for every fixed set* (α) *in* A *the family* (4.1) *defines a Mayer field over* $\mathfrak{F}$. *The slope functions and multipliers of the field are given by the equations*

$$\begin{aligned} p_i(x, y, \alpha) &= y_{ix}[x, a(x, y, \alpha), \alpha], \\ l_0 = 1, \qquad l_\beta(x, y, \alpha) &= \lambda_\beta[x, a(x, y, \alpha), \alpha], \end{aligned} \tag{4.3}$$

where the values $a_i(x, y, \alpha)$ *are determined by the first* n *of the equations* (4.1). *The arc* E_0 *is an extremal of the field determined by the set* $(\alpha)=(0)$.

In order to establish this result let

$$\begin{aligned} y_i &= Y_i(x, b_1, \cdots, b_n, c_1, \cdots, c_n) \\ \lambda_0 &= 1, \qquad \lambda_\beta = \Lambda_\beta(x, b_1, \cdots, b_n, c_1, \cdots, c_n) \end{aligned} \tag{4.4}$$

be a $2n$-parameter family of extremals containing E_0 for values $b_i=b_{i0}$, $c_i=c_{i0}$, $x_0^1 \leqq x \leqq x_0^2$ and such that the parameters b_i, c_i are the values of the functions Y_i, $Z_i=F_{y_i'}(x, Y, Y_x, \Lambda)$ at $x=x_0^1$, the functions Y_i, Y_{ix}, Z_i, Λ_β having continuous first and second partial derivatives in a neighborhood of the values (x, b, c) belonging to E_0. The $(n+r)$-parameter family (4.1) obtained from the family (4.4) by setting

$$\begin{aligned} b_i &= b_{i0} + \eta_{ik}(x_0^1)a_k + u_{ih}(x_0^1)\alpha_h, \\ c_i &= c_{i0} + \zeta_{ik}(x_0^1)a_k + v_{ih}(x_0^1)\alpha_h \end{aligned} \tag{4.5}$$

can now be shown to have the properties described in the lemma. For, by differentiating the identities

$$b_i = y_i(x_0^1, a, \alpha), \qquad c_i = z_i(x_0^1, a, \alpha),$$

where $z_i=F_{y_i'}$, with respect to a_k and α_h it is found that the relations (4.2) hold at $x=x_0^1$ and hence also along E_0, since along E_0 the right and left mem-

bers of equations (4.2) are solutions of the accessory equations (2.5) with the same initial values at $x=x_0^1$. The determinant $|y_{i\alpha_k}|$ is accordingly different from zero along E_0. It follows that the first n equations (4.1) have unique solutions $a_i(x, y, \alpha)$ which are continuous and have continuous first and second partial derivatives for all (x, y, α) with (x, y) in a neighborhood $\mathfrak{F}$ of E_0 and (α) in a neighborhood A of $(\alpha)=(0)$. For each fixed set (α) in A the family (4.1) not only simply covers $\mathfrak{F}$ but also defines a Mayer field over $\mathfrak{F}$ with the slope functions and multipliers (4.3). This follows because in view of equations (4.5), $\alpha_h=$const., (2.6) the Hilbert integral takes the form

$$\int F_{y_i'}dy_i = \int c_i db_i = \frac{1}{2}\int d[(2c_i - \zeta_{ij}a_j)\eta_{ik}a_k]$$

on the hyperplane $x=x_0^1$ and is accordingly independent of the path on $x=x_0^1$ and hence also in $\mathfrak{F}$ (I, p. 733). This completes the proof of Lemma 4.1.

A second lemma is the following. (Cf. VII, p. 110.)

LEMMA 4.2. *Let E_0 be an extremal arc having the properties described in Lemma* 4.1. *The function $W(\alpha)$ defined by equations* (3.6) *and* (4.3) *will have a proper relative minimum at $(\alpha)=(0)$ if E_0 satisfies the transversality condition* (2.2) *and the condition $Q(w)>0$ for every set $(w)\neq(0)$, where*

$$(4.6)\quad Q(w) = b_{hl}w_hw_l + [c_i(\zeta_{ik}d_k + v_i) + (c_i - u_i)(v_i - \zeta_{ik}e_k)]_1^2$$

$$(4.7)\quad u_i = u_{ih}w_h, \quad v_i = v_{ih}w_h, \quad c_i = c_{ih}w_h, \quad d_i = d_{ih}w_h, \quad e_i = e_{ih}w_h,$$
$$(h, l = 1, \cdots, r;\ i, k = 1, \cdots, n)$$

b_{hl} are constants defined by equations (2.3), *and $c_{ih}(x)$, $d_{ih}(x)$, $e_{ih}(x)$ are functions satisfying the conditions* ($s=1, 2$)

$$(4.8)\quad c_{ih}(x_0^s) = c_{ih}^s, \quad \eta_{ik}d_{kh} = c_{ih} - u_{ih}, \quad \eta_{ik}e_{kh} = u_{ih}.$$

It is well known that the function $W(\alpha)$ will have a proper minimum at $(\alpha)=(0)$ whenever the conditions $dW=0$, $d^2W>0$ hold at $(\alpha)=(0)$ for all differentials $d\alpha_1, \cdots, d\alpha_r$ not all zero. The lemma will accordingly be established if we show that the equations

$$(4.9)\quad dW = 0, \quad d^2W = Q(d\alpha)$$

are identities in $d\alpha_1, \cdots, d\alpha_r$. To this end let δ denote the differential with respect to the variables $\alpha_1, \cdots, \alpha_r$ of a function of (x, y, α). Thus

$$\delta p_i = p_{i\alpha_h}d\alpha_h, \quad \delta F_{y_i'} = d\alpha_h\left(\frac{\partial}{\partial\alpha_h}\right)F_{y_i'}(x, y, p, l).$$

We then have

$$dW = dg + [(F - p_iF_{y_i'})dx + F_{y_i'}dy_i]_1^2 + \int_{(x^1,y^1)}^{(x^2,y^2)} (dy_i - p_idx)\delta F_{y_i'},$$

where here and elsewhere the differentials dx^1, dy_i^1, dx^2, dy_i^2 are to be evaluated in terms of $d\alpha_h$ by means of equations (1.3). This expression vanishes identically at $(\alpha) = (0)$, since E_0 satisfies the transversality condition (2.2) and the differential equations $y_i' = p_i(x, y, 0)$ of the field determined by $(\alpha) = (0)$. This proves the first relation (4.9). In order to prove the second relation we note first that at $(\alpha) = (0)$

$$d^2W = d^2g + [(F - p_iF_{y_i'})d^2x + F_{y_i'}d^2y_i + d(F - p_iF_{y_i'})dx + dF_{y_i'}dy_i]_1^2$$
$$+ [(dy_i - p_idx)\delta F_{y_i'}]_1^2 + \int_{E_0} \{-\delta p_i\delta F_{y_i'}dx + (dy_i - p_idx)\delta^2F_{y_i'}\}.$$

With the help of equations (2.3),

$$d(F - p_iF_{y_i'}) = F_xdx + F_{y_i}dy_i - p_idF_{y_i'},$$

and the relations $y_i' = p_i$ along E_0 it is found that

$$d^2W = b_{hl}d\alpha_hd\alpha_l + [(dy_i - p_idx)(dF_{y_i'} - F_{y_i}dx + \delta F_{y_i'})]_1^2 - \int_{E_0} \delta p_i\delta F_{y_i'}dx.$$

It is clear that this expression for d^2W will reduce to the form $Q(d\alpha)$ if we establish the relations

$$[(dy_i - p_idx)(dF_{y_i'} - F_{y_i}dx)]_1^2 = [c_i(\zeta_{ik}d_k + v_i)]_1^2,$$

$$[(dy_i - p_idx)\delta F_{y_i'}]_1^2 = [c_i(v_i - \zeta_{ik}e_k)]_1^2, \tag{4.10}$$

$$\int_{E_0} \delta p_i\delta F_{y_i'}dx = [u_i(v_i - \zeta_{ik}e_k)]_1^2,$$

where the functions c_i, d_i, e_i, u_i, v_i are defined by equations (4.7) with $w_h = d\alpha_h$. As a first step in the proof of these equations we note that in view of equations (2.4) and $y_i' = p_i$ along E_0 we have at $(\alpha) = (0)$

$$dy_i^s - p_i(x^s, y^s, 0)dx^s = c_{ih}^s d\alpha_h = c_i(x^s) \qquad (s = 1, 2), \tag{4.11}$$

where x^s, y_i^s and their differentials are to be evaluated by means of equations (1.3). Moreover if we set $\delta a_i = a_{i\alpha_h}d\alpha_h$ and let da_i^s be the differential of the function $a_i[x^s(\alpha), y^s(\alpha), \alpha]$, then along E_0 with $(\alpha) = (0)$

$$\delta a_i = -e_i, \qquad da_i^s = d_i(x^s) \qquad (s = 1, 2), \tag{4.12}$$

where $d_i(x)$, $e_i(x)$ are defined as above. For, by differentiating the identity

$$y_i = y_i[x, a(x, y, \alpha), \alpha] \tag{4.13}$$

for α_h at $(\alpha)=(0)$, it is found with the help of equations (4.2), (4.7) that along E_0

$$0 = y_{ia_k}\delta a_k + y_{ia_h}d\alpha_h = \eta_{ik}\delta a_k + u_i,$$

and hence that $\delta a_i = -e_i$ by virtue of the definition of e_i. Similarly from equations (4.13) with x, y_i replaced by $x^s(\alpha)$, $y_i^s(\alpha)$ it follows that at $(\alpha)=(0)$

$$dy_i^s - y_{ix}dx^s = y_{ia_k}da_k^s + y_{ia_h}d\alpha_h = \eta_{ik}da_k^s + u_i,$$

and hence by equations (4.11) and $y_{ix}=p_i$ along E_0 that

$$\eta_{ik}(x^s)da_k^s = c_i(x^s) - u_i(x^s) \qquad (s = 1, 2).$$

This is clearly only possible in case $d_i(x^s)=da_i^s$ by virtue of equations (4.7) and (4.8) with $w_h=d\alpha_h$. Finally by the use of the identity

$$F_{y_i'}(x, y, p, l) = z_i[x, a(x, y, \alpha), \alpha]$$

and the Euler-Lagrange equations $z_{ix}=F_{y_i}$ along the extremal E_0 one finds with the help of equations (4.2), (4.7), (4.12), and $w_h=d\alpha_h$ that

$$\begin{aligned} dF_{y_i'} - F_{y_i}dx &= z_{ia_k}da_k + z_{ia_h}d\alpha_h = \zeta_{ik}da_k + v_i, \\ \delta F_{y_i'} &= z_{ia_k}\delta a_k + z_{ia_h}d\alpha_h = -\zeta_{ik}e_k + v_i. \end{aligned} \tag{4.14}$$

The first two equations (4.10) are now an easy consequence of equations (4.11), (4.14), and (4.12). In order to establish the last of these equations we need the further relation along E_0

$$\delta p_i = y_{ixa_k}\delta a_k + y_{ixa_h}d\alpha_h = -\eta_{ik}'e_k + u_i' = \eta_{ik}e_k' \tag{4.15}$$

which follows readily from equations (4.3), (4.2), and (4.12), the last equation being obtained by differentiating the equation $u_i=\eta_{ik}e_k$. Moreover along E_0 the expression $\eta_{ik}\delta F_{y_i'}$ is a constant. For, by means of equations (4.14) and $u_i=\eta_{ij}e_j$ it is found that along E_0

$$\eta_{ik}\delta F_{y_i'} = (\eta_{ik}v_i - u_i\zeta_{ik}) + (\eta_{ij}\zeta_{ik} - \eta_{ik}\zeta_{ij})e_j.$$

In this expression the second term on the right is zero by virtue of equations (2.6). The first term is a constant since (I, p. 738) the sets η_{ik}, ζ_{ik} and u_i, v_i are solutions of the accessory equations (2.5). The expression $\eta_{ik}\delta F_{y_i'}$ is accordingly constant along E_0 and hence by equations (4.15) we have

$$(d/dx)(u_i\delta F_{y_i'}) = (d/dx)(\delta F_{y_i'}\eta_{ik}e_k) = \delta F_{y_i'}\eta_{ik}e_k' = \delta p_i\delta F_{y_i'}.$$

From this relation and equation (4.14) one readily obtains the last equation (4.10). This completes the proof of Lemma 4.2.

5. **Proof of Theorem 2.1.** In this section we shall consider as admissible arc E_0 satisfying the conditions described in Theorem 2.1 with a set of multipliers $\lambda_0=1$, $\lambda_\beta(x)$. It is well known (I, p. 735) that E_0 forms with these multipliers a non-singular extremal arc. The following lemma will be useful in the proof of Theorem 2.1.

LEMMA 5.1. *There exists for the extremal E_0 a conjugate system η_{ik}, ζ_{ik} and a set of r solutions u_{ih}, v_{ih} $(h=1, \cdots, r)$ of the accessory equations* (2.5) *such that the determinant $|\eta_{ik}(x)|$ is different from zero on E_0 and the quadratic form $Q(w)$ described in Lemma* 4.2 *is positive definite on E_0.*

If we accept the truth of Lemma 5.1 the proof of Theorem 2.1 can be made as follows: By virtue of Lemmas 5.1, 4.1, and 4.2 there exist neighborhoods $\mathfrak{F}$ of E_0 and A of $(\alpha)=(0)$ and a set of slope functions and multipliers (4.3) defining a Mayer field over $\mathfrak{F}$ for each set (α) in A and such that $W(\alpha)>W(0)$ for every set $(\alpha)\neq(0)$ in A. The arc E_0 is an extremal of the field determined by $(\alpha)=(0)$. Moreover by virtue of the condition $\Pi'_{\mathfrak{N}}$ for E_0 the condition (3.7) of Theorem 3.2 will be satisfied if we take the neighborhoods $\mathfrak{F}$ and A so small that the elements (x, y, p, l) will lie in the neighborhood $\mathfrak{N}$ of the elements (x, y, y', λ) on E_0 prescribed by the condition $\Pi'_{\mathfrak{N}}$. Theorem 2.1 now follows from Theorem 3.2.

In order to establish Lemma 5.1 we set, for convenience, $x^1=x_0^1$, $x^2=x_0^2$ and prove first the further lemma.

LEMMA 5.2. *If $c_i(x)$ $(x^1\leqq x\leqq x^2)$ is an arc satisfying the condition*

$$\zeta_i(x^2)c_i(x^2) - \zeta_i(x^1)c_i(x^1) = 0 \qquad (i = 1, \cdots, n)$$

for every solution η_i, ζ_i of the accessory equations (2.5) *having $\eta_i\equiv 0$ on x^1x^2, then the end values of $c_i(x)$ can be joined by a solution $\eta_i=u_i(x)$, $\zeta_i=v_i(x)$ of the accessory equations.*

This result follows at once from the proof of "Lemma 7.2" of Hestenes (IV, p. 809), since the end points of E_0 are not conjugate; that is, there is no solution η_i, ζ_i of the accessory equations having $\eta_i(x^1)=\eta_i(x^2)=0$ and $\eta_i\not\equiv 0$ on x^1x^2. This follows because along such an arc one would have $J_2(\eta, 0)=0$ contrary to the condition IV′. This lemma differs from "Lemma 7.2" of Hestenes in that we do not require the arc $c_i(x)$ to be admissible.

Returning to the proof of Lemma 5.1 we first note that by virtue of the condition IV′ there exists a conjugate system η_{ik}, ζ_{ik} of solutions of the accessory equations having its determinant $|\eta_{ik}(x)|$ different from zero on E_0 (IV, p. 811). Secondly, let $U_{i\gamma}$, $V_{i\gamma}$ $(\gamma=1, \cdots, q)$ be a maximal set of solutions of equations (2.5) such that $U_{i\gamma}\equiv 0$ on x^1x^2 and the matrix

$$\|V_{i\gamma}(x^2)c^2_{ih} - V_{i\gamma}(x^1)c^1_{ih}\| \qquad (\gamma = 1, \cdots, q;\ h = 1, \cdots, r)$$

has rank q. In fact we can suppose that

$$V_{i\gamma}(x^2)c^2_{ih} - V_{i\gamma}(x^1)c^1_{ih} = \delta_{\gamma h} \qquad (\gamma = 1, \cdots, q;\ h = 1, \cdots, r), \tag{5.1}$$

where $\delta_{\gamma\gamma}=1$, $\delta_{\gamma h}=0$ $(\gamma \neq h)$, since this result can be brought about by a non-singular linear transformation on the parameters $\alpha_1, \cdots, \alpha_r$ and by taking suitable linear combinations of the solutions $U_{i\gamma}$, $V_{i\gamma}$. Let $c_{ih}(x)$ be a set of arcs having $c_{ih}(x^s)=c^s_{ih}$ $(s=1, 2;\ h=1, \cdots, r)$. For each value σ $(q<\sigma\leqq r)$ the arc $c_{i\sigma}$ satisfies the conditions described in Lemma 5.2 by virtue of equations (5.1). Its end values can therefore be joined by a solution $u_{i\sigma}, v_{i\sigma}$ of the accessory equations. We may accordingly suppose that $c_{i\sigma}=u_{i\sigma}$. Let

$$u_{i\gamma} = \epsilon U_{i\gamma} \equiv 0, \qquad v_{i\gamma} = \epsilon V_{i\gamma} \qquad (\gamma = 1, \cdots, q), \tag{5.2}$$

where ϵ is a constant chosen in the manner described below, and let $d_{ih}(x)$, $e_{ih}(x)$ $(h=1, \cdots, r)$ be solutions of equations (4.8). Since $u_{i\gamma}\equiv 0$ $(\gamma=1, \cdots, q)$, $c_{i\sigma}=u_{i\sigma}$ $(\sigma=q+1, \cdots, r)$, we have also $e_{i\gamma}\equiv 0$, $d_{i\sigma}\equiv 0$. If now we set

$$\begin{aligned}
A_{\beta\gamma} &= b_{\beta\gamma} + (1/2)\left[c_{i\beta}\zeta_{ik}d_{k\gamma} + c_{i\gamma}\zeta_{ik}d_{k\beta}\right]^2_1 && (\beta, \gamma = 1, \cdots, q),\\
2B_{\gamma\tau} &= 2b_{\gamma\tau} + \left[c_{i\tau}\zeta_{ik}d_{k\gamma} - c_{i\gamma}(\zeta_{ik}e_{k\tau} - 2v_{i\tau})\right]^2_1,\\
C_{\sigma\tau} &= b_{\sigma\tau} + (1/2)\left[u_{i\sigma}v_{i\tau} + u_{i\tau}v_{i\sigma}\right]^2_1 && (\sigma, \tau = q+1, \cdots, r),
\end{aligned}$$

then by the use of equations (5.1), (5.2), $e_{i\gamma}\equiv 0$, $d_{i\sigma}\equiv 0$ it is found that the quadratic form $Q(w)$ of Lemma 4.2 can be written in the form

$$Q(w) = (A_{\beta\gamma} + 2\epsilon\delta_{\beta\gamma})w_\beta w_\gamma + 2B_{\gamma\tau}w_\gamma w_\tau + C_{\sigma\tau}w_\sigma w_\tau,$$

where $\delta_{\beta\beta}=1$, $\delta_{\beta\gamma}=0$ $(\beta\neq\gamma)$. The values $A_{\beta\gamma}=A_{\gamma\beta}$, $B_{\gamma\tau}$, $C_{\sigma\tau}=C_{\tau\sigma}$ are independent of ϵ. If we set

$$\eta_i = u_{i\sigma}w_\sigma,\ \zeta_i = v_{i\sigma}w_\sigma,\ \bar{w}_\gamma = 0,\ \bar{w}_\sigma = w_\sigma \ (\gamma = 1, \cdots, q;\ \sigma = q+1, \cdots, r),$$

we have $\eta_i(x^s)=c^s_{ih}\bar{w}_h$ $(s=1, 2;\ h=1, \cdots, r)$ and

$$C_{\sigma\tau}w_\sigma w_\tau = b_{hl}\bar{w}_h\bar{w}_l + \left[\eta_i\zeta_i\right]^2_1 = J_2(\eta, \bar{w}), \tag{5.3}$$

the last equation being obtained by integration by parts with the help of equations (2.5). By the condition IV′ we have $J_2(\eta, \bar{w})>0$ unless $\eta_i\equiv 0$, $\bar{w}_h=0$, that is, unless $w_\sigma=0$. The quadratic form (5.3) is accordingly positive definite. From the theory of quadratic forms it now follows that $Q(w)$ will also be positive definite if the constant ϵ is sufficiently large. For example, this can be seen by showing that the matrix

$$\left\| \begin{matrix} A_{\beta\gamma} + 2\epsilon\delta_{\beta\gamma} & B_{\beta\tau} \\ B_{\gamma\sigma} & C_{\sigma\tau} \end{matrix} \right\|$$

has a sequence of positive principal minors beginning in the lower right-hand corner. This proves Lemma 5.1. The proof of Theorem 2.1 is now complete.

6. **Relation to the methods of Mayer, Hahn, and Bliss.** The methods used in the last two sections are closely related to the methods of Mayer and Hahn as presented by Bliss (II) when properly interpreted. The earlier method seems to be applicable only in case E_0 is normal on the interval $x_0^1 x_0^2$. In making his sufficiency proof Bliss essentially establishes a result which when phrased in the terminology of the present paper can be stated as follows:

THEOREM 6.1. *If the arc E_0 is normal on $x_0^1 x_0^2$ and satisfies the conditions described in Theorem* 2.1, *there exist neighborhoods $\mathfrak{F}$ of E_0 and N of the end values of E_0 and a set of slope functions and multipliers* (3.1) *having the properties described in Theorem* 3.1 *and such that every point (x^1, y^1, x^2, y^2) in N is a set of end values of an extremal of the field determined by it. Moreover the function $W(x^1, y^1, x^2, y^2)$ is the extremal integral; that is, it represents the value of J along the extremal in $\mathfrak{F}$ whose end values are given by (x^1, y^1, x^2, y^2).*

Thus we see that in this case the inequality (3.4) can be written in the form

$$J(C) \geqq J(E) \geqq J(E_0), \tag{6.1}$$

where E is the extremal in $\mathfrak{F}$ joining the ends of C. The relation (6.1) and the last theorem indicate clearly the connection between our work and that of Mayer, Hahn, and Bliss.

REFERENCES

I. Bliss, *The problem of Lagrange in the calculus of variations*, American Journal of Mathematics, vol. 52 (1930), pp. 673–744.

II. Morse, *Sufficient conditions in the problem of Lagrange with variable end points*, American Journal of Mathematics, vol. 53 (1931), pp. 517–596.

III. Bliss, *The problem of Bolza in the calculus of variations*, Annals of Mathematics, (2), vol. 33 (1932), pp. 261–274.

IV. Hestenes, *Sufficient conditions for the problem of Bolza in the calculus of variations*, these Transactions, vol. 36 (1934), pp. 793–818.

V. Morse, *Sufficient conditions in the problem of Lagrange without assumptions of normalcy*, these Transactions, vol. 37 (1935), pp. 147–160.

VI. Reid, *The theory of the second variation for the non-parametric problem of Bolza*, American Journal of Mathematics, vol. 57 (1935), pp. 573–586.

VII. Bliss, *The problem of Bolza in the calculus of variations*, Lectures delivered at the University of Chicago during the Winter Quarter, 1935.

VIII. Hestenes, *On sufficient conditions in the problems of Lagrange and Bolza*, Annals of Mathematics, (2), vol. 37 (1936), pp. 543–551.

IX. Reid, *Sufficient conditions by expansion methods for the problem of Bolza in the calculus of variations in parametric form*, Bulletin of the American Mathematical Society, vol. 41 (1935), p. 788, abstract 41-11-379.

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A NEW CLASS OF TRANSCENDENTAL NUMBERS*

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1. **Introduction.** Every algebraic number except zero and unity (e^0), when raised to a power which is an irrational algebraic number gives a transcendental number. This was conjectured by Hilbert, and proposed as a problem in 1900.† It was only recently that Gelfond,‡ and independently at about the same time, Schneider,§ succeeded in proving this theorem. G. Ricci‖ has shown that the transcendentality persists if the algebraic numbers are replaced by products of algebraic numbers and a restricted type of Liouville number. The question we consider here is the more general one where the algebraic numbers are replaced by numbers capable of approximation by sequences of algebraic numbers in an appropriate manner.

These form a restricted type of certain generalized Liouville transcendental numbers very closely related to those treated by Ore.¶ They bear a relation to the numbers of an algebraic field similar to that of the ordinary Liouville numbers to the rational numbers.

Our principal result is that if X and Y are each suitably approximable by sequences of algebraic numbers, then X^Y and $\log X/\log Y$ are transcendental. Some numerical examples are given in §7.

Though the transcendental numbers found by us have the power of the continuum, they form a set of zero measure as we show.

My interest in this subject was stimulated by a report on the Hilbert-Gelfond-Schneider theorem which I gave to the seminar of Professor H. Weyl at Princeton, and in particular by a conjecture of Professor J. von Neumann that natural generalizations of these results would involve approximations by algebraic numbers, rather than by rational numbers. I also wish to acknowledge my appreciation of the opportunity to work without distraction on this field, made possible by the courtesy of the Massachusetts Institute of Technology and the Institute for Advanced Study.

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† For reference to the earlier work on this and related questions concerning transcendental numbers, the reader may consult J. F. Koksma, *Diophantische Approximationen*, Ergebnisse der Mathematik, vol. 4, No. 4, Berlin, 1936, pp. 58–65.

‡ A. Gelfond, Bulletin de l'Académie des Sciences de l'U.R.S.S., vol. 7 (1934), p. 623.

§ T. Schneider, Crelle's Journal, vol. 172 (1934), p. 65.

‖ G. Ricci, Annali delle R. Scuola Normale Superiore di Pisa, vol. 4 (1935), p. 341.

¶ O. Ore, Avhandlinger, Norske Videnskaps-Akademie, Oslo, Matematisk-naturvidenskapelig Klasse, vol. 1 (1925), p. 11.

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2. **The lemmas.** We shall adopt the following notation for our lemmas. K_ν will mean an algebraic field of degree ν, fixed throughout. For a concise presentation of the concepts and theory on algebraic fields used here, see, e.g., Landau, *Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale*, Leipzig, 1927. The symbols α, β, and η will denote algebraic numbers belonging to this field. We will attach subscripts to indicate that we have an infinite sequence of such numbers. However, the conjugates c^c these numbers in K_ν will all be uniformly bounded. Since, except for repetitions, a number has the same conjugates in any field containing it, the specification of K_ν here is not essential. We indicate this by writing

$$\|\alpha\| < V, \quad \|\beta\| < V, \quad \|\eta\| < V. \tag{2.1}$$

Moreover, $|\log \beta|$ and its reciprocal will be uniformly bounded by V.

The letters c, q, s, t, S, T will denote positive integers. In particular, c is such that $c\alpha$, $c\beta$, and $c\eta$ are all algebraic integers. Usually s and t will take on the values $0, 1, 2, \cdots, S-1$ and $0, 1, 2, \cdots, T-1$, respectively, unless otherwise indicated.

We use γ or γ_i as a generic symbol for any positive quantity which depends only on K_ν and V.

Finally we define

$$g(x) = \sum_{k=0}^{q-1} \sum_{l=0}^{q-1} C_{kl} \alpha^{kx} \beta^{lx}, \tag{2.2}$$

where the C_{kl} are algebraic integers belonging to K_ν, more precisely specified later.

We also define

$$g_s(x) = (\log \beta)^s \sum_{k=0}^{q-1} \sum_{l=0}^{q-1} C_{kl} (k\eta + l)^s \alpha^{kx} \beta^{lx}. \tag{2.3}$$

It may be noticed that, if η were equal to $\log \alpha / \log \beta$, this would be the sth derivative of $g(x)$.

LEMMA I.* *If $q^2 \geqq 2ST$, there exists a $g(x)$ such that $g_s(t) = 0$, with coefficients C_{kl} not all zero satisfying the inequality*

$$\log \|C_{kl}\| < S[\log (\gamma c) + \log q] + 2qT \log (\gamma c). \tag{2.4}$$

To prove this, we note that the ST equations $g_s(t) = 0$ are linear in the q^2 variables C_{kl}, so that, if we denote the coefficients by a_{klst}, we may write them in the form

* Cf. Schneider, loc. cit., and C. Siegel, Abhandlungen der Preussischen Akademie der Wissenschaften, No. 1, 1929.

$$\sum a_{klst} C_{kl} = 0. \tag{2.5}$$

If we multiply the equations by a suitable power of c before doing this the coefficients a_{klst} will be algebraic integers in K_ν, and we shall assume this done. Furthermore, we may deduce bounds for the conjugates of these coefficients from those for α, β, and η. Thus:

$$\|a_{klst}\| < (\gamma_1 c)^{S+2q(T-1)} q^{S-1} \equiv \mu. \tag{2.6}$$

Now let ρ_i, $i=1, 2, \cdots, \nu$ denote an integral basis for K_ν. Suppose that $\|\rho_i\| < R$. Put

$$y_{st} = \sum a_{klst} x_{kl}, \tag{2.7}$$

and

$$x_{kl} = \sum_{i=1}^{\nu} B_{kli} \rho_i. \tag{2.8}$$

If the B_{kli} take on the values $0, \pm 1, \pm 2, \cdots, \pm h$, it follows that

$$\|x_{kl}\| < \nu R h \equiv \gamma_2 h. \tag{2.9}$$

Hence

$$\|y_{st}\| < q^2 \mu \gamma_2 h. \tag{2.10}$$

Next we put

$$y_{st} = \sum_{i=1}^{\nu} B_{sti} \rho_i. \tag{2.11}$$

We may write similar equations, with the same B_{sti} but with y_{st} and ρ_i replaced by their conjugates in K_ν. These may be solved, since the ρ_i formed a basis, and the determinant of them and their conjugates depends essentially only on K_ν. Thus we will have

$$|B_{sti}| < q^2 \mu \gamma_3 h \equiv \delta. \tag{2.12}$$

We now observe that there are $(2h+1)^{\nu q^2}$ choices of B_{kli} giving rise to sets of y_{st}, and there are only $(2\delta+1)^{\nu ST}$ choices of B_{sti} giving distinct sets of y_{st}. Hence two sets of y_{st} arising from two distinct sets of x_{kl} must agree if

$$(2h+1)^{\nu q^2} > (2\delta+1)^{\nu ST}, \tag{2.13}$$

or, since $\nu q^2 \geqq 2\nu ST$, if

$$(2h+1)^2 > 4h^2 + 1 > 2\delta + 1. \tag{2.14}$$

That is, if

$$4h^2 > 2h\gamma_3(\gamma_1 c)^{S+2q(T-1)}q^{S+1}, \tag{2.15}$$

or

$$2h > \gamma_3(\gamma_1 c)^{S+2q(T-1)}q^{S+1}. \tag{2.16}$$

But, on taking the difference of these two sets of x_{kl}, we will have a set of C_{kl} of the kind required by the lemma, with

$$\|C_{kl}\| < 2\|x_{kl}\| < 2\gamma_2 h. \tag{2.17}$$

That is, not all zero solutions exist with

$$\|C_{kl}\| < (\gamma_4 c)^{S+2qT}q^S. \tag{2.18}$$

This proves Lemma I.

The bound for the coefficients readily leads to a bound for the function, expressed in

LEMMA II.* *The $g_s(x)$ of Lemma* I *satisfies the inequality*

$$\log |g_s(x)| < \log \|C_{kl}\| + \gamma q |x| + (s+2)\log q + \gamma s. \tag{2.19}$$

This is proved directly from the definition of $g_s(x)$, which gives

$$|\bar{g}_s(x)| < \|C_{kl}\|\gamma_1^{s+2qx}q^{s+2}|\log \beta|^s. \tag{2.20}$$

A lower bound for the functions $g_s(t)$ is given by

LEMMA III.* *Except when $g_s(t)=0$, it satisfies the inequality*

$$\log |g_s(t)| \geqq -\gamma \log \|C_{kl}\| - \gamma(s+2)\log q - \gamma qt - \gamma s \log c - \gamma qt \log c. \tag{2.21}$$

This lemma is proved by consideration of the algebraic integer

$$c^{s+2qt}g_s(t)(\log \beta)^{-s} = \sum_{k=0}^{q-1}\sum_{l=0}^{q-1} C_{kl}c^{s+2qt}(k\eta + l)^s \alpha^{kt}\beta^{lt}. \tag{2.22}$$

As bounds for the conjugates of this number we have

$$\|c^{s+2qt}g_s(t)(\log \beta)^{-s}\| < q^2 c^{s+2qt}\|C_{kl}\|\gamma^{s+2qt}q^s. \tag{2.23}$$

Since the norm of an algebraic integer not zero must be at least unity in absolute value, it follows from this that, unless $g_s(t)=0$,

$$|g_s(t)c^{s+2qt}(\log \beta)^{-s}| \cdot \|c^{s+2qt}g_s(t)(\log \beta)^{-s}\|^{\nu-1} \geqq 1. \tag{2.24}$$

Hence we have

* Cf. Gelfond, loc. cit.

$$(2.25)\qquad |g_s(t)| \geqq \|C_{kl}\|^{-\nu+1}[c^{s+2qt}]^{-\nu}[\gamma^{s+2qt}]^{-\nu+1}[q^{s+2}]^{-\nu+1}|\log\beta|^{s\nu},$$

which establishes Lemma III.

Our next lemma concerns the polynomial of interpolation determined by its value and that of its derivatives at certain places. Specifically it reads:

LEMMA IV.* *If $S>T$, and $|P^s(t)|<M$, the polynomial of degree at most $ST-1$ determined by the ST values $P^s(t)$, when $|x|\geqq T$, satisfies the inequality*

$$(2.26)\qquad |P(x)| < M\exp[-ST\log T + 2ST + 2S\log S]\cdot|x(x-1)\cdots(x-T+1)|^S.$$

Let us first derive some formulas concerning the polynomial $P(x)$ of degree at most n, determined by the values of it and its first s_i-1 derivatives at the points a_i, where $\sum_{i=1}^{\tau}s_i=n+1$. We write

$$(2.27)\qquad \phi(x) = (x-a_1)^{s_1}(x-a_2)^{s_2}\cdots(x-a_\tau)^{s_\tau} = (x-a_i)^{s_i}/\psi_i(x).$$

Then

$$(2.28)\qquad P(x) = \sum_{i=1}^{\tau}\left[\frac{\sum_{k=0}^{s_i-1}A_k(x-a_i)^k}{(x-a_i)^{s_i}}\right]\phi(x)$$

is a polynomial of degree at most n. Also, for $x=a_i$, $k=0, 1, \cdots, s_i-1$, we find that

$$(2.29)\qquad \frac{d^k}{dx^k}\left[\frac{P(x)}{\phi(x)}\cdot(x-a_i)^{s_i}\right]_{a_i} = \frac{d^k}{dx^k}\left[\sum_{k=0}^{s_i-1}A_k(x-a_i)^k\right]_{a_i} = k!A_k.$$

Hence we have

$$(2.30)\qquad A_k k! = \frac{d^k}{dx^k}[P(x)\cdot\psi_i(x)]_{a_i} = \sum_{a=0}^{k}\frac{k!}{a!(k-a)!}P^a(a_i)\psi_i^{k-a}(a_i),$$

and

$$(2.31)\qquad P(x) = \left[\sum_{i=1}^{\tau}\sum_{k=0}^{s_i-1}\sum_{a=0}^{k}\frac{P^a(a_i)}{a!}\frac{\psi_i^{k-a}(a_i)}{(k-a)!}(x-a_i)^{k-s_i}\right]\phi(x).$$

But, if the prime indicates that $j\neq i$, $\psi_i(x)=\prod'(x-a_j)^{-s_j}$, and

$$(2.32)\qquad \begin{aligned}\frac{d^h}{dx^h}\psi_i(x) &= h!\sum_{\Sigma l_j=h}\prod'\frac{1}{l_j!}\frac{d^{l_j}}{dx^{l_j}}(x-a_j)^{-s_j}\\ &= h!(-1)^h\sum_{\Sigma l_j=h}\prod'\frac{(s_j+l_j-1)!}{l_j!(s_j-1)!}(x-a_j)^{-s_j-l_j}.\end{aligned}$$

* Cf. Ricci, loc. cit.; Hermite, Crelle's Journal, 1878, pp. 70–79.

Hence, putting $b=s_i-1-k$, a, b and the l_j all run from 0 to s_i-1, subject only to the condition that $a+b+\sum l_j=s_i-1$, and we have finally,

$$\frac{P(x)}{\phi(x)}=\sum_{i=1}^{\tau}\sum_{a+b+\Sigma l_j=s_i-1}(x-a_i)^{-b-1}\frac{P^a(a_i)}{a!}\cdot\prod{}'\frac{(s_j+l_j-1)!}{l_j!(s_j-1)!}(a_i-a_j)^{-s_j-l_j}(-1)^{\Sigma l_j}. \tag{2.33}$$

We now specialize the values by putting $a_1, a_2, \cdots, a_\tau=0, 1, 2, \cdots, T-1$ and $s_1=s_2=\cdots=s_\tau=S$. We also require $|x|\geqq T$. Then we have

$$|x-a_i|\geqq 1 \quad\text{and}\quad |(x-a_i)^{-b-1}|\leqq 1. \tag{2.34}$$

Again

$$a!\geqq 1 \quad\text{and}\quad \frac{1}{a!}\leqq 1. \tag{2.35}$$

Next we notice that

$$\prod{}'|a_i-a_j|^{-s_j-l_j}\leqq\prod{}'|a_i-a_j|^{-S}. \tag{2.36}$$

When T is odd, $T_1=2m+1$, the least $\prod'|a_i-a_j|$ is $m!\,m!$, but when T is even, $T_2=2m$, the least $\prod'|a_i-a_j|$ is $m!(m-1)!$. But*

$$\log x!=(x+\tfrac{1}{2})\log x-x+\log(2\pi)^{1/2}+\frac{\theta}{12x}\qquad(0<\theta<1), \tag{2.37}$$

or

$$\log x!=\log\frac{(x+1)!}{x+1}=(x+\tfrac{1}{2})\log(x+1)-x-1+\log(2\pi)^{1/2}+\frac{\theta}{12(x+1)}. \tag{2.38}$$

For the odd case, we obtain from the second expression (2.38):

$$\begin{aligned}2\log m!&=(2m+1)\log(2m+2)-(2m+1)\log 2-2m-2\\&\quad+2\log(2\pi)^{1/2}+\frac{\theta}{6(m+1)}>(2m+1)\log(2m+1)\\&\quad-4m-2\geqq T_1\log T_1-2T_1.\end{aligned} \tag{2.39}$$

For the even case, using both expressions (2.37) and (2.38):

* See, for example, Whittaker and Watson, *Modern Analysis*, 4th edition, Cambridge, 1927, p. 251 ff., §12.33.

$$\log m! + \log (m-1)! = 2m \log 2m - 2m \log 2 - 2m + 2 \log (2\pi)^{1/2} + \frac{\theta}{6m} > 2m \log 2m - 4m \geqq T_2 \log T_2 - 2T_2. \tag{2.40}$$

This shows that in all cases

$$\Pi' | a_i - a_j | > \exp [T \log T - 2T] \quad \text{and} \tag{2.41}$$
$$\Pi' | a_i - a_j |^{-S} < \exp [-ST \log T + 2ST].$$

Again, since after the first term the numerators are all less than $2S$, and the denominators are greater than or equal to 2, we have

$$\left| \frac{(s_j + l_j - 1)!}{l_j!(s_j - 1)!} \right| = \left| \frac{s_j}{1} \cdot \frac{s_j + 1}{2} \cdots \frac{s_j + l_j - 1}{l_j} \right| < S^{l_j}. \tag{2.42}$$

In consequence of this we have

$$\left| \Pi' \frac{(s_j + l_j - 1)!}{l_j!(s_j - 1)!} \right| < S^{\Sigma l_j} \leqq S^{S-1}. \tag{2.43}$$

To estimate the number of terms, we note that i has T values, while a, b and the $(T-1)$ l_j have at most S values. This gives TS^{T+1} as an upper limit to the number of terms. Consequently, by combining all our appraisals, we find that

$$\left| \frac{P(x)}{\phi(x)} \right| < [\max | P^a(a_i) |] \exp [-ST \log T + 2ST] TS^{S+T}. \tag{2.44}$$

Since $S > T$, and both are integers, $S \geqq T+1$, so that

$$TS^{S+T} < S^{S+T+1} \leqq S^{2S}. \tag{2.45}$$

Recalling that M is a bound for $|P^a(t)|$, or $|P^a(a_i)|$, we have

$$| P(x) | < M \exp [-ST \log T + 2ST + 2S \log S] \cdot | x(x-1) \cdots (x-T+1) |^S, \tag{2.46}$$

which is the inequality of the lemma.

Our final lemma is Darboux's mean value theorem* for functions of a complex variable.

LEMMA V. *For any function of a complex variable analytic on the straight line segment joining z_1 and z_2, we have*

$$\left| \frac{f(z_2) - f(z_1)}{z_2 - z_1} \right| \leqq \max | f'(z) |, \tag{2.47}$$

the maximum meaning maximum on the segment.

* G. Darboux, Liouville, (3), vol. 2 (1876), p. 291.

The lemma follows at once from the fact that

$$f(z_2) - f(z_1) = \int_{z_1}^{z_2} f'(z)dz, \tag{2.48}$$

so that

$$|f(z_2) - f(z_1)| \leqq \max |f'(z)| \cdot |z_2 - z_1|. \tag{2.49}$$

3. **The general theorem.** Our principal theorem may be formulated as follows:

THEOREM I. *Let α_i, β_i, and η_i, irrational (in particular $\neq 0$ or ∞), be three sequences of algebraic numbers in a fixed field, K_ν, with uniformly bounded conjugates. Let c_i be a sequence of integers, becoming infinite, such that $c_i\alpha_i$, $c_i\beta_i$, and $c_i\eta_i$ are algebraic integers. Then, if the three sequences approach limits A, B, both distinct from zero and unity, and H in such a way that*

$$|A - \alpha_i|, \quad |B - \beta_i|, \quad |H - \eta_i| < c_i^{-(\log c_i)^k}, \tag{3.1}$$

and

$$H = \frac{\log A}{\log B}, \tag{3.2}$$

it is impossible to have $k > 6$.

We shall prove the theorem by assuming it false and deducing a contradiction. For simplicity of writing, we shall generally omit the subscript i on the terms of the sequences. Consider then a particular set α, β, η, and c. Writing, as usual, $[x]$ to denote the greatest integer contained in x, put

$$q = [(\log c)^\kappa], \quad S = [q^{5/3}], \quad T = [\tfrac{1}{2}q^{1/3}], \quad \left(\frac{1}{\kappa} = \frac{1}{3} - 2\epsilon, \ \epsilon > 0\right). \tag{3.3}$$

Since these values are such that $q^2 > 2ST$, we may apply Lemma I to find a function

$$g(x) = \sum_{k=0}^{q-1} \sum_{l=0}^{q-1} C_{kl}\alpha^{kx}\beta^{lx} \tag{3.4}$$

with coefficients C_{kl} algebraic integers in K_ν, not all zero, and such that

$$g_s(t) = (\log \beta)^s \sum_{k=0}^{q-1} \sum_{l=0}^{q-1} C_{kl}(k\eta + l)^s\alpha^{kt}\beta^{lt} = 0 \tag{3.5}$$

for

$$s = 0, 1, 2, \cdots, S-1; t = 0, 1, 2, \cdots, T-1. \tag{3.6}$$

Furthermore, these coefficients will satisfy the inequality

$$\begin{aligned} \log \|C_{kl}\| &< S[\log(\gamma c) + \log q] + 2qT \log(\gamma c) \\ &< q^{5/3}[\log \gamma + (q+1)^{1/3-\epsilon} + \log q] \\ &\quad + 2q \cdot \tfrac{1}{2} q^{1/3} \cdot [\log \gamma + (q+1)^{1/3-\epsilon}] \end{aligned} \tag{3.7}$$

$$< q^{2-\epsilon} + o(q^{2-\epsilon}). \tag{3.8}$$

Here, as later, we write $o(\ )$ to mean terms of lower order in q than those written explicitly in the parentheses.

Next, using these same coefficients C_{kl}, form the function

$$f(x) = \sum_{k=0}^{q-1} \sum_{l=0}^{q-1} C_{kl} A^{kx} B^{lx}, \tag{3.9}$$

and construct

$$F(x) = f(x) - P(x), \tag{3.10}$$

with

$$F^s(t) = 0, \tag{3.11}$$

by subtracting off a polynomial with

$$P^s(t) = f^s(t). \tag{3.12}$$

We can show that this polynomial is small by Lemma IV, provided the $f^s(t)$ are small. These will be small if they approximate the $g_s(t)$, since the latter are zero. To guarantee this, we must at this point introduce an assumption as to the degree of approximation of α, β, and η to A, B, and H. Accordingly we put

$$\alpha - A = \Delta A, \quad \beta - B = \Delta B, \quad \eta - H = \Delta H, \tag{3.13}$$

and assume that

$$|\Delta A|, \ |\Delta B|, \text{ and } |\Delta H| \text{ are all } < e^{-q^{7/3}}. \tag{3.14}$$

As a first step in estimating the size of $f^s(t)$, we consider the crude estimate obtained from Lemma II, which shows that

$$\log |f^s(t)| < \log \|C_{kl}\| + \gamma q t + (s+2) \log q + \gamma s \tag{3.15}$$

$$< q^{2-\epsilon} + o(q^{2-\epsilon}). \tag{3.16}$$

Since we wish to compare the polynomial

$$f^s(t) = \sum\sum C_{kl}(kH + l)^s (\log B)^s A^{kt} B^{lt}, \tag{3.17}$$

with

$$G_s(t) = (\log B)^s g_s(t)(\log \beta)^{-s} = \sum\sum C_{kl}(k\eta + l)^s(\log B)^s \alpha^{kt}\beta^{lt}, \tag{3.18}$$

let us consider a general polynomial in three complex variables x, y, and z with $x_2 - x_1 = \Delta x$, $y_2 - y_1 = \Delta y$, $z_2 - z_1 = \Delta z$,

$$P(x, y, z) = \sum C_{pqr} x^p y^q z^r. \tag{3.19}$$

If C, P, Q, R, X, Y, and Z are upper bounds for C_{pqr}, p, q, r, x, y, and z respectively, and X, Y, and Z exceed unity, a crude estimate for the size of the polynomial, of the type just written for $f^s(t)$ is

$$U = PQRCX^P Y^Q Z^R. \tag{3.20}$$

But, on applying Lemma V to an individual term three times we find that

$$\begin{aligned} &| x_2^p y_2^q z_2^r - x_1^p y_1^q z_1^r | \\ &\quad < p\Delta X X^{p-1} Y^q Z^r + q\Delta Y X^p Y^{q-1} Z^r + r\Delta Z X^p Y^q Z^{r-1} \\ &\quad < X^P Y^Q Z^R (P\Delta X + Q\Delta Y + R\Delta Z). \end{aligned} \tag{3.21}$$

Consequently, we have

$$| P(x_2, y_2, z_2) - P(x_1, y_1, z_1) | < U(P\Delta X + Q\Delta Y + R\Delta Z). \tag{3.22}$$

By reasoning of the same kind, we find that

$$\begin{aligned} | G_s(t) - f^s(t) | &< U(qT\Delta A + qT\Delta B + S\Delta H) \\ &< \exp\left[q^{2-\epsilon} + o(q^{2-\epsilon})\right](q^{4/3} + q^{5/3}) e^{-q^{7/3}}, \end{aligned} \tag{3.23}$$

by making use of the value of U, our crude estimate for $f^s(t)$, (3.16), the values of S and T and our restriction on ΔA, ΔB, and ΔH, (3.14).

But, since $g_s(t) = 0$, we also have $G_s(t) = 0$, so that the inequality just proved amounts to

$$\log | f^s(t) | < - q^{7/3} + o(q^{7/3}), \tag{3.24}$$

which is the refined estimate for $f^s(t)$ we were seeking.

For the product which occurs in Lemma IV, since $T < q^{1/3}$, $S < q^{5/3}$, if we add the restriction $|x| \leqq q$, we have

$$| x(x-1) \cdots (x - T + 1) |^S < (2q)^{ST} < (2q)^{q^2} \leqq \exp q^2 \log 2q. \tag{3.25}$$

We may now apply Lemma IV, which, in view of the inequalities (3.24) and (3.25) gives

$$\begin{aligned} \log | P(x) | &< - q^{7/3} + o(q^{7/3}) + 2q^2 + 4q^2 \log^2 q + q^2 \log 2q \\ &< - q^{7/3} + o(q^{7/3}). \end{aligned} \tag{3.26}$$

While the lemma, as written, required $|x| \geqq T$, as soon as the product is replaced by a quantity independent of x, this restriction is no longer essential, from the maximum properties of polynomials. Thus the relation just found holds for all $|x| \leqq q$.

Let us next consider $F(x)$ for values of $|x| \leqq q$. For these values we find from Lemma II, (2.19) with $s=0$,

$$\begin{aligned} \log |f(x)| &< \log \|C_{kl}\| + \gamma_1 q |x| + 2 \log q \\ &< \gamma q^2 + o(q^2), \end{aligned} \tag{3.27}$$

in view of (3.8). It follows at once from (3.10), (3.26), and (3.27) that

$$\log |F(x)| < \gamma q^2 + o(q^2). \tag{3.28}$$

If we write

$$\phi(x) = [x(x-1)\cdots(x-T+1)]^S, \tag{3.29}$$

in consequence of the way in which we constructed $F(x)$, all the zeros of $\phi(x)$ are zeros of $F(x)$ to a multiplicity at least as high. Consequently the function

$$E(x) = \frac{F(x)}{\phi(x)} \tag{3.30}$$

is an entire function. We compare its value for a fixed x_1, and x_2 varying on a circle such that

$$|x_1| = q^{1-2\epsilon}, \qquad |x_2| = q \qquad (\epsilon > 0). \tag{3.31}$$

From the relation

$$|E(x_1)| \leqq \max |E(x_2)|, \tag{3.32}$$

we deduce

$$\begin{aligned} |F(x_1)| &\leqq \max |F(x_2)\phi(x_1)/\phi(x_2)| \\ &< \exp [\gamma q^2 + o(q^2)] \frac{(2q^{1-2\epsilon})^{ST}}{(q/2)^{ST}} \\ &< \exp [-\epsilon q^2 \log q + o(q^2 \log q)]. \end{aligned} \tag{3.33}$$

Since the estimate for $P(x)$ in (3.26) is smaller than this, from (3.10) follows

$$\log |f(x_1)| < -\epsilon q^2 \log q + o(q^2 \log q). \tag{3.34}$$

We proceed from this to the derivatives by the Cauchy integral formula

$$f^s(t_1) = \frac{s!}{2\pi i} \int \frac{f(x)dx}{(x-t_1)^{s+1}}, \tag{3.35}$$

where the path of integration may be taken as a circle about the origin of radius $|x_1| = q^{1-2\epsilon}$. We find from this, (3.3), and (3.34) that

$$(3.36)\quad |f^s(t_1)| < s^s |f(x_1)| < (q^{5/3})^{q^{5/3}} \exp\left[-\epsilon q^2 \log q + o(q^2 \log q)\right],$$

if $|t_1| < |x_1| - 1$. This will be true, for q large enough, if we require $|t_1| < q^{2/3}$. We are particularly interested in the values

$$(3.37)\qquad t_1 = 0, 1, 2, \cdots, T_1 - 1;\ T_1 = [q^{2/3-\epsilon}].$$

For these values, we have

$$(3.38)\qquad \log |f^s(t_1)| < -\epsilon q^2 \log q + o(q^2 \log q).$$

To go from this to $g_s(t_1)$ we must again use Lemma V. We begin by noting that the new value of T_1 has no effect on the crude estimate, (3.16), so that we still have for the value of U

$$(3.39)\qquad \log |f^s(t_1)| < \log U \leqq 2q^2 + o(q^2).$$

Consequently, we now have in place of (3.23)

$$(3.40)\quad \begin{aligned} |G_s(t_1) - f^s(t_1)| &< U(qT_1\Delta A + qT_1\Delta B + S\Delta H) \\ &< \exp\left[2q^2 + o(q^2)\right](2q^{5/3} + q^{5/3})e^{-q^{7/3}}. \end{aligned}$$

Thus the analogue of (3.24) is now

$$(3.41)\qquad \log |G_s(t_1) - f^s(t_1)| < -q^{7/3} + o(q^{7/3}).$$

From this, and from (3.38), we find

$$(3.42)\qquad \log |G_s(t_1)| < -\epsilon q^2 \log q + o(q^2 \log q).$$

Since

$$(3.43)\quad \log |g_s(t_1)| = \log |G_s(t_1)| + s \log |\log \beta| - s \log |\log B|,$$

we also have

$$(3.44)\qquad \log |g_s(t_1)| < -\epsilon q^2 \log q + o(q^2 \log q).$$

But, by Lemma III, unless $g_s(t_1) = 0$, it satisfies the inequality

$$(3.45)\quad \begin{aligned} \log |g_s(t_1)| > &-\gamma \log \|C_{kl}\| - \gamma(s+2) \log q - \gamma q t_1 \\ &- \gamma s \log c - \gamma q t_1 \log c, \end{aligned}$$

so that, in view of (3.8), (3.3), and (3.37) we have

$$(3.46)\quad \begin{aligned} \log |g_s(t_1)| &> -2\gamma q^2 - \gamma q^{5/3} \log q - \gamma q^{5/3} - \gamma q^2 - \gamma q^2 - o(q^2) \\ &> -\gamma q^2 - o(q^2). \end{aligned}$$

Since for sufficiently large q this is in contradiction with (3.44), by Lemma III, it proves that

$$g_s(t_1) = 0, \text{ for the } t_1 \text{ of } (3.37). \tag{3.47}$$

We may now repeat our procedure, making use of the additional zeros whose existence has just been established. That is, we construct

$$F_1(x) = f(x) - P_1(x), \tag{3.48}$$

with

$$F_1^s(t_1) = 0, \tag{3.49}$$

by subtracting off a polynomial with

$$P_1^s(t_1) = f^s(t_1). \tag{3.50}$$

As before, we wish to estimate the size of this polynomial by Lemma IV. We begin by using the vanishing of $g_s(t_1)$, and hence $G_s(t_1)$, in conjunction with Lemma V to get an estimate for $f^s(t_1)$. The application of Lemma V has already been made in (3.40), so that we have merely to set $G_s(t_1) = 0$ in (3.41) to obtain

$$\log |f^s(t_1)| < -q^{7/3} + o(q^{7/3}). \tag{3.51}$$

We must next consider the product, and find as in (3.25) that

$$\begin{aligned} |x(x-1)\cdots(x-T_1+1)|^S &< (2q)^{ST_1} < (2q)^{7/3-\epsilon} \\ &< \exp[q^{7/3-\epsilon}\log 2q]. \end{aligned} \tag{3.52}$$

We are now in a position to apply Lemma IV, and find from it, using (3.51) and (3.52),

$$\begin{aligned} \log |P_1(x)| &< -q^{7/3} + o(q^{7/3}) + 2q^{7/3-\epsilon} \\ &\quad + 4q^2 \log q + q^{7/3-\epsilon}\log 2q \\ &< -q^{7/3} + o(q^{7/3}). \end{aligned} \tag{3.53}$$

From this, (3.27), and (3.48), we conclude that for $|x| \leqq q$,

$$\log |F_1(x)| < \gamma q^2 + o(q^2). \tag{3.54}$$

If we write

$$\phi_1(x) = [x(x-1)\cdots(x-T_1+1)]^S. \tag{3.55}$$

we may now form the entire function

$$E_1(x) = \frac{F_1(x)}{\phi_1(x)}. \tag{3.56}$$

We compare its values for a fixed x_1, and x_2 varying on a circle where

$$|x_1| = q^{1-\epsilon}, \quad |x_2| = q, \quad (\epsilon > 0). \tag{3.57}$$

From

$$|E_1(x)| \leqq \max |E_1(x_2)|, \tag{3.58}$$

we deduce

$$\begin{aligned} |F_1(x_1)| &\leqq \max |F_1(x_2)\phi(x_1)/\phi_1(x_2)| \\ &\leqq \exp [\gamma q^2 + o(q^2)] \frac{(2q^{1-\epsilon})^{ST_1}}{(q/2)^{ST_1}} \\ &< \exp [-\epsilon q^{7/3-\epsilon} \log q + o(q^{7/3-\epsilon} \log q)]. \end{aligned} \tag{3.59}$$

As this exceeds the estimate for $P_1(x)$ in (3.53), it follows from (3.48) that

$$\log |f(x_1)| < -\epsilon q^{7/3-\epsilon} \log q + o(q^{7/3-\epsilon} \log q). \tag{3.60}$$

We use this to estimate the derivatives at the origin. We have from Cauchy's integral formula

$$f^s(0) = \frac{s!}{2\pi i} \int \frac{f(x)dx}{x^{s+1}}, \tag{3.61}$$

where the path of integration may be taken as a circle of radius $|x_1| = q^{1-\epsilon}$ with center at the origin. We find from this

$$|f^s(0)| < s^s |f(x_1)|. \tag{3.62}$$

If we take

$$s_1 = 0, 1, 2, \cdots, S_1 - 1, \quad S_1 = [q^2 \log q], \tag{3.63}$$

we have

$$\log (s_1^{s_1}) < S_1 \log S_1 < q^2 \log q(2 \log q + \log \log q). \tag{3.64}$$

Combining (3.60), (3.62), and (3.64) we have

$$\log |f^{s_1}(0)| < -\epsilon q^{7/3-\epsilon} \log q + o(q^{7/3-\epsilon} \log q). \tag{3.65}$$

We next apply Lemma V in the usual way to go from this to $g_{s_1}(0)$. We first get a value to serve as U from Lemma II, namely:

$$\log |g_{s_1}(0)| < q^2(\log q)^2 + o(q^2\{\log q\}^2). \tag{3.66}$$

We then use this, (3.43), (3.14), and the reasoning used for (3.23) to derive

$$\log |G_{s_1}(0) - f^{s_1}(0)| < -q^{7/3} + o(q^{7/3}). \tag{3.67}$$

From this, (3.43) and (3.65), we find that

$$\log \left| g_{s_1}(0) \right| < -\epsilon q^{7/3-\epsilon} \log q + o(q^{7/3-\epsilon} \log q). \tag{3.68}$$

But, by Lemma III, unless $g_{s_1}(0) = 0$, it satisfies the inequality

$$\log \left| g_{s_1}(0) \right| > -\gamma q^{7/3-2\epsilon} \log q + o(q^{7/3-2\epsilon} \log q). \tag{3.69}$$

Since this is in contradiction with (3.68), when q is sufficiently large, it follows from Lemma III that

$$g_{s_1}(0) = 0. \tag{3.70}$$

But from the expression for S_1 in (3.63), we see that for q sufficiently large, S_1 exceeds q^2. For such a value of q, we may regard the equations

$$g_{s_1}(0) = 0 \tag{3.71}$$

as a system of linear equations in the q^2 variables,

$$u_{kl} = \alpha^{k0}\beta^{l0} = 1. \tag{3.72}$$

Eventually, $\alpha \neq 0$, $\beta \neq 0$, since otherwise $H = \infty$ excluded, or $H = 0$ excluded by the conditions on A and B. Some of the coefficients C_{kl} may be zero. In this case we omit the corresponding terms and variables u_{kl}. The coefficients C_{kl} are known to be not all zero. If there are Q_1 distinct from zero, the values

$$s = 0, 1, 2, \cdots, Q_1 - 1 \tag{3.73}$$

gives a system of Q_1 equations of the form

$$\sum \sum C_{kl} v_{kl}^{s} u_{kl} = 0, \tag{3.74}$$

where

$$v_{kl} = (k\eta + l). \tag{3.75}$$

Since the u_{kl} are not all zero, the determinant of their coefficients must vanish. On factoring out the C_{kl}, none of which vanish, there is left a Vandermonde determinant, which reduces to

$$\prod_{i,j} (v_{k_i l_i} - v_{k_j l_j}) = 0. \tag{3.76}$$

Thus, for some pair of subscripts $(k_i l_i)$, $(k_j l_j)$ distinct from one another, we must have a vanishing factor,

$$k_i\eta + l_i - (k_j\eta + l_j) = 0. \tag{3.77}$$

This is impossible, since the coefficients are integers, and we have expressly ruled out the possibility $\eta = 0$, ∞, or a rational number.

This contradiction has been derived from the assumptions that

$$(3.78)\quad q = [(\log c)^k],\ \frac{1}{k} = \frac{1}{3} - 2\epsilon,\ \text{and}\ |\Delta A|,\ |\Delta B|,\ |\Delta H| < e^{-q^{7/3}},$$

for indefinitely large q. The assumption of the theorem was that

$$(3.79)\quad |\Delta A|,\ |\Delta B|,\ |\Delta H| < c^{-(\log c)^k},\qquad k > 6.$$

If we put

$$(3.80)\quad k = 6 + \eta,$$

and define

$$(3.81)\quad q = [(\log c)^\kappa],\ \frac{1}{\kappa} = \frac{1}{3} - 2\epsilon,\ \text{so that}\ \log c \geqq q^{1/\kappa},$$

it follows that

$$(3.82)\quad \log \{c^{-(\log c)^k}\} = -(\log c)^{7+\eta} \leqq -q^{(7+\eta)(1/3-2\epsilon)} < -q^{7/3}$$

provided that ϵ is sufficiently small. That is, when (3.79) hold for a particular k, a suitable ϵ and q can be found such that the proof as given leads to a contradiction. This proves the theorem.

COROLLARY. *If any two of the three numbers A, B, and H satisfy the hypothesis of Theorem* I, *i.e., the statement as far as* (3.1), *and $k>6$, then the third number given by*

$$(3.83)\quad H = \frac{\log A}{\log B},\qquad A = B^H$$

is necessarily transcendental.

In particular, we note that one or both of the two numbers we started with may be actually algebraic.

4. **The first special theorem.** If some of our quantities are actually algebraic, instead of merely approximable by algebraic numbers, a slightly weaker condition suffices. Thus we have

THEOREM II. *Let α and β be fixed algebraic numbers both distinct from zero and unity. Let η_i, irrational (in particular not 0 or ∞), be a sequence of algebraic numbers in a fixed field K_r with uniformly bounded conjugates. Let c_i be a sequence of integers, becoming infinite, such that c_i, η_i are algebraic integers. Then, if the sequence approaches a limit H in such a way that*

$$(4.1)\quad |H - \eta_i| < c_i^{-(\log c_i)^k}$$

and

$$H = \frac{\log \alpha}{\log \beta}, \tag{4.2}$$

it is impossible to have $k > 4$.

We shall prove this theorem by the same method as that used for Theorem I. To facilitate reference to the earlier argument, we shall use similar numbers for corresponding equations. Thus these do not run consecutively. The omitted numbers correspond to such equations in §3 as we refer to, without repeating explicitly. Let then η and c be particular values, and put

$$q = [(\log c)^k], \quad S = [q^{3/2}], \quad T = [\tfrac{1}{2}q^{1/2}] \qquad \left(\frac{1}{k} = \frac{1}{2} - 4\epsilon, \epsilon > 0\right). \tag{4.3}$$

Apply Lemma I to find a function $g(x)$, (3.4), with coefficients C_{kl} algebraic integers in K_ν, not all zero, and such that $g_s(t) = 0$, (3.5) and (3.6). Furthermore, the coefficients will now satisfy the inequality

$$\log \|C_{kl}\| < S[\log(\gamma c) + \log q] + 2qT \log \gamma. \tag{4.7}$$

To justify the omission of the term $2qT \log c$, we turn back to the proof of Lemma I, and note that this term entered because of the term $c^{2q(T-1)}$ in (2.6), necessitated by the multiplier which made $\alpha^{k(T-1)}$ and $\beta^{k(T-1)}$ algebraic integers. As the α and β are here fixed, so is the multiplier for them, and we may replace $c^{2q(T-1)}$ by $\gamma^{2q(T-1)}$, which may be incorporated with a term already present in (2.6). From (4.7) and (4.3) we conclude that

$$\log \|C_{kl}\| < q^2 + o(q^2). \tag{4.8}$$

We next work with $f(x)$, which is here

$$f(x) = g(x) = \sum_{k=0}^{q-1} \sum_{l=0}^{q-1} C_{kl} \alpha^{kx} \beta^{lx}. \tag{4.9}$$

However, since the derivatives $f^s(x)$ are different from $g_s(x)$, we must again construct $F(x)$ by using (3.10), (3.11), and (3.12).

With regard to the degree of approximation, we assume that for the ΔH defined in (3.13) we have

$$|\Delta H| < e^{-q^{k/2}}. \tag{4.14}$$

The application of Lemma II to obtain a crude estimate proceeds as before. From (3.15), (4.3), and (4.8) we have

$$\log |f^s(t)| < \log U \leqq q^2 + o(q^2). \tag{4.16}$$

For the comparison of

$$f^s(t) = \sum\sum C_{kl}(kH + l)^s(\log \beta)^s \alpha^{kt}\beta^{lt}, \tag{4.17}$$

with

$$G_s(t) = g_s(t) = \sum\sum C_{kl}(k\eta + l)^s \alpha^{kt}\beta^{lt}, \tag{4.18}$$

we use (3.22) to deduce

$$| g_s(t) - f^s(t) | < US\Delta H < \exp [q^2 + o(q^2)]q^{3/2}e^{-q^{5/2}}, \tag{4.23}$$

from which we obtain the refined estimate

$$\log | f^s(t) | < - q^{5/2} + o(q^{5/2}). \tag{4.24}$$

Since the values in (4.3) satisfy $ST < q^2$, the inequality (3.25) holds here, and we may combine it with (4.24) and Lemma IV to deduce

$$\log | P(x) | < - q^{5/2} + o(q^{5/2}) \text{ for } | x | \leqq q. \tag{4.26}$$

To proceed to $F(x)$, we first use (4.8) and Lemma II to deduce (3.27). Then this, (3.10), and (4.26) may be combined to give (3.28). From this, as before, (3.33) and, in view of (3.10) and (4.26), (3.34) follow. We then focus our attention on the values

$$t_1 = 0, 1, 2, \cdots, T_1 - 1; \; T_1 = [q^{1-3\epsilon}]. \tag{4.37}$$

For these values, using (3.35), we readily deduce (3.38). Then, with only slight modifications of the previous reasoning, we conclude that (3.44) holds.

We then apply Lemma III, which here becomes

If not 0, $g_s(t_1)$ *satisfies the inequality*

$$\log | g_s(t_1) | > - \gamma \log \|C_{kl}\| - \gamma(s + 2) \log q - \gamma q t_1 - \gamma s \log c. \tag{4.45}$$

We justify the omission of the term $-\gamma q t_1 \log c$ by examining the proof of Lemma III. This term arises from c^{2qt} which first appears in (2.22). Here, since α and β are fixed, we may take a fixed multiplier for them, and so use γ^{2qt}, which in the final result replaces $-\gamma q t_1 \log c$ by $-\gamma q t_1$, which in turn may be incorporated with the term of this type already present.

From (4.45), in conjunction with (4.8), (4.3), and (4.37) we may conclude that (3.46) holds. As before, the contradiction of this and (3.44) proves that

$$g_s(t_1) = 0, \text{ for the } t_1 \text{ of (4.37)}. \tag{4.47}$$

We now come to the second application of the process. We construct $F_1(x)$, defined by (3.48), to satisfy (3.49) and (3.50). In place of (3.51), we now have by similar reasoning

$$\log | f^s(t_1) | < - q^{5/2} + o(q^{5/2}). \tag{4.51}$$

For the product our new values give

$$(4.52) \qquad |x(x-1)\cdots(x-T_1+1)|^S < (2q)^{ST_1} < \exp\left[q^{5/2-3\epsilon}\log 2q\right].$$

These last two inequalities enable us to apply Lemma IV to obtain

$$(4.53) \qquad \log|P_1(x)| < -q^{5/2}+o(q^{5/2}).$$

From this, (3.27), and (3.48), we see that (3.54) again holds. The argument based on the zeros leads from this to the inequality

$$(4.59) \qquad \log|F_1(x_1)| < -\epsilon q^{5/2-3\epsilon}\log q + o(q^{5/2-3\epsilon}\log q),$$

and hence to

$$(4.60) \qquad \log|f(x_1)| < -\epsilon q^{5/2-3\epsilon}\log q + o(q^{5/2-3\epsilon}\log q).$$

Then using (3.61) and the argument which follows with the value of S_1 again given by (3.63), we find that

$$(4.65) \qquad \log|f^{s_1}(0)| < -\epsilon q^{5/2-3\epsilon}\log q + o(q^{5/2-3\epsilon}\log q).$$

Since (3.66) may be again used, and

$$(4.67) \qquad \log|g_{s_1}(0)-f^{s_1}(0)| < -q^{5/2}+o(q^{5/2}),$$

we may derive

$$(4.68) \qquad \log|g_{s_1}(0)| < -\epsilon q^{5/2-3\epsilon}\log q + o(q^{5/2-3\epsilon}\log q).$$

But the modified inequality of Lemma III given in (4.45) shows that

$$(4.69) \qquad \log|g_{s_1}(0)| > -\gamma q^{5/2-4\epsilon}\log q + o(q^{5/2-4\epsilon}),$$

unless $g_{s_1}(0)=0$. Since the last two inequalities are contradictory, we must have the second possibility, (3.70). From here on the earlier argument applies without change, and shows that the assumptions

$$(4.78) \qquad q=[(\log c)^k], \quad \frac{1}{k}=\frac{1}{2}-4\epsilon, \quad \text{and} \quad |\Delta H| < e^{-q^{5/2}},$$

for indefinitely large q, lead to a contradiction.

The assumption of Theorem II was that

$$(4.79) \qquad |\Delta H| < c^{-(\log c)^k}, \qquad k>4.$$

If we put

$$(4.80) \qquad k=4+\eta,$$

and define

$$(4.81) \qquad q = [(\log c)^k], \quad \frac{1}{k} = \frac{1}{2} - 4\epsilon, \quad \text{so that } \log c \geqq q^{1/k},$$

it follows that

$$(4.82) \qquad \log \left\{c^{-(\log c)^k}\right\} = -(\log c)^{5+\eta} \leqq -q^{(5+\eta)(1/2-4\epsilon)} < -q^{5/2},$$

provided that ϵ is sufficiently small. That is, when (4.79) holds for a particular k, a suitable ϵ and q can be found such that proof leads to a contradiction. This proves Theorem II.

COROLLARY. *If β and H satisfy the hypothesis of Theorem* II, *i.e., the statement as far as* (4.1) *and $k>4$, then the number given by*

$$(4.83) \qquad \alpha = \beta^H$$

is necessarily transcendental.

5. **The second special theorem.** We next consider the case where the exponent is actually an algebraic number. The theorem here is

THEOREM III. *Let η be a fixed irrational algebraic number (in particular distinct from 0 and ∞). Let α_i and β_i be two sequences of algebraic numbers in a fixed field K_ν, with uniformly bounded conjugates. Let c_i be a sequence of integers, becoming infinite, such that $c_i\alpha_i$, $c_i\beta_i$ are algebraic integers. Then, if these sequences approach limits A and B, each distinct from zero and unity, in such a way that*

$$(5.1) \qquad |A - \alpha_i|, \ |B - \beta_i| < c_i^{-(\log c_i)^k}$$

and

$$(5.2) \qquad \eta = \frac{\log A}{\log B},$$

it is impossible to have $k>1$.

We again use the form of exposition of the preceding section. Let, then α, β, and c be particular values, and put

$$(5.3) \qquad q = [(\log c)^k], \quad S = [q^{2-\epsilon}], \quad T = [\tfrac{1}{2}q^\epsilon], \quad \left(\frac{1}{k} = 1 - 2\epsilon, \epsilon > 0\right).$$

Then apply Lemma I to find a function $g(x)$ of (3.4), for which (3.5) and (3.6) hold. The inequality satisfied by the coefficients in this case may be written

$$(5.7) \qquad \log \|C_{kl}\| < S[\log \gamma + \log q] + 2qT \log (\gamma c).$$

Here we have omitted the term $S \log c$, since this comes from the term c^S in (2.6). As η is now fixed, we may take a fixed multiplier for it and so replace the term c^S by γ^S, which may be combined with the term of this type already present in (2.6).

It follows from (5.7) and (5.3) that

$$\log \|C_{kl}\| < q^2 + o(q^2). \tag{5.8}$$

We now form the function $f(x)$ of (3.9), and the $F(x)$ of (3.10), (3.11), and (3.12). This time we take as our assumption on the degree of approximation the following restriction on the size of the ΔA and ΔB defined by (3.13)

$$|\Delta A|, \quad |\Delta B| < e^{-q^{2+2\epsilon}}. \tag{5.14}$$

The crude estimate is obtained as before by applying Lemma II. Thus from (3.15), (5.3), and (5.8) we deduce

$$\log |f^s(t)| < \log U \leqq q^2 + o(q^2). \tag{5.16}$$

For the comparison of

$$f^s(t) = \sum \sum C_{kl}(k\eta + l)^s (\log B)^s A^{kt} B^{lt}, \tag{5.17}$$

with the $G_s(t)$ of (3.18), we use (3.22) to deduce

$$|G_s(t) - f^s(t)| < U(qT\Delta A + qT\Delta B) < e^{q^2 + o(q^2)} q^s e^{-q^{2+2\epsilon}}. \tag{5.23}$$

This leads to the refined estimate

$$\log |f^s(t)| < -q^{2+2\epsilon} + o(q^{2+2\epsilon}). \tag{5.24}$$

Since the values in (5.3) satisfy $ST < q^2$, the inequality (3.25) holds here, and may be combined with (5.24) and Lemma IV to yield

$$\log |P(x)| < -q^{2+2\epsilon} + o(q^{2+2\epsilon}), \text{ for } |x| \leqq q. \tag{5.26}$$

We proceed to $F(x)$, by first using (5.8) and Lemma II to deduce (3.27), and then combining this with (3.10) and (5.26) to give (3.28). From this, as before (3.33) and, in view of (3.10) and (5.26), (3.34) follow.

We then consider the particular values

$$t_1 = 0, 1, 2, \cdots, T_1 - 1; \; T_1 = [q^{2\epsilon}]. \tag{5.37}$$

For these values we may deduce (3.38) by using (3.35). Then, making a few obvious modifications in the earlier argument, we show that (3.44) holds.

Then we apply Lemma III. The inequality here becomes

$$\log |g_s(t)| > -\gamma \log \|C_{kl}\| - \gamma(s+2) \log q - \gamma q t_1 - \gamma s - q t_1 \log c. \tag{5.45}$$

The only modification is the replacement of $-\gamma s \log c$ by the term $-\gamma s$.

This is permissible, since the multiplier for η is now fixed, and so the factor c^s of (2.22) may be replaced by γ^s. From (5.45), in conjunction with (5.8), (5.3), and (5.37) we may conclude that (3.46) holds. As before, the contradiction of this and (3.44) proves that

$$g_s(t_1) = 0, \text{ for the } t_1 \text{ of (5.37)}. \tag{5.47}$$

We are now at that stage of the proof where our process is repeated. We construct $F_1(x)$, defined by (3.48), to satisfy (3.49) and (3.50). In place of (3.51), we now have, similarly,

$$\log |f^s(t_1)| < -q^{2+2\epsilon} + o(q^{2+2\epsilon}). \tag{5.51}$$

For the product of our new values gives

$$|x(x-1)\cdots(x-T_1+1)|^S < (2q)^{ST_1} < \exp[q^{2+\epsilon}\log 2q]. \tag{5.52}$$

These last two inequalities enable us to apply Lemma IV, and so obtain

$$\log |P_1(x)| < -q^{2+2\epsilon} + o(q^{2+2\epsilon}). \tag{5.53}$$

From this, (3.27), and (3.48), we see that (3.54) again holds. The argument on entire functions leads from this to

$$\log |F_1(x_1)| < -\epsilon q^{2+\epsilon}\log q + o(q^{2+\epsilon}\log q), \tag{5.59}$$

and hence to

$$\log |f(x_1)| < -q^{2+\epsilon}\log q + o(q^{2+\epsilon}\log q). \tag{5.60}$$

Then using (3.61) and the argument which follows with the value of S_1 again given by (3.63), we find that

$$\log |f^{s_1}(0)| < -\epsilon q^{2+\epsilon}\log q + o(q^{2+\epsilon}\log q). \tag{5.65}$$

Since we may again use (3.66), and

$$\log |G_{s_1}(0) - f^{s_1}(0)| < -q^{2+2\epsilon} + o(q^{2+2\epsilon}), \tag{5.67}$$

we may, using (3.43), derive the inequality

$$\log |g_{s_1}(0)| < -\epsilon q^{2+\epsilon}\log q + o(q^{2+\epsilon}\log q). \tag{5.68}$$

But the modified inequality of Lemma III given in (5.45) shows that

$$\log |g_{s_1}(0)| > -q^2(\log q)^2 + o(q^2\{\log q\}^2), \tag{5.69}$$

unless $g_{s_1}(0) = 0$. Since the last two inequalities contradict one another, we must have the second alternative, (3.70). From here on the earlier argument applies with no change, and shows that the assumptions

$$q = [(\log c)^k], \quad |\Delta A| \text{ and } |\Delta B| < e^{-q^{2+2\epsilon}}, \tag{5.78}$$

for indefinitely large q, lead to a contradiction.

The assumption of Theorem III was that

$$|\Delta A|,\ |\Delta B| < c^{-(\log c)^k},\quad k > 1. \tag{5.79}$$

If we put

$$k = 1 + \eta, \tag{5.80}$$

and define

$$q = [(\log c)^k],\quad \frac{1}{k} = 1 - 2\epsilon,\quad \text{so that } \log c \geqq q^{1/k}, \tag{5.81}$$

it follows that

$$\log (c^{-(\log c)^k}) = -(\log c)^{2+\eta} \leqq -q^{(2+\eta)(1-2\epsilon)} < -q^{2+2\epsilon} \tag{5.82}$$

provided that ϵ is sufficiently small. That is, when (5.79) holds for a particular k, a suitable ϵ and q can be found such that the proof leads to a contradiction. This proves Theorem III.

COROLLARY. *If η and B satisfy the hypothesis of Theorem* III, *i.e., the statement as far as* (5.2) *and $k>1$, then the number given by*

$$A = B^{\eta} \tag{5.83}$$

is necessarily transcendental.

6. **Generalized Liouville numbers.** If the numbers approximable by algebraic numbers used in Theorems I, II, and III were algebraic, our theorems would merely be complicated restatements of that due to Gelfond and Schneider. That this is never the case, follows from the theorems of this section.

THEOREM IV. *Let α_i be a sequence of algebraic numbers all belonging to a fixed field K_ν, and with their conjugates uniformly bounded. Let an infinite number of them be distinct from one another. Let c_i and λ_i be two sequences of integers becoming infinite, such that the quantities $c_i\alpha_i$ are algebraic integers, and the numbers α_i approach a limit A in such a way that*

$$|\alpha_i - A| < \frac{1}{c_i^{\lambda_i}}. \tag{6.1}$$

Then A is a transcendental number.

For, suppose A were an algebraic number, and that

$$P(x) \equiv p_n x^n + p_{n-1}x^{n-1} + \cdots + p_1 x + p_0 = 0 \tag{6.2}$$

were the irreducible equation it satisfied. Let V be the bound for $\|\alpha_i\|$ and $|A|$, and put $\sum_{i=0}^{n} p_i = s$. Write α for a particular α_i. Then, by Lemma V, we have

$$\left| \frac{P(A) - P(\alpha)}{A - \alpha} \right| \leqq P'(\xi) < nsV, \tag{6.3}$$

so that, since $P(A) = 0$, we have

$$|P(\alpha)| < |\alpha - A| \, nsV^{n-1}. \tag{6.4}$$

We next notice that

$$\|c^n P(\alpha)\| < c^n sV^n, \tag{6.5}$$

and since the norm of an algebraic integer not zero is numerically at least unity,

$$1 \leqq |c^n P(\alpha)| \, (\|c^n P(\alpha)\|)^{\nu-1} \leqq c^{n\nu} s^{(\nu-1)} V^{n(\nu-1)} |P(\alpha)|. \tag{6.6}$$

The possibility that $P(\alpha) = 0$ offers no difficulty, since we have assumed that there are an infinite number of distinct α_i. Consequently, we may omit those equal to A or to one of its conjugates, and still have an infinite sequence left to use in the proof.

Now from (6.4) and (6.6) we have

$$|\alpha - A| > \frac{1}{c^{n\nu} n s^{\nu} V^{n\nu-1}} > \frac{1}{\gamma_1 c^{\gamma_2}}. \tag{6.7}$$

In this relation α and A are two distinct non-conjugate algebraic numbers, and the γ depend only on A, the algebraic field K_ν containing α and the bounds for $\|\alpha\|$.

From this last result, Theorem IV follows at once, since (6.7) contradicts (6.1), if

$$c_i > \gamma_1 \quad \text{and} \quad \lambda_i > \gamma_2 + 1. \tag{6.8}$$

Like their subclass, the ordinary Liouville numbers, the generalized Liouville numbers do not constitute a large fraction of the transcendental. In fact we shall establish

THEOREM V. *The generalized Liouville numbers correspond to a set of points of zero measure in the complex plane, and those on the real axis approximable by real algebraic numbers have zero measure on the line.*

First, consider a fixed field K_ν, and the algebraic numbers in it, α, such that

$$\|\alpha\| < V. \tag{6.9}$$

If c is the smallest integer for which $c\alpha$ is an algebraic integer, and we write, as in Lemma I,

$$c\alpha = \sum_{j=1}^{\nu} B_j \rho_j, \tag{6.10}$$

it follows from $\|c\alpha\| < \|cV\|$ that

$$|B_i| < \gamma cV. \tag{6.11}$$

Consequently, in the field K_ν, subject to the bound V, the number of α is at most

$$(2\gamma_1 cV + 1)^\nu < \gamma_2 c^\nu, \text{ for a particular } c. \tag{6.12}$$

Next select a sequence of positive numbers ϵ_m, decreasing to zero, and a sequence of positive integers μ_m increasing to infinity, as m increases. Then for each value of α_i, whose multiplier is c_i we draw a circle about its representative point of radius ϵ_m/c_i^m. Call S_m the set of points inside any of these circles, and define the set T by

$$T = S_1 \cdot S_2 \cdot \cdots \cdot S_m \cdot \cdots . \tag{6.13}$$

For any m, the generalized transcendental number A, with bound V, satisfying (6.1) will be in S_m. For, since c_i and λ_i are becoming infinite, for a sufficiently large i, we shall have

$$c_i^{\lambda_i - m} > \frac{1}{\epsilon_m}, \text{ and hence } \frac{1}{c_i^{\lambda_i}} < \frac{\epsilon_m}{c_i^m}. \tag{6.14}$$

Thus T includes all such numbers A, related to K and V.

But the set T has a measure not exceeding that of S_m, and this last, by (6.12) and the definition of S_m, is at most

$$\sum_{c_i=1}^{\infty} \gamma_2 c_i^\nu \pi \left(\frac{\epsilon_m}{c_i^m}\right)^2 = \gamma_2 \pi \epsilon_m^2 \sum c_i^{\nu - 2m}. \tag{6.15}$$

When $2m$ exceeds $\nu+2$, the series converges to a sum <2, so that the measure of S_m approaches zero with ϵ_m. Thus the measure of T is zero. Let V be a positive integer. Then there are only an enumerable number of choices for K_ν and V. Hence each generalized Liouville number is included in one of an enumerable number of sets, each of zero measure. This proves the first part of Theorem V.

The statement about the one-dimensional measure of the points on the real axis approximable by real algebraic numbers is proved similarly by en-

closing them in intervals. In this case the series corresponding to that in (6.15) is $\sum c^{\nu-m}$, which converges when m exceeds $\nu+2$.

In view of our general theorems, the measure of the B^A is of interest. We have

THEOREM VI. *The set of numbers B^A, where A and B are generalized Liouville numbers with sequences α_i, β_i having a common c_i is of measure zero in the complex plane. The subset with α_i, β_i, $\beta_i^{\alpha_i}$ real has zero measure on the real axis.*

We begin by finding the relation between the degree of the approximation of A and B to that of the power. We have

$$B^A - \beta^\alpha = B^A - \beta^A + \beta^A - \beta^\alpha, \quad \text{and} \tag{6.16}$$

$$|B^A - \beta^A| \leqq \max |Az^{A-1}| \cdot |\Delta B|, \tag{6.17}$$

by Lemma V, where $\Delta B = \beta - B$.

If we restrict $|A|$, $|B|$ and $|\alpha_i|$, $|\beta_i|$ as well as their reciprocals, and $|\log B|$, $|\log \beta_i|$, to be all less than M, necessarily >1, we have from this

$$|B^A - \beta^A| < M^{M+2} |\Delta B|. \tag{6.18}$$

Again, by Lemma V,

$$|\beta^A - \beta^\alpha| \leqq \max |\beta^z \log \beta| \cdot |\Delta A|, \tag{6.19}$$

so that

$$|\beta^A - \beta^\alpha| < M^{M+1} |\Delta A|. \tag{6.20}$$

Next, assume that sequences c_i and λ_i exist, such that simultaneously

$$|\alpha_i - A| < \frac{1}{{c_i'}^{\lambda_i}} \quad \text{and} \quad |\beta_i - B| < \frac{1}{{c_i'}^{\lambda_i}}, \tag{6.21}$$

where the c_i' are multipliers for both α_i and β_i. Then, from (6.16), (6.18), (6.20), and (6.21) we deduce that

$$|B^A - \beta^\alpha| < M_1/{c_i'}^{\lambda_i} \leqq \frac{\epsilon_m}{c_i^m} \frac{M_1}{\epsilon_m c_i^{\lambda_i - m}}, \tag{6.22}$$

where M_1 is a bound depending only on M. For m, M, and hence M_1 fixed, when i becomes infinite, the second factor approaches zero, so that eventually,

$$|B^A - \beta^\alpha| < \epsilon_m/{c_i'}^m < \epsilon_m/c_i^m, \tag{6.23}$$

where c_i is the least multiplier for both α_i and β_i, and therefore less than or equal to c_i'.

This situation enables us to use the reasoning applied to prove the last theorem. We surround each point β^α with least common multiplier c_i for α_i

and β_i by a circle of radius ϵ_m/c_i^m, and call S_m the set of points inside any of these circles. The set T is then defined by

$$T = S_1 \cdot S_2 \cdot \cdots \cdot S_m \cdot \cdots . \tag{6.24}$$

The inequality (6.23) shows that the point B^A is in each of the sets S_m, and therefore in the set T. To estimate the measure of S_m we observe that c_i is at least as great as the multipliers for α_i or β_i individually, and hence by applying (6.12) to the number of α_i and β_i for a given c_i, we find that the number of points $\beta_i^{\alpha_i}$ for a given c_i is at most $\gamma c^{2\nu}$. Thus the measure of S_m is at most

$$\sum_{c_i=1}^{\infty} \gamma c_i^{2\nu} \pi \left(\frac{\epsilon_m}{c_i^m}\right)^2 \leqq \gamma \pi \epsilon_m^2 \sum c_i^{2\nu-2m}. \tag{6.25}$$

Since the series converges for ν greater than $m+1$, to a sum less than 2, the measure of S_m approaches zero with ϵ_m, and the measure of T is zero.

This shows that when the restriction involving M holds, and if M is taken integral, there are only an enumerable number of choices for M and K_ν, so that the set discussed in the theorem has been shown to be the sum of an enumerable number of sets, each of measure zero.

For the one-dimensional case, we use intervals in place of the circles.

It is worth noting that there are no theorems like those of this section on transcendentality and measure for numbers merely known to be approximable by sequences of numbers $\beta_i^{\alpha_i}$, without α_i and β_i separately approaching limits. The first fact is shown by the example

$$\alpha_i = -\frac{2^{1/2}}{c_i}, \qquad \beta_i = \frac{p_i}{c_i}, \qquad c_i = i,$$

where p_i is an integer so chosen that

$$\left| 2^{i/2^{1/2}} - \frac{p_i}{i} \right| < \frac{1}{i}, \text{ so that } \beta_i = 2^{i/2^{1/2}}\left(1 + \frac{\theta}{i 2^{i/2^{1/2}}}\right) \text{ with } |\theta| < 1.$$

Under these conditions we have

$$\left| \tfrac{1}{2} - \beta_i^{\alpha_i} \right| < \frac{1}{c^{\lambda_i}}, \quad \text{where} \quad \lambda_i = \left[\frac{i \log 2}{2^{1/2} \log i}\right]$$

and so becomes infinite with i. Similar sequences may be constructed with any number in place of $\frac{1}{2}$, so that they do not lead to sets of zero measure.

7. **Conclusions and examples.** A simple example of an ordinary Liouville number is obtained by using a series

$$\sum_{n=1}^{\infty} \frac{p_n}{N_n}, \tag{7.1}$$

where the p_n are uniformly bounded integers, and the N_n are integral values increasing with n sufficiently fast. In particular, by taking the p_n less than 10, and the $N_n = 10^{n!}$, Liouville found one whose decimal expansion could readily be written down. In place of this, we might use

$$N_n = 10^{(10^n \log n)} \quad \text{or} \quad a^{(b^E)} \tag{7.2}$$

where a and b are fixed integers greater than unity, and E is a function of n increasing faster than n, i.e., such that $E/n \to \infty$.

If we take as the p_n algebraic integers from a fixed field K_ν, with uniformly bounded conjugates, and again use (7.1) and (7.2), we obtain simple examples of the generalized transcendental numbers of Theorem IV.

To obtain simple examples of the restricted generalized Liouville numbers used in Theorems I, II, and III, we use (7.1) with the p_n algebraic integers from a fixed field K_ν, with uniformly bounded conjugates, and put

$$N_n = 10^{(10^{(10^n)})} \quad \text{or} \quad a^{(b^{(K^n)})}, \tag{7.3}$$

where in the second form, a, b, and K are fixed integers, a and b exceed unity, and K exceeds $k+1$, for the k of the condition (3.1).

Suppose, to be specific, we put x_i equal to the square root of the digit in the ith place of the decimal part of π, and y_i equal to the square root of the digit in the ith place of the decimal part of e. Then write

$$X = \sum_{n=1}^{\infty} x_n 2^{-(2^{(8^n)})}, \qquad Y = \sum_{n=1}^{\infty} y_n 2^{-(2^{(8^n)})}. \tag{7.4}$$

Then, by the corollary to Theorem I, we may assert the transcendentality of the following numbers

$$2^X,\ X^2,\ X^Y,\ \frac{\log X}{\log Y},\ \frac{\log X}{\log 2}. \tag{7.5}$$

The first of these also illustrates the corollary to Theorem II, and would continue to do so if we replaced the 8 in (7.4) by 6. The second of the numbers is an illustration of the corollary to Theorem III, and in this case we could replace the 8 in (7.4) by 3.

We note that if we take the x_n all equal to the 2, and the y_n all equal to 3 in (7.4), the transcendentality of the first three numbers in (7.5) would follow from the theorems of G. Ricci.

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A DIRECT EXPANSION PROOF OF SUFFICIENT CONDITIONS FOR THE NON-PARAMETRIC PROBLEM OF BOLZA*

BY

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1. **Introduction.** In a recent paper† the author has proved by expansion methods a sufficiency theorem for the general problem of Bolza that had been established previously by Hestenes‡ using the classical field method. This proof had in common with the previous proof of Hestenes the property that the theorem was not established directly for the general problem of Bolza with non-separated end conditions; instead, such a problem was first transformed into an equivalent one in a greater number of dependent variables and with separated end conditions. In a paper appearing in the present volume of these Transactions, Hestenes§ has devised a direct proof by field methods for the general problem of Bolza. He has utilized the notion of families of fields of extremals, and the proof is a generalization of a method which he has previously used in treating the problem of Mayer.

It is the purpose of the present note to give a direct expansion proof of sufficient conditions for the general non-parametric problem of Bolza. In this direct expansion proof the auxiliary theorem of §3 of **R** is replaced by Theorem A of §2 of the present paper. This latter theorem is an extension of the results of §5 of **H** for the special case of a problem having the form of the second variation, and thus may be considered as an extension of the transformation of Clebsch.

It is to be remarked that there already existed direct sufficiency proofs by field methods for the general problem of Bolza in case suitable normality conditions were satisfied.‖ Hestenes obtains in **H** a direct proof not involving normality assumptions. On the other hand, the results of the author's paper **R** do not yield a direct expansion proof of sufficient conditions for a problem with non-separated end conditions that involves no differential side condi-

* Presented to the Society, April 10, 1937; received by the editors January 20, 1937.

† *Sufficient conditions by expansion methods for the problem of Bolza in the calculus of variations*, to appear in a forthcoming number of the Annals of Mathematics. This paper will be referred to as **R**.

‡ These Transactions, vol. 36 (1934), pp. 793–818.

§ *A direct sufficiency proof for the problem of Bolza in the calculus of variations*, these Transactions, vol. 42 (1937), pp. 141–154. This paper will be referred to as **H**.

‖ See, in particular, Morse, American Journal of Mathematics, vol. 53 (1931), pp. 517–546, and Bliss, Annals of Mathematics, vol. 33 (1932), pp. 261–274.

tions, and which is thus identically normal. This fact is illustrated by the example given at the end of §3 of **R**. In view of this difference between the the generalization of Hestenes' paper **H** and the extension of the present note over the previous paper **R**, it is somewhat surprising that essentially the same auxiliary theorem is involved in these two direct proofs by quite different methods of the same sufficient conditions for the general problem of Bolza.

2. **Auxiliary theorems.** The form of the problem of Bolza here considered is that of finding in a class of arcs

$$y_i(x) \qquad (i = 1, \cdots, n;\ x_1 \leqq x \leqq x_2)$$

satisfying differential equations and end-conditions

$$\phi_\beta[x, y, y'] = 0 \qquad (\beta = 1, \cdots, m < n), \tag{2.1}$$

$$\psi_\mu[x_1, y(x_1), x_2, y(x_2)] = 0 \qquad (\mu = 1, \cdots, p \leqq 2n + 2), \tag{2.2}$$

one which minimizes a given functional

$$J = g[x_1, y(x_1), x_2, y(x_2)] + \int_{x_1}^{x_2} f[x, y, y']dx. \tag{2.3}$$

The notation and terminology of **R** *will be used throughout.*

Suppose that $E: y_i(x)$, $\lambda_0 = 1$, $\lambda_\beta(x)$ $(x_1 \leqq x \leqq x_2)$ is a non-singular extremal which satisfies with constants e_μ conditions I and IV$_*'$. This latter condition is that along E the second variation

$$J_2[\xi, \eta] \equiv 2\gamma[\xi_1, \eta(x_1), \xi_2, \eta(x_2)] + \int_{x_1}^{x_2} 2\omega[x, \eta, \eta']dx \tag{2.4}$$

be positive for all non-identically vanishing variations ξ_1, ξ_2, $\eta_i(x)$ which satisfy the linear differential equations and end-conditions

$$\Phi_\beta[x, \eta, \eta'] \equiv \phi_{\beta y_i}\eta_i + \phi_{\beta r_i}\eta_i' = 0 \qquad (\beta = 1, \cdots, m),$$

$$\Psi_\mu[\xi_1, \eta(x_1), \xi_2, \eta(x_2)]$$

$$\equiv \Psi_{\mu\xi_1}\xi_1 + \Psi_{\mu\eta_{i1}}\eta_i(x_1) + \Psi_{\mu\xi_2}\xi_2 + \Psi_{\mu\eta_{i2}}\eta_i(x_2) = 0 \quad (\mu = 1, \cdots, p).$$

As usual, it is assumed that the matrix $(\Psi_{\mu\xi_1}; \Psi_{\mu\eta_{i1}}; \Psi_{\mu\xi_2}; \Psi_{\mu\eta_{i2}})$ is of rank p. As a consequence of IV$_*'$ there exists a family $\eta_i = u_{ik}(x)$, $\zeta_i = v_{ik}(x)$ $(k = 1, \cdots, n)$ of mutually conjugate solutions of the canonical accessory equations with $|u_{ik}(x)| \neq 0$ on x_1x_2.†

† See Reid, American Journal of Mathematics, vol. 57 (1935), pp. 573–586; also Morse, these Transactions, vol. 37 (1935), pp. 147–160, and Hestenes, Annals of Mathematics, vol. 37 (1936), pp. 543–551.

Now suppose that b_θ^1, $a_{i\theta}^1$, b_θ^2, $a_{i\theta}^2$ $(\theta=1, \cdots, 2n+2-p)$ are linearly independent solutions of the equations

$$\Psi_{\mu\xi_1}b_\theta^1 + \Psi_{\mu\eta_{i1}}a_{i\theta}^1 + \Psi_{\mu\xi_2}b_\theta^2 + \Psi_{\mu\eta_{i2}}a_{i\theta}^2 = 0 \qquad (\mu = 1, \cdots, p).$$

Let q denote the maximum number of linearly independent accessory extremals of the form $Y_{il}(x)\equiv 0$, $Z_{il}(x)$ $(l=1, \cdots, q)$ such that for every non-identically vanishing linear combination

$$\eta_i = Y_{il}(x)c_l \equiv 0, \qquad \zeta_i = Z_{il}(x)c_l$$

the equations

$$\zeta_i(x_1)a_{i\theta}^1 - \zeta_i(x_2)a_{i\theta}^2 = 0 \qquad (\theta = 1, \cdots, 2n + 2 - p)$$

are not all satisfied. Without loss of generality we may then suppose that the sets b_θ^s, $a_{i\theta}^s$ $(s=1, 2; \theta=1, \cdots, 2n+2-p)$ are so chosen that

$$Z_{il}(x_1)a_{i\theta}^1 - Z_{il}(x_2)a_{i\theta}^2 = -\delta_{l\theta} \quad (l = 1, \cdots, q; \theta = 1, \cdots, 2n + 2 - p).$$

Then for an arbitrary accessory extremal $\eta_i(x)\equiv 0$, $\zeta_i(x)$ we have

$$\zeta_i(x_1)a_{i\nu}^1 - \zeta_i(x_2)a_{i\nu}^2 = 0 \quad (\nu = q + 1, \cdots, 2n + 2 - p)$$

and there exists a unique accessory extremal $\eta_i = U_{i\nu}(x)$, $\zeta_i = V_{i\nu}(x)$ such that

$$U_{i\nu}(x_s) = a_{i\nu}^s \qquad (s = 1, 2; \nu = q + 1, \cdots, 2n + 2 - p).$$

Let

$$U_{il}(x) = \rho Y_{il}(x) \equiv 0, \quad V_{il}(x) = \rho Z_{il}(x) \qquad (l = 1, \cdots, q),$$

where ρ is a real constant whose value will be determined shortly. The multipliers corresponding to u_{ik}, v_{ik} and $U_{i\theta}$, $V_{i\theta}$ will be denoted by $l_{\beta k}(x)$ and $L_{\beta\theta}(x)$ $(k=1, \cdots, n; \theta=1, \cdots, 2n+2-p)$, respectively.

Suppose ξ_1, ξ_2 are constants and $\eta_i(x)$ are arbitrary functions of class D' on X_1X_2. Then there exist unique constants c_μ', c_θ $(\mu=1, \cdots, p; \theta=1, \cdots, 2n+2-p)$ such that

$$(2.5) \qquad \xi_s = \Psi_{\mu\xi_s}c_\mu' + b_\theta^s c_\theta, \quad \eta_i(X_s) = \Psi_{\mu\eta_{is}}c_\mu' + a_{i\theta}^s c_\theta \qquad (s = 1, 2),$$

and corresponding to these constants c_θ there are determined for X_1, X_2 sufficiently near x_1, x_2 unique functions $h_i(x)$ of class D' and such that on X_1X_2

$$(2.6) \qquad \eta_i(x) = u_{ik}(x)h_k(x) + U_{i\theta}(x)c_\theta.$$

For such a set ξ_1, ξ_2, $\eta_i(x)$ define

$$u_i'(x) = u_{ik}'(x)h_k(x) + U_{i\theta}'(x)c_\theta, \qquad v_i(x) = u_{ik}(x)h_k'(x),$$
$$l_\beta(x) = l_{\beta k}(x)h_k(x), \qquad L_\beta(x) = L_{\beta\theta}(x)c_\theta, \qquad \mu_\beta = l_\beta + L_\beta,$$

and consider the expression

$$(2.7) \qquad 2\gamma[\xi_1, \eta(X_1), \xi_2, \eta(X_2)] + \int_{X_1}^{X_2} (2\Omega[x, \eta, \eta'; \mu] - F_{r_i r_k} v_i v_k) dx.$$

In this expression it is to be understood that the coefficients of γ are the values determined by the end-values of E, and do not depend upon X_1 and X_2. For X_1, X_2 near x_1, x_2 the expression (2.7) is well-defined. The integral in (2.7) is readily seen to be the Hilbert integral for the second variation involving slope functions and multipliers determined by the $3n-p$ parameter family of accessory extremals given above, and evaluated along the curve $\eta = [\eta_i(x)]$ $(X_1 \leqq x \leqq X_2)$. One may also verify directly by the use of Taylor's formula and the transformation of Clebsch that for X_1, X_2 near x_1, x_2 the expression (2.7) is a quadratic form in $[\xi_1, \eta_i(X_1), \xi_2, \eta_i(X_2)]$ whose coefficients depend upon the end-values of u_{ik}, $U_{i\theta}$, v_{ik}, $V_{i\theta}$ at X_1 and X_2. For brevity this form will be denoted by $2Q[\xi_1, \eta_i(X_1), \xi_2, \eta_i(X_2) | X_1, X_2]$. Now it may be proved that the parameter ρ involved in the definition of U_{il}, V_{il} $(l=1, \cdots, q)$ can be chosen so large that $Q[\xi_1, \eta_i(x_1), \xi_2, \eta_i(x_2) | x_1, x_2] > 0$ for every non-zero set $[\xi_1, \eta_i(x_1), \xi_2, \eta_i(x_2)]$ satisfying the equations $\Psi_\mu[\xi_1, \eta(x_1), \xi_2, \eta(x_2)] = 0$ $(\mu = 1, \cdots, p)$ [see **H**, §5]. *In the following discussion we shall suppose that ρ has been so chosen.* It is to be emphasized that the arc $\eta = [\eta_i(x)]$ has not been assumed to be differentially admissible, and hence ξ_1, ξ_2, $\eta_i(X_1)$, $\eta_i(X_2)$ are independent variables. By continuity it then follows that there exists a positive δ such that for $|X_s - x_s| < \delta$ $(s=1, 2)$ and an arbitrary non-zero set $[\xi_1, \eta_{i1}, \xi_2, \eta_{i2}]$ satisfying

$$|\Psi_\mu[\xi_1, \eta_{i1}, \xi_2, \eta_{i2}]| < \delta[\xi_1^2 + \xi_2^2 + \eta_{i1}\eta_{i1} + \eta_{i2}\eta_{i2}]^{1/2}$$

we have

$$Q[\xi_1, \eta_{i1}, \xi_2, \eta_{i2} | X_1, X_2] > 0.$$

Consequently, by the use of the integral form of the remainder in the expansions of the functions ψ_μ about the values $[x_1, y_i(x_1), x_2, y_i(x_2)]$ (see **R**, §3) we obtain the following theorem:

THEOREM A. *Suppose that for a problem of Bolza $E: y_i(x)$, $\lambda_0 = 1$, $\lambda_\beta(x)$ $(x_1 \leqq x \leqq x_2)$ is a non-singular extremal which satisfies with constants e_μ conditions* I and IV$_*'$. *Using the accessory extremals u_{ik}, v_{ik}, $U_{i\theta}$, $V_{i\theta}$ and terminology introduced above, there exists a neighborhood $\mathfrak{N}_0$ of the end-values $[x_1, y_i(x_1), x_2, y_i(x_2)]$ such that if $[X_1, Y_{i1}, X_2, Y_{i2}]$ is in $\mathfrak{N}_0$ the extremal $y_i(x)$, $\lambda_0 = 1$, $\lambda_\beta(x)$ is defined and non-singular on X_1X_2 while $|u_{ik}| \neq 0$ on this interval; moreover, there exists a $\kappa > 0$ such that if $[X_1, Y_{i1}, X_2, Y_{i2}]$ is in $\mathfrak{N}_0$ and terminally admissible we have*

$$(2.8)\quad Q[X_1 - x_1, \eta_{i1}, X_2 - x_2, \eta_{i2} \mid X_1, X_2] \geqq \kappa[(X_1 - x_1)^2 + (X_2 - x_2)^2 + \eta_{i1}\eta_{i1} + \eta_{i2}\eta_{i2}],$$

where $\eta_{is} = Y_{is} - Y_i(X_s)$ $(s = 1, 2)$.

By relatively slight modifications of the proof of Theorem 4.3 of **R** one may establish the following corresponding result.

THEOREM B. *Suppose that* $E: y_i(x)$, $\lambda_0 = 1$, $\lambda_\beta(x)$ $(x_1 \leqq x \leqq x_2)$ *is an extremal satisfying conditions* II'_N, III', *and that* $u_{ik}(x)$, $v_{ik}(x)$, $U_{i\theta}(x)$, $V_{i\theta}(x)$ $(k = 1, \cdots, n;\ \theta = 1, \cdots, 2n + 2 - p)$ *are associated accessory extremals with* $|u_{ik}(x)| \neq 0$ *on* $x_1 x_2$. *For arbitrary sets* (h_i, c_θ) *let*

$$y_i(x, h, c) = y_i(x) + u_{ik}(x)h_k + U_{i\theta}(x)c_\theta,$$
$$r_i(x, h, c) = y_i'(x) + u_{ik}'(x)h_k + U_{i\theta}'(x)c_\theta,$$
$$\lambda_\beta(x, h, c) = \lambda_\beta(x) + l_{\beta k}(x)h_k + L_{\beta\theta}(x)c_\theta,$$

where $l_{\beta k}$ *and* $L_{\beta\theta}$ *are the multipliers for the accessory extremals* u_{ik}, v_{ik} *and* $U_{i\theta}$, $V_{i\theta}$, *respectively. Then for every* $\epsilon > 0$ *there exists a* $d_{1\epsilon} > 0$ *and a neighborhood* $\mathfrak{F}_{1\epsilon}$ *in* xy*-space of the* $[x, y(x)]$ *on* E *such that if* $0 \leqq \|(c_\theta)\| \leqq d_{1\epsilon}$, *while* $[x, y(x, h, c)]$ *is in* $\mathfrak{F}_{1\epsilon}$ *and* $[x, y(x, h, c), \bar{r}]$ *is differentially admissible, we have*

$$(2.9)\quad \mathcal{E}[x, y(x, h, c), r(x, h, c), \lambda(x, h, c); \bar{r}] \geqq \tau R[\|\bar{r} - r(x, h, c)\|] - \epsilon\|(h_i, c_\theta)\|^2,$$

where τ *is a positive constant independent of* ϵ.†

As in Theorem 4.3 of **R** we may choose $\tau = \tau_3/4$, where τ_3 is the constant appearing in Lemma 4.3. In the above expression (2.8) $R[t]$ is the convex function $(1+t^2)^{1/2} - 1$ for $t \geqq 0$.

Finally, we shall use Theorem 5.1 of **R**, which for completeness we shall state.

THEOREM C. *If* $z_t(x)$ $(t = 1, \cdots, N)$ *are absolutely continuous functions on* $X_1 \leqq x \leqq X_2$, *and* $\|z\| \leqq \delta$ *on this interval, then*

$$(2.10)\quad \int_{X_1}^{X_2} \|z\| \cdot \|z'\| dx \leqq d_1\left[\int_{X_1}^{X_2} R[\|z'\|]dx + \|z(X_1)\|^2\right],$$

$$(2.11)\quad \int_{X_1}^{X_2} \|z\|^2 dx \leqq d_2\left[\int_{X_1}^{X_2} R[\|z'\|]dx + \|z(X_1)\|^2\right],$$

where

$$d_1 = 4((1 + \delta^2)^{1/2} + 1)\max\{1, X_2 - X_1\}, \quad d_2 = 3(X_2 - X_1)d_1.$$

† If $z \equiv (z_t)$ $(t = 1, \cdots, N)$, $\|z\| = \|(z_t)\|$ is used to denote the positive square root of $z_1^2 + \cdots + z_N^2$.

3. **The sufficiency theorem.** We shall now indicate briefly how the direct proof of the following sufficiency theorem differs from the proof previously given in **R**.

THEOREM 3.1. *Suppose that* $E: y_i(x)$, $\lambda_0 = 1$, $\lambda_\beta(x)$ $(x_1 \leqq x \leqq x_2)$ *is an extremal arc satisfying with constants* e_μ *conditions* I, II'_N, III′ and IV_*'. *Then there exists a neighborhood* $\mathfrak{F}$ *of* E *in* xy*-space and a neighborhood* $\mathfrak{N}$ *of the ends of* E *in* $[x_1, y_{i1}, x_2, y_{i2}]$*-space such that* $J[C] > J[E]$ *for every admissible arc* C *in* $\mathfrak{F}$ *with ends in* $\mathfrak{N}$ *and not identical with* E.

Suppose that $C: Y_i(x)$ $(X_1 \leqq x \leqq X_2)$ is an admissible arc with end-points $[X_1, Y_i(X_1), X_2, Y_i(X_2)]$ in the $\mathfrak{N}_0$ neighborhood of $[x_1, y_i(x_1), x_2, y_i(x_2)]$ defined by Theorem A. Let $\eta_i(x) = Y_i(x) - y_i(x)$ on X_1X_2, and determine constants c_μ', c_θ $(\mu = 1, \cdots, p;\ \theta = 1, \cdots, 2n+2-p)$ by equations (2.5) for $\eta_i(X_s)$ thus defined and $\xi_s = X_s - x_s$ $(s = 1, 2)$. The corresponding functions $h_i(x)$ determined by (2.6) are of class D' on X_1X_2. If for a given arc C of this type we set

$$u_i'(x) = u_{ik}'(x)h_k(x) + U_{i\theta}'(x)c_\theta, \qquad v_i(x) = u_{ik}(x)h_k'(x)$$

we have, in the notation of Theorem B,

$$Y_i(x) = y_i(x, h[x], c), \qquad y_i'(x) + u_i'(x) = r_i(x, h[x], c),$$
$$\lambda_\beta(x) + \mu_\beta(x) = \lambda_\beta(x, h[x], c).$$

Finally, let m_0, M_0 be positive constants such that for every element $[X_1, Y_{i1}, X_2, Y_{i2}]$ in $\mathfrak{N}_0$ we have $m_0\|h\| \leqq \|(u_{ik}[x]h_k)\| \leqq M_0\|h\|$ for arbitrary sets $h = (h_i)$.

We write for brevity

$$\Delta g = g[X_1, Y(X_1), X_2, Y(X_2)] - g[x_1, y(x_1), x_2, y(x_2)].$$

Then

$$J[C] - J[E] = \Delta g + \int_{X_1}^{X_2} F[x, Y, Y', \lambda + \mu]dx - \int_{x_1}^{x_2} F[x, y, y', \lambda]dx$$
$$= \Delta g + J^0 - J^1 - J^2, \tag{3.1}$$

where

$$J^s = (-1)^s \int_{X_s}^{x_s} F[x, y, y', \lambda]dx \qquad (s = 1, 2), \tag{3.2}$$

$$J^0 = \int_{X_1}^{X_2} \mathcal{E}[x, Y, y' + u', \lambda + \mu; Y']dx + F_{r_i}(X_s)\eta_i(X_s)\Big|_{s=1}^{s=2} \tag{3.3}$$

$$+\frac{1}{2}\int_{X_1}^{X_2}(2\Omega[x,\eta,\eta',\mu]-F_{r_ir_k}v_iv_k)dx+\int_{X_1}^{X_2}(B+v_iB_i^*)dx,$$

and for every $\epsilon>0$ there exists a corresponding $d_\epsilon>0$ such that if $\|\eta\|\leqq d_\epsilon$

$$|B(x)|\leqq\epsilon\|\{h_i(x),c_\theta\}\|^2,\quad \|B^*(x)\|\leqq(\epsilon/M_0)\|\{h_i(x),c_\theta\}\|.$$

The proofs of these relations are similar to those of (6.1), (6.2), (6.3) of **R**. As in **R**, one may likewise show that for a given $\epsilon>0$ there is a bounded neighborhood $\mathfrak{N}_{2\epsilon}$ of the end-points of E which is interior to the $\mathfrak{N}_0$ neighborhood of Theorem A, and such that if C is admissible with end-points in $\mathfrak{N}_{2\epsilon}$, then

$$\begin{aligned}\Delta g-J^1-J^2+F_{r_i}(X_s)\eta_i(X_s)\Big|_{s=1}^{s=2}&\\ \geqq\gamma[X_1-x_1,\eta_i(X_1),X_2-x_2,\eta_i(X_2)]&\\ -\epsilon[(X_1-x_1)^2+(X_2-x_2)^2+\|\eta(X_1)\|^2+\|\eta(X_2)\|^2].&\end{aligned}\tag{3.4}$$

Let $\mathfrak{F}_{2\epsilon}$ be a neighborhood of E such that if C is in $\mathfrak{F}_{2\epsilon}$ then $\|\eta(x)\|\leqq d_\epsilon$ on X_1X_2. Consequently, if C is an admissible arc in $\mathfrak{F}_{2\epsilon}$ with end-points in $\mathfrak{N}_{2\epsilon}$ we have†

$$\begin{aligned}J[C]-J[E]\geqq Q[X_1-x_1,\eta_i(X_1),X_2-x_2,\eta_i(X_2)\mid X_1,X_2]&\\ +\int_{X_1}^{X_2}\mathcal{E}[x,Y,y'+u',\lambda+\mu;Y']dx&\\ -\epsilon\Big[(X_1-x_1)^2+(X_2-x_2)^2+\|\eta(X_1)\|^2+\|\eta(X_2)\|^2&\\ +\int_{X_1}^{X_2}(\|\{h_i(x),c_\theta\}\|^2+\|\{h_i(x),c_\theta\}\|\cdot\|\{h_i'(x),c_\theta'\}\|)dx\Big].&\end{aligned}\tag{3.5}$$

For a given $\epsilon>0$ let $\mathfrak{F}_\epsilon$ denote a bounded neighborhood of E in xy-space interior to both the neighborhood $\mathfrak{F}_{1\epsilon}$ of Theorem B and the neighborhood $\mathfrak{F}_{2\epsilon}$ defined above; moreover, suppose that $\mathfrak{N}_\epsilon$ is a neighborhood of the ends of E interior to both $\mathfrak{N}_0$ and $\mathfrak{N}_{2\epsilon}$, and such that if the end-values of C are in $\mathfrak{N}_\epsilon$ the corresponding constants c_θ satisfy $\|(c_\theta)\|\leqq d_{1\epsilon}$, where $d_{1\epsilon}$ is as in Theorem B. Suppose $C:Y_i(x)$ $(X_1\leqq x\leqq X_2)$ is an admissible arc in $\mathfrak{F}_\epsilon$ with end-points in $\mathfrak{N}_\epsilon$. Since $\mathfrak{N}_\epsilon$ is interior to $\mathfrak{N}_0$ inequality (2.8) is applicable to the quadratic form Q. We also see that $\mathcal{E}(x,Y,y'+u',\lambda+\mu;Y']$ satisfies

† Since c_θ $(\theta=1,\cdots,2n-p)$ are constants,

$$\|v\|\leqq M_0\|h'(x)\|=M_0\|\{h_i'(x),c_\theta'\}\|,$$

and by Cauchy's inequality

$$|v_iB_i^*|\leqq\epsilon\|\{h_i'(x),c_\theta'\}\|\cdot\|\{h_i(x),c_\theta\}\|.$$

(2.9). The functions $h_i(x)$ determined by C are absolutely continuous, and $\|h'(x)\| = \|\{h_i'(x), c_\theta'\}\|$, $R[\|h'(x)\|]$, $R[\|v(x)\|]$ are Riemann integrable on X_1X_2. Since $m_0\|h'\| \leqq \|v\|$, we have $R[\|h'\|] \leqq d_3 R[\|v\|]$, where $d_3 = (1/m_0)\max\{1, 1/m_0\}$ (see **R**, Lemma A of §4). Moreover, since c_θ $(\theta = 1, \cdots, 2n+2-p)$ are linear in $X_s - x_s$, $\eta_i(X_s)$ $(s = 1, 2)$, there is a constant m' such that

$$\|\{h_i(X_1), c_\theta\}\|^2 \leqq m'[(X_1 - x_1)^2 + (X_2 - x_2)^2 + \|\eta(X_1)\|^2 + \|\eta(X_2)\|^2].$$

Applying (2.10) and (2.11) to the functions $(z_t) = (h_i, c_\theta)$ and combining the relations indicated above, it follows that for an admissible arc C in $\mathfrak{F}_\epsilon$ with end-points in $\mathfrak{N}_\epsilon$, we have

$$\begin{aligned} J[C] - J[E] \geqq (\kappa - \epsilon\{1 + [2d_2 + d_1]m'\}) \sum_{s=1}^{2} [(X_s - x_s)^2 + \|\eta(X_s)\|^2] \\ + (\tau - \epsilon[2d_2 + d_1]d_3) \int_{X_1}^{X_2} R[\|v(x)\|]dx. \end{aligned} \tag{3.6}$$

Now d_1 and d_2 depend upon the particular arc C in that they involve $(X_2 - X_1)$ and a quantity δ such that $\|\{h_i(x), c_\theta\}\| < \delta$ on X_1X_2. These quantities, however, are uniformly bounded when ϵ is restricted to a bounded set of values, and C is in the $\mathfrak{F}_\epsilon$ neighborhood of E with end-points in the $\mathfrak{N}_\epsilon$ neighborhood of the end-points of E. Hence the quantities $(\kappa - \epsilon\{1 + [2d_2 + d_1]m'\})$, $(\tau - \epsilon[2d_2 + d_1]d_3)$ are positive for ϵ sufficiently small. Let $\mathfrak{F} = \mathfrak{F}_\epsilon$, $\mathfrak{N} = \mathfrak{N}_\epsilon$ for such a value of ϵ. Then for an arbitrary admissible arc C in $\mathfrak{F}$ with end-points in $\mathfrak{N}$ we have $J[C] \geqq J[E]$, and the equality sign holds if and only if

$$0 = \int_{X_1}^{X_2} R[\|v(x)\|]dx = X_s - x_s = \|\eta(X_s)\| \qquad (s = 1, 2),$$

that is, if and only if C is identical with E. We have established, therefore, the conclusion of Theorem 3.1.

As indicated in **R**, the class of comparison arcs C may be enlarged by interpreting the integrals involved as Lebesgue integrals.

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ON A CLASSIFICATION OF INTEGRAL FUNCTIONS BY MEANS OF CERTAIN INVARIANT POINT PROPERTIES. A SUPPLEMENT*

BY

M. T. BIRD

In a recent paper† Carmichael, Martin, and Bird noted that the existence or non-existence of their α-sequences had not been established. The purpose of this note is to establish the impossibility of α-sequences. In other words, Theorem I of the paper cited above may be replaced by the following theorem:

In order that $\{t_n\}$ *shall be an I-sequence it is necessary that the following condition shall be satisfied*:

$$\lim_{n=\infty} (t_n)^{1/n} = \infty .$$

It is sufficient for the proof of the theorem to show that the inferior limit of $(t_n)^{1/n}$ cannot be zero. For this purpose we assume that the inferior limit of $(t_n)^{1/n}$ is zero and show that a contradiction arises.

The positive integers can be arranged in ascending order in two infinite sequences $\{m_i\}$ and $\{n_i\}$ such that

$$t_{m_i} > 1, \quad i = 1, 2, 3, \cdots ; \qquad t_{n_i} \leqq 1, \quad i = 1, 2, 3, \cdots .$$

Let β, γ be a pair of integers such that

$$\gamma \geqq \beta \geqq 2.$$

We wish to show that an infinite subsequence $\{\nu_i\}$ of the sequence $\{n_i\}$ exists such that for an infinite subsequence $\{\mu_i\}$ of the sequence $\{m_i\}$ we have

$$\beta\mu_i \leqq \nu_i \leqq \gamma^2\mu_i, \qquad i = 1, 2, 3, \cdots .$$

We take ν_1 to be the least member of the sequence $\{n_i\}$ such that

$$\beta m_1 \leqq \nu_1 .$$

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† Carmichael, Martin, and Bird, *On a classification of integral functions by means of certain invariant point properties*, these Transactions, vol. 40 (1936), pp. 462–473.

Let us write ν_1 in the form $\nu_1 = x\gamma^2 m_1$. If x does not exceed 1 we take μ_1 to be equal to m_1. If x exceeds 1 we define μ_1 as the greatest integer which does not exceed $x\gamma m_1$. In either case μ_1 is a member of the sequence $\{m_i\}$ such that we have

$$\beta\mu_1 \leqq \nu_1 \leqq \gamma^2\mu_1.$$

We take m_j' to be the least member of the sequence $\{m_i\}$ such that

$$\beta m_j' > \nu_{j-1}, \qquad j = 2, 3, 4, \cdots.$$

We define ν_j and μ_j with respect to m_j' in the same manner that we defined ν_1 and μ_1 with respect to m_1. In this way we define the monotonically increasing subsequences $\{\nu_i\}$ and $\{\mu_i\}$ of the sequences $\{n_i\}$ and $\{m_i\}$, respectively, which are such that

$$\beta\mu_i \leqq \nu_i \leqq \gamma^2\mu_i, \qquad i = 1, 2, 3, \cdots.$$

We proceed to show that the function defined by the series

$$E_1(x) = \sum_{i=0}^{\infty} x^{n_i},$$

which converges for $|x| < 1$, disputes the relation (1) of the paper cited above. We observe

$$\limsup_{n=\infty} |t_n E_1^{(n)}(0)/n!|^{1/n} \leqq 1.$$

Furthermore, we have

$$E_1^{(\mu)}(a)/\mu! \geqq a^{\nu-\mu} C_{\nu,\mu}, \qquad 0 < a < 1,$$

where ν and μ are corresponding members of the sequences $\{\nu_i\}$ and $\{\mu_i\}$, respectively.

It is easy to prove the inequality

$$C_{\nu,\mu} \geqq \frac{1}{\nu+1}\left(\frac{\nu}{\nu-\mu}\right)^{\nu}\left(\frac{\nu-\mu}{\mu}\right)^{\mu}.$$

This leads us to the inequality

$$E_1^{(\mu)}(a)/\mu! \geqq \frac{1}{\nu+1}\left(\frac{a\nu}{\nu-\mu}\right)^{\nu}\left(\frac{\nu-\mu}{a\mu}\right)^{\mu}.$$

Let us take a to be equal to $1-\gamma^{-2}$. Then we have the inequalities

$$t_\mu E_1^{(\mu)}(a)/\mu! \geqq (\gamma^2\mu+1)^{-1}(\beta-1)^{\mu}a^{-\mu} \geqq (\gamma^2\mu+1)^{-1}a^{-\mu}.$$

Hence we have

$$\liminf_{i=\infty} \left| t_{\mu_i} E^{(\mu_i)}(a)/\mu_i! \right|^{1/\mu_i} \geqq \sigma^{-1} > 1$$

and, consequently,

$$\limsup_{n=\infty} \left| t_n E^{(n)}(a)/n! \right|^{1/n} > \limsup_{n=\infty} \left| t_n E^{(n)}(0)/n! \right|^{1/n}.$$

This contradicts relation (1) of the paper cited above and leads us to conclude that the limit of $(t_n)^{1/n}$ must be infinite.

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ON THE SPACE* (BV)

BY

C. RAYMOND ADAMS AND ANTHONY P. MORSE

1. **Introduction.** For the space (AC) of functions $x(t)$ absolutely continuous on the interval $0 \leqq t \leqq 1$ it is natural, since (AC) then becomes isomorphic with the space (L) of Lebesgue integrable functions, to employ the metric

$$(x, y) = | x(0) - y(0) | + T_0^1(x - y), \tag{1}$$

where $T_0^1(x-y)$ stands for the total variation of $x(t)-y(t)$ on $0 \leqq t \leqq 1$. Until recently the few writers† who have considered the space (BV) of functions of bounded variation have simply carried over to this larger space the metric of (AC). Although with this metric (BV) is a Banach space, it is not separable; perhaps it is partly on that account that (BV) was so little studied.

In I Adams introduced for (BV) the new metric

$$(x, y) = \int_0^1 | x(t) - y(t) | \, dt + | T_0^1(x) - T_0^1(y) |, \tag{2}$$

giving reasons for this choice. These reasons will not be repeated here, but by way of partial motivation one might make two observations: (i) an arbitrary function in (AC) can be approximated (arbitrarily closely) in the metric (1) by an inscribed polygonal function, but no function in the class $(BV)-(AC)$ can be so approximated;‡ (ii) the metric (2) permits such approximation to an arbitrary function in (BV). Hereinafter, unless otherwise specified, (BV) shall always be understood to be metrized with (2); although not then a Banach space, it is boundedly compact and contains a countable dense set of polygonal functions (see I).

Presumably the most important subsets of (BV) are (CBV), the set of continuous functions of bounded variation, and (AC). With the metric (1), (AC) is a closed linear manifold in (CBV) which in turn is a like manifold in (BV), so it may be inferred at once that (AC) is of first category in (CBV) and of second category in itself, while (CBV) is of first category in (BV) and of second category in itself. When the metric (2) is employed, however, the

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† For references see Adams, *The space of functions of bounded variation and certain general spaces*, these Transactions, vol. 40 (1936), pp. 421–438; later referred to as I.

‡ For an indication of this fact see Adams and Lewy, *On convergence in length*, Duke Mathematical Journal, vol. 1 (1935), pp. 19–26; especially p. 23. This paper will be referred to as AL.

questions of category appear to be less trivial, and one of the main objects of this paper is to answer them.

As a preliminary we set forth in §2 a lemma on semi-continuity of the functional $T(x)$. In §1 of I we compared several kinds of convergence and indicated by example that certain kinds do not imply certain others;* in each case the example was one in which the limit function has an external saltus. That only such an example will serve this purpose will appear among the consequences of the lemma in §2, and of an analogous lemma in §5. In §3 we consider the questions of category just mentioned. With (DBV) designating the set of discontinuous functions of bounded variation and (CS) the set of continuous singular functions,† the critical issues in §3 are met by showing that (DBV) is the sum of sets each closed and non-dense in (BV) and that $(CBV)-(CS)$ is the sum of sets each closed and non-dense in (CBV). In §4 we determine a base for (BV). Finally, in §5, we examine similar questions which arise when $T(x)$ is replaced by $L(x)$, the Peano length of $x(t)$.

2. **Semi-continuity of $T(x)$, with applications.** A short time ago Adams and Clarkson‡ observed the rather obvious but none the less fundamental semi-continuity relation: $\lim_{n\to\infty} x_n(t) = x(t)$ *everywhere on*§ $(0, 1)$ *implies*‖ $\liminf_{n\to\infty} T(x_n) \geqq T(x)$. Although it was not expressly so stated, this result holds for $T(x) = \infty$ as well as for $T(x) < \infty$. Our first object here is to obtain a similar result when $x_n(t)$ is assumed to converge to $x(t)$ only on a dense set in $0 \leqq t \leqq 1$.

To avoid possible misunderstanding let us first set up the

DEFINITION. An arbitrary function $x(t)$ will be said to have *no external saltus* if and only if at each point t_1, $0 \leqq t_1 \leqq 1$, $x(t_1)$ satisfies the condition $\liminf_{t\to t_1} x(t) \leqq x(t_1) \leqq \limsup_{t\to t_1} x(t)$.

The class of functions having no external saltus will be designated by (N); its intersection with (BV), by (BVN). Clearly $x(t)\varepsilon(BVN)$ implies continuity of $x(t)$ at $t=0$ and $t=1$.

LEMMA 1. *The relation* $x_n(t)\to x(t)$ *on a dense set in* $(0, 1)$, *with* $x(t)\varepsilon(BVN)$, *implies* $\liminf_{n\to\infty} T(x_n) \geqq T(x)$.

The reader should have little difficulty in establishing this lemma.

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* See p. 423 of I.

† See, for example, Saks, *Théorie de l'Intégrale*, Warsaw, 1933, pp. 11 ff.

‡ Adams and Clarkson, *On convergence in variation*, Bulletin of the American Mathematical Society, vol. 40 (1934), pp. 413–417; to be referred to as AC.

§ The notation (a, b) for an interval will always mean the *closed* interval $a \leqq t \leqq b$.

‖ In general we employ the symbol $T_a^b(x)$ for the total variation of $x(t)$ on $a \leqq t \leqq b$; in particular the limits a, b will be omitted when it seems that no confusion can arise therefrom.

Although we do not intend to make use of it here, we may assert for the case $T(x)=\infty$

LEMMA 1′. *The relation* $x_n(t)\to x(t)$ *on a dense set in* $(0, 1)$, *when* $T(x)=\infty$ *but* $x(t)\varepsilon(N)$ *and the set* E *of its discontinuities of the second kind is either vacuous or possesses an isolated point,* implies* $\lim_{n\to\infty}T(x_n)=\infty$.

Proof. For Case 1, E vacuous, a proof can be constructed essentially in the same manner as for Lemma 1. In Case 2, E not vacuous, let t_1 be an isolated point of E. If for example $x(t_1-0)$ fails to exist, there must be an interval (t_0, t_1) which contains no discontinuity of the second kind other than t_1; and $T_{t_0}^{t_1}(x)=\infty$. If $T_{t_2}^{t_1}(x)=\infty$ for any t_2, $t_0<t_2<t_1$, the conclusion follows from Case 1; otherwise, for arbitrary M there exists t_2 such that $T_{t_2}^{t_1}(x)>M$ and the conclusion is a consequence of Lemma 1.

That Lemma 1 fails to hold when $x(t)$ is permitted to have an external saltus at even a single point is clear from the simplest examples. That Lemma 1′ ceases to be valid when E has no isolated point is shown by $x_n(t)=0$ for all t, all n; $x(t)$ the characteristic function of the rationals.

The two following lemmas are immediate consequences.

LEMMA 2. *The relation* $\int_0^1|x_n(t)-x(t)|\,dt\to 0$, *with* $x(t)\varepsilon(BVN)$, *implies* $\liminf_{n\to\infty}T(x_n)\geqq T(x)$.

Proof. If the conclusion were false there would exist $\epsilon>0$ and a subsequence $x_p(t)$ such that $T(x_p)<T(x)-\epsilon$ for all p. This subsequence, however, must contain a subsequence $x_q(t)$ convergent almost everywhere to $x(t)$; hence by Lemma 1 we have $\liminf_{q\to\infty}T(x_q)\geqq T(x)$, a contradiction.

LEMMA 2′. *The relation* $\int_0^1|x_n(t)-x(t)|\,dt\to 0$, *when* $T(x)=\infty$ *but* $x(t)$ *is qualified as in Lemma* 1′, *implies* $\lim_{n\to\infty}T(x_n)=\infty$.

DEFINITION. We shall employ $x_n(t)-dv\to x(t)$ as an abbreviation for the two conditions: $x_n(t)\to x(t)$ on a dense set in $(0, 1)$, and $T(x_n)\to T(x)$.

THEOREM 1. *The relation* $x_n(t)-dv\to x(t)$ *on* $(0, 1)$ $[(x_n, x)\to 0]$, *with* $x(t)\varepsilon(BVN)$, *implies that relation*† *for every subinterval whose end-points are points of continuity of* $x(t)$.

Proof. We need only show that $T_0^a(x_n)\to T_0^a(x)$ whenever a is a point of continuity of $x(t)$. By Lemma 1 [Lemma 2] we have $\liminf T_0^a(x_n)\geqq T_0^a(x)$, $\liminf T_a^1(x_n)\geqq T_a^1(x)$; and $\limsup T_0^a(x_n)>T_0^a(x)$ would imply

* From the proof it is evident that this lemma can be strengthened a little by considering right- and left-hand limits.

† We repeat that (x, y) always refers to the metric (2) unless otherwise specified.

$$\lim T_0^1(x_n) \geqq \limsup T_0^a(x_n) + \liminf T_a^1(x_n) > T_0^a(x) + T_a^1(x) = T_0^1(x),$$

in contradiction to hypothesis.

That this theorem breaks down for subintervals having an end-point at a discontinuity of $x(t)$ is apparent from simple examples.

In I we remarked that "if $(x_n, x)\to 0$ and $x(t)$ is discontinuous at even a single point, $x_n(t)$ may fail to converge to $x(t)$ at every point." This can happen, however, only when $x(t)$ has an external saltus, as we see from

THEOREM 2. *The relation* $x_n(t) - dv \to x(t)$ *on* (0, 1) $[(x_n, x)\to 0]$, *with* $x_0(t)\varepsilon(BVN)$, *implies* $x_n(t)\to x(t)$ *at every point of continuity of* $x(t)$.

Proof. Let a be any such point of continuity. No loss of generality results from assuming $a>0$; for if $a=0$, we can deal with the interval (0, 1) as we now deal with $(0, a)$. Denial of the conclusion implies the existence of $\epsilon>0$ and a subsequence $x_p(t)$ such that $|x_p(a)-x(a)|>\epsilon$ for all p. But there exists $\delta>0$ such that

$$|\, x(t) - x(a)\,| \leqq T_t^a(x) = T_0^a(x) - T_0^t(x) < \epsilon/2 \text{ for } a-\delta < t < a;$$

and $a-\delta<t<a$ contains a point t_1 where $x(t)$ is continuous. Hence, in view of Theorem 1, we have $T_{t_1}^a(x_p)\to T_{t_1}^a(x)<\epsilon/2$. But the interval $t_1<t<a$ contains a point t_2 for which $x_p(t_2)\to x(t_2)$, and we have for p sufficiently large

$$T_{t_1}^a(x_p) \geqq T_{t_2}^a(x_p) \geqq |\, x_p(t_2) - x_p(a)\,| > \epsilon/2.$$

From this contradiction the theorem is to be inferred.

COROLLARY 1. *For* $x(t)\varepsilon(BVN)$, *the relations* $x_n(t) - dv \to x(t)$ *and* $(x_n, x)\to 0$ *are equivalent.*

That the first condition implies the second is seen by aid of Theorem 2, the uniform boundedness of $x_n(t)$, and Lebesgue's convergence theorem. The reverse implication is an immediate consequence of Theorem 2.

DEFINITION. For $x(t)\varepsilon(BV)$ we define an *associated function* $\bar{x}(t)$ thus: $\bar{x}(t)=x(t+0)$ for $0\leqq t<1$, $\bar{x}(1)=x(1-0)$.

DEFINITION. For $x(t)\varepsilon(BV)$ we define the *sum of its external salluses* as $S(x)=T(x)-T(\bar{x})$.

If at an interior point t_1 of (0, 1) the function $x(t)$ has an external saltus, the magnitude of the "external part" of the saltus (or briefly, the external saltus) is commonly defined* as

$$\min\,[\,|\, x(t_1) - x(t_1 - 0)\,|,\ |\, x(t_1) - x(t_1 + 0)\,|\,].$$

* See Hobson, *The Theory of Functions of a Real Variable*, 3d edition, vol. 1, Cambridge, 1927, p. 334.

Using two results given by Hobson* one may show without difficulty that

$$S(x) = 2\sum(\text{external saltuses at interior points}) + |x(0) - x(0+0)| + |x(1) - x(1-0)|.$$

THEOREM 3. *The relation* $(x_n, x)\to 0$, *with* $x(t)\varepsilon(BVN)$, *implies* $(\bar{x}_n, x)\to 0$ *and* $S(x_n)\to 0$.

Proof. Since $\bar{x}_n(t) = x_n(t)$ almost everywhere, we have $\int_0^1 |\bar{x}_n(t) - x(t)|\,dt \to 0$; whence, by Lemma 2, $\liminf T(\bar{x}_n) \geqq T(x)$. But from $T(x_n) = T(\bar{x}_n) + S(x_n) \to T(x)$, $S(x_n) \geqq 0$, we infer $T(\bar{x}_n)\to T(x)$ and $S(x_n)\to 0$.

That $(x_n, x)\to 0$ with $x(t)\varepsilon(BV) - (BVN)$ in general fails to imply $S(x_n)\to S(x)$ is evident from simple examples.

THEOREM 4. *The relation* $(x_n, x)\to 0$, *with* $S(x_n) \geqq k > 0$ *for all* n, *implies* $S(x) \geqq k$.

Proof. As in the proof of Theorem 3 we have $\liminf T(\bar{x}_n) \geqq T(\bar{x})$. But the relations

$$T(x) = T(\bar{x}) + S(x) = \lim T(x_n) \geqq \liminf T(\bar{x}_n) + \liminf S(x_n) \geqq \liminf T(\bar{x}_n) + k$$

then imply $k \leqq S(x) + T(\bar{x}) - \liminf T(\bar{x}_n) \leqq S(x)$.

3. **Questions of category.** Let us recall from I that when $(x, y) = 0$ we say that x is *metrically equal* to y and write $x \cong y$. An element of the metric space (BV) is therefore a class of functions any two of which are metrically equal; for $x\varepsilon(CBV)$ the class consists of only one function, but for $x\varepsilon(DBV)$ it contains an infinite number of functions. Since category is essentially a metric property, it seems desirable now to consider (BV) merely as a *metric* space rather than as a *vector metric* space, the elements of which are single functions (see I).

From the definition of $\bar{x}(t)$ given above in §2 it is easily seen that if two functions $x(t)$, $y(t)$ satisfy the condition $(x, y) = 0$, then $\bar{x}(t) \equiv \bar{y}(t)$ on $(0, 1)$ and $S(x) = S(y)$. Hence it is proper to set up the following

DEFINITION. If $x\varepsilon(BVN)$ and $x(t)$ is an arbitrary function of the class constituting x, $\bar{x}(t)$ will be called the *representative of the element* x. If $x\varepsilon(DBV) - (BVN)$ and $x(t)$ is an arbitrary function of the class constituting x, any function $x(t) = \bar{x}(t)$ for $0 \leqq t \leqq 1$, $t \neq t_1$; $x(t_1) = \bar{x}(t_1) + k$, where t_1 is any point of $(0, 1)$ and k is so chosen that $|x(t_1) - x(t_1 - 0)| + |x(t_1) - x(t_1 + 0)|$ exceeds the saltus of $\bar{x}(t)$ at t_1 by $S(x)$, will be called a *representative of* x.

LEMMA 3. *If* $(x_n, x)\to 0$ *and for each* n *some representative* $x_n(t)$ *of* x_n *has a*

* Hobson, loc. cit., p. 335.

saltus $\geqq k>0$ *at one or more points of* (0, 1), *then some representative* $x(t)$ *of* x *will have at one or more points a saltus* $\geqq k$.

Proof. Let t_n be a point where $x_n(t)$ has a saltus $\geqq k$; let l be a limit point of the sequence t_n; and let the subsequence $t_p \to l$. In case $S(x)>0$ let $x(t)$ be the representative of x obtained by taking t_1 in the foregoing definition as l. Then, δ being positive but arbitrarily small, set $a_\delta = \underset{t}{E}(l-\delta \leqq t \leqq l+\delta) \cdot \underset{t}{E}(0 \leqq t \leqq 1)$ and designate by b_δ the closure of the complement of a_δ with respect to (0, 1). Since b_δ consists either of one closed subinterval of (0, 1), or of two disjoint intervals of this nature, on each of which $x(t)$ has no external saltus, we have $\liminf_{p\to\infty} T_{b_\delta}(x_p) \geqq T_{b_\delta}(x)$ by Lemma 1. Hence the relation

$$\begin{aligned}\limsup T_{a_\delta}(x_p) + \liminf T_{b_\delta}(x_p) &\leqq \lim \left[T_{a_\delta}(x_p) + T_{b_\delta}(x_p)\right] = \lim T_0^1(x_p) \\ &= T_0^1(x) = T_{a_\delta}(x) + T_{b_\delta}(x)\end{aligned}$$

implies

$$\limsup T_{a_\delta}(x_p) \leqq T_{a_\delta}(x) + T_{b_\delta}(x) - \liminf T_{b_\delta}(x_p) \leqq T_{a_\delta}(x).$$

But since $t_p \to l$ we have $\limsup T_{a_\delta}(x_p) \geqq k$; whence $T_{a_\delta}(x) \geqq k$ for all $\delta>0$ and $\lim_{\delta\to 0} T_{a_\delta}(x) \geqq k$. And this last limit is precisely the saltus of $x(t)$ at l.

THEOREM 5. (*DBV*) *is the sum of a countable number of sets each closed and non-dense in* (*BV*).

Proof. Let E_m be the set of points $x \varepsilon (DBV)$ each of which has a representative $x(t)$ possessing at some point a saltus $\geqq 1/m$. Clearly $(DBV) = \sum_{m=1}^{\infty} E_m$. By Lemma 3 each E_m is closed in (*BV*). That each E_m is non-dense in (*BV*) now follows from the fact (see I, §1) that (*CBV*) is dense in (*BV*).

Since (*BV*), being complete (see I, §1), is of second category in itself, (*CBV*) is of second category in (*BV*), and we have in the terminology of Denjoy

COROLLARY 2. (*CBV*) *is a residual set in* (*BV*).

COROLLARY 3. (*CBV*) *is of second category in itself.*

COROLLARY 4. (*DBV*) *is of first category in itself.*

Otherwise, for the decomposition $(DBV) = \sum_{m=1}^{\infty} E_m$ employed in proving Theorem 5, the closure in (*DBV*) of at least one set E_m would contain a sphere $K(x_0, r) \subset (DBV)$. Since (*DBV*) is dense in (*BV*), the closure in (*BV*) of this set E_m would then contain the corresponding sphere $K_1(x_0, r) \subset (BV)$, thus yielding a contradiction.

We next consider the set (*CS*) of continuous singular functions; i.e., of

functions $x(t)\varepsilon(CBV)$ for which the derivative $x'(t)$ vanishes almost everywhere* on the interval $(0, 1)$.

THEOREM 6. *$(CBV)-(CS)$ is the sum of a countable number of sets each closed and non-dense in (CBV).*

Proof. For brevity we call the sum of a finite number of disjoint closed intervals $a_i \leqq t \leqq b_i$ $(i=1, 2, \cdots, p)$ an elementary figure, designate such a set generically by R and its measure by $|R|$, and let $T_R(x)$ stand for $\sum_{i=1}^{p} T_{a_i}^{b_i}(x)$.

For each integer $m>0$ we define the subset A_m of (AC) as follows: $\alpha(t)\varepsilon A_m$ if and only if we have

$$T_0^1(\alpha) \geqq 1/m, \qquad \text{and} \qquad T_R(\alpha) \leqq \tfrac{1}{2} T_0^1(\alpha)$$

for every elementary figure $R \subset (0, 1)$ with $|R| \leqq 1/m$. Then the subset E_m of $(CBV)-(CS)$ we define thus: $x(t)\varepsilon E_m$ if and only if

$$\alpha(t)\varepsilon A_m, \qquad \text{where} \qquad \alpha(t) = \int_0^t x'(s)ds.$$

Denoting by $\overline{E}_m$ $(m=1, 2, 3, \cdots)$ the closure of E_m in (CBV), we propose to show that

$$(CBV) - (CS) = \sum_{m=1}^{\infty} \overline{E}_m.$$

It is clear that each point of $(CBV)-(CS)$ belongs to E_m for some m. From the fact (see I) that the polygonal functions are dense in (CBV), it is easily apparent that (CS) is likewise. Therefore if for some $m>0$, $\overline{E}_m$ were not in $(CBV)-(CS)$, there would exist a point $y\varepsilon(CS)$ and a sequence $x_n\varepsilon E_m$ such that $(x_n, y)\to 0$. Let $\alpha_n(t)=\int_0^t x_n'(s)ds$ and $\beta_n(t)=x(t)-\alpha_n(t)$. The relation $y\varepsilon(CS)$ implies the existence of an elementary figure R such that

$$T_R(y) \geqq T_0^1(y) - \frac{1}{4m}, \qquad |R| \leqq \frac{1}{m}.$$

In view of the corollary to Theorem 2 of AC and of the fact that $T(x_n) = T(\alpha_n)+T(\beta_n)$ on every elementary figure, we should then have

$$2T_0^1(y) - \frac{1}{2m} \leqq 2T_R(y) = \lim 2T_R(x_n) = \lim \left[2T_R(\alpha_n) + 2T_R(\beta_n)\right]$$
$$\leqq \liminf \left[T_0^1(\alpha_n) + 2T_0^1(\beta_n)\right]$$

* See, for example, Saks, loc. cit. If one chooses to exclude the functions $x(t)=$const. from the set (CS), the proof of Theorem 6 can easily be modified to show that $(CBV)-(CS)$ is still the sum of non-dense closed sets in (CBV).

$$\leqq \limsup\,[2T_0^1(\alpha_n) + 2T_0^1(\beta_n)] - \liminf\, T_0^1(\alpha_n)$$

$$= \limsup\, 2T_0^1(x_n) - \liminf\, T_0^1(\alpha_n) \leqq 2T_0^1(y) - \frac{1}{m},$$

a contradiction which implies the decomposition above stated. That $\overline{E}_m$ $(m=1, 2, 3, \cdots)$ is non-dense in (CBV) follows from the fact that (CS) is dense in (CBV).

From Theorem 6 and Corollary 3 follows

COROLLARY 5. *(CS) is a residual set in (CBV).*

COROLLARY 6. *(CS) is of second category in itself.*

An immediate consequence of Theorem 6 is

COROLLARY 7. *(AC) is of first category in (CBV).*

Since (AC) is dense in (CBV), reasoning of the sort used in proving Corollary 4 may be employed to prove

COROLLARY 8. *(AC) is of first category in itself.*

A complete classification of functions of bounded variation according to external saltus, continuity, absolute continuity, and singularity leads to a decomposition of (BV) into the following seven disjoint sets each of which is easily seen to be dense in (BV): $(DBV)\cdot(S)\cdot(N)$, $(DBV)\cdot(BV-S)\cdot(N)$, $(DBV)\cdot(S)\cdot(BV-BVN)$, $(DBV)\cdot(BV-S)\cdot(BV-BVN)$, (AC), (CS), and $(CBV-AC-CS)$. Combining these sets in all possible ways one obtains 126 distinct proper subsets of (BV) concerning each of which several questions of category and Borel character can be raised. Many of these combinations appear decidedly artificial;* nevertheless it may be worthwhile to point out that all questions of category concerning them can now be answered. For in proving Corollary 4 we have essentially established the lemma: $Q \subset R$ and Q dense in R imply that the category of Q in R is the same as the category of Q in itself. By repeated application of this lemma we infer from the above results that (CS) is a residual set in (BV). Hence *any one of the* 126 *subsets which contains* [*does not contain*] (CS) *is of second* [*first*] *category in* (BV), *in itself, and in any other of the subsets in question of which it is a part.*

That (AC) is not a G_δ in (CBV) may readily be inferred; but whether it is an F_σ remains among the numerous open questions concerning Borel character. We may remark, however, that not only (CBV) but also (CS) is a G_δ in

* Curiously enough, our first application of the results of this paper will involve one of these "artificial" combinations.

(BV) and is therefore homeomorphic to a complete metric space, according to a theorem of Alexandroff.*

4. **Bases.** A sequence of elements x_i of a vector metric space S is called a base for S if and only if to an arbitrary element $x \in S$ there corresponds a unique sequence of constants c_i such that $(\sum_{i=1}^{n} c_i x_i, x) \to 0$ with $1/n$. For the space (C) of continuous functions Schauder† has constructed a base consisting of a sequence of polygonal functions. His argument makes clear that *this sequence also constitutes a base for both* (AC) *and* (CBV) with our metric; that it is *not* a base for either (BV) or (BVN), however, may readily be seen.

In contrast with the point of view in §3, we must now regard (BV) and (BVN) as *vector metric* spaces, whose elements are single functions. As a first step toward the determination of a base for (BV) we shall prove that Haar's system of orthogonal functions‡ serves as a base for (BVN). We recall that each Haar function $x_i(t)$ is continuous at the end points of $(0, 1)$ and that otherwise $x_i(t) = [x_i(t-0) + x_i(t+0)]/2$, whence $x_i \in (BVN)$ and any linear combination $\sum_{i=1}^{n} c_i x_i \in (BVN)$.

Now let x be an arbitrary element of (BVN). The function $x(t)$ is one of a set which constitutes an element of the space (L) of Lebesgue integrable functions. Since the Haar system provides a base for (L), there exists a unique sequence of constants c_i satisfying the condition, $\int_0^1 |\sum_{i=1}^{n} c_i x_i - x| \, dt \to 0$ with $1/n$; and these c_i are the coefficients in the Fourier expansion of $x(t)$ in the Haar functions.§ But the partial sum $s_n(t)$ of this expansion is a step-function which, on each of its intervals of constancy, is the mean value of $x(t)$ on that interval.‖ From the following lemma the fact that $T(s_n) \to T(x)$, whence $(\sum_{i=1}^{n} c_i x_i, x) \to 0$, is to be inferred.

LEMMA 4. *If* $x(t) \in (BVN)$ *and* α_{pq} *is any set of numbers satisfying the conditions* $m_{pq} \leqq \alpha_{pq} \leqq M_{pq}$ $(p = 1, 2, \cdots, 2^q; q = 0, 1, 2, \cdots)$, *where*

$$\left.\begin{aligned} m_{pq} &= \inf\ x(t) \\ M_{pq} &= \sup\ x(t) \end{aligned}\right\} \text{ for } (p-1)/2^q \leqq t \leqq p/2^q;$$

then $\sum_{p=1}^{2^q} |\alpha_{p+1,q} - \alpha_{pq}| \to T_0^1(x)$ *as* $q \to \infty$.

This is a consequence of a theorem of Hobson, loc. cit., p. 332.

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* See, for example, Kuratowski, *Topologie* I, Warsaw, 1933, p. 200.

† Schauder, *Zur Theorie stetiger Abbildungen in Funktionalräumen*, Mathematische Zeitschrift, vol. 26 (1927), pp. 47–65; especially pp. 48–49.

‡ Haar, *Zur Theorie der orthogonalen Funktionensysteme*, Mathematische Annalen, vol. 69 (1910), pp. 331–371; especially pp. 361 ff.

§ See Schauder, loc. cit., pp. 50–51.

‖ See Haar, loc. cit., p. 367.

If to Haar's system of orthogonal functions we add the function $x_0(t)=0$ *for* $0\leqq t<1$, $=1$ *for* $t=1$, *we obtain a base for* (BV). For an arbitrary function $x(t)\varepsilon(BV)$ is metrically equal to $\bar{x}(t)+s(t)$, where $s(t)=0$ for $0\leqq t<1$, $=S(x)$ for $t=1$. From the above reasoning it is clear that constants c_i $(i=1, 2, 3, \cdots)$ can be uniquely determined so that $(\sum_{i=1}^{n} c_i x_i, \bar{x})\to 0$ with $1/n$, and from the relation $T(\sum_{i=0}^{n} c_i x_i)=T(\sum_{i=1}^{n} c_i x_i)+T(c_0 x_0)=T(\sum_{i=1}^{n} c_i x_i)+c_0$ it follows that with and only with $c_0=S(x)$ shall we have $(\sum_{i=0}^{n} c_i x_i, x)\to 0$. The uniqueness of the entire sequence c_i $(i=0, 1, 2, \cdots)$ may now be deduced without difficulty.

The question as to whether (BV), or perhaps more likely (BVN), has a base consisting wholly of continuous functions remains open.

5. **Concerning the functional** $L(x)$. We now consider the question of analogues of the above results when $T(x)$ is replaced by $L(x)$, the Peano length of $x(t)$, and the metric (2) by

$$(3)\qquad (x, y)=\int_0^1 |\, x(t)-y(t)\,|\, dt+|\, L_0^1(x)-L_0^1(y)\,|.$$

The reader should have no considerable difficulty in establishing for $L(x)$ the analogue of Lemma 1, which provides a basis for drawing conclusions parallel to those described in §§2, 3 prior to Theorem 6. Although the employment of the metric (3) leaves unaltered the category and Borel properties set forth in Theorem 5 and its corollaries, it does effect a change on the subsets of (CBV). Employing as in the proof of Theorem 6 the notation

$$x_n=\alpha_n+\beta_n,\quad x=\alpha+\beta;\quad \alpha_n,\ \alpha\varepsilon(AC);\quad \beta_n,\ \beta\varepsilon(CS),$$

we have

LEMMA 5. *The relation* $(x_n, x)\to 0$, *with* $T(\beta_n)\geqq k>0$ *for all* n, *implies* $T(\beta)\geqq k$.

Proof. According to I, §1 we have in the notation of "convergence in length" $x_n-l\to x$, $\alpha-l\to\alpha$; from a theorem of Morse* one may then infer $(x_n-\alpha)-l\to(x-\alpha)$. By Theorem 1 of AL it follows that $(x_n-\alpha)-v\to\beta$, in the notation of "convergence in variation." Hence we have $T(x_n-\alpha)=T(\alpha_n-\alpha)+T(\beta_n)\to T(\beta)$, and from $T(\alpha_n-\alpha)\geqq 0$ the conclusion follows.

THEOREM 7. $(CBV)-(AC)$ *is the sum of a countable number of sets each closed and non-dense in* (CBV).

Proof. Allowing E_m to denote the set of points $x\varepsilon(CBV)-(AC)$ each of

* Morse, *Convergence in variation and related topics*, these Transactions, vol. 41 (1937), pp. 48–83, Theorem 5.4.

which satisfies the condition $T(\beta) \geqq 1/m$, we clearly have $(CBV) - (AC) = \sum_{m=1}^{\infty} E_m$. By Lemma 5 each E_m is closed; since with the metric (3) the set (AC) is dense* in (CBV), each E_m is also non-dense.

COROLLARY 9. *(AC) is a residual set in (CBV), and is of second category in itself.*

COROLLARY 10. *(CS) is of first category in (CBV), and is of second category in itself.*

Proof. In fact (CS) is a closed non-dense set† in (CBV). For, by Theorem 1 of AL the relation $x_n - l \to x$ implies $x_n - v \to x$; therefore, if $x_n \epsilon (CS)$ for all n, we have for all t

$$L_0^t(x_n) = L_0^t(0) + T_0^t(x_n) = t + T_0^t(x_n) \to L_0^t(x) = t + T_0^t(x),$$

whence $\{1 + [x'(t)]^2\}^{1/2} = 1 + |x'(t)|$ almost everywhere and $x'(t) = 0$ almost everywhere; so that if $x \epsilon (CBV)$, it also is in (CS).

It is easily seen by example, however, that (CS) has limit points in (DBV). Let $(\overline{CS})$ be the closure of (CS) in (BV). Since $(\overline{CS})$ is a closed set in a complete space (see I, §1), it is of second category in itself. We consider then $(\overline{CS}) - (CS) \subset (DBV)$. Let $(DBV) = \sum_{m=1}^{\infty} E_m$ be the decomposition used in showing (DBV) to be of first category in (BV), i.e., the analogue for $L(x)$ of Theorem 5, and let $G_m = E_m \cdot [(\overline{CS}) - (CS)]$. Since for each m the set E_m has no limit point in (CBV), G_m has no limit point in (CS). But (CS) is dense in $(\overline{CS})$; therefore $\overline{G}_m$, the closure of G_m in $(\overline{CS})$, contains no sphere of $(\overline{CS})$. Thus $(\overline{CS}) - (CS)$ is of first category in $(\overline{CS})$, and (CS) is a residual set in $(\overline{CS})$. (CS) is therefore of second category in itself.

By aid of Theorem 5.2 of Morse (loc. cit.) it can readily be proved that (CS) is dense in (S), so that $(\overline{CS}) = (\overline{S})$; reasoning similar to that of the preceding paragraph then shows that (S) is of second category in itself. That (S) is of first category in (BV) is easily seen. It may be remarked that (AC) is a residual set in (BV); that, of the 126 sets mentioned at the close of §3, any combination including (AC) [not including (AC) and not contained in (S)] is of second [first] category in (BV), in itself, and in any other combination of which it is a part; and that most, if not all, of the questions concerning the category of (S) and its subsets can be answered by the aid of results now in hand. We observe also that not only (CBV) but also each of the sets (AC), (CS), and $(CBV) - (CS)$ is a G_δ in (BV) and therefore homeomorphic to a complete metric space.

* See I, §1.

† By aid of Theorem 8 below one may show similarly that (SN) is a closed non-dense set in (BVN).

With the metric (3), Schauder's base for (*C*) is again a base for (*AC*) and for (*CBV*); and by aid of Theorem 5.2 of Morse (loc. cit.) it follows that the base obtained above in §4 for (*BV*) with the metric (2) is also a base for (*S*) with the metric (3). But Haar's system of orthogonal functions is not a base for (*BVN*), and the question of bases for (*BVN*) and for (*BV*) we leave open.

One may use the notation $x_n(t) - dl \rightarrow x(t)$ on (0, 1) to stand for the pair of conditions $x_n(t) \rightarrow x(t)$ on a dense set in (0, 1) and $L(x_n) \rightarrow L(x)$. It is natural to inquire whether $x_n - dl \rightarrow x$ implies $x_n - dv \rightarrow x$. This question may be answered with the aid of the following lemma, whose proof will be left to the reader.

LEMMA 6. *For $x(t) \in (BVN)$ it is possible to approximate $x(t)$ in length by an inscribed broken line whose "corners" are at points of continuity of $x(t)$.*

THEOREM 8. *The relation $x_n(t) - dl \rightarrow x(t)$ on (0, 1), with $x(t) \in (BVN)$, implies $x_n(t) - dv \rightarrow x(t)$.*

Proof. By Lemma 1 we have $\liminf T(x_n) \geqq T(x)$. For given $\epsilon > 0$ there exists a broken line B of the sort specified in Lemma 6 such that $L(B) > L(x) - \epsilon$. By aid of the analogue for $L(x)$ of Theorem 1 we may now pursue the argument used in establishing Theorem 1 of AL to complete the proof here.

In Theorem 8 *the qualification $x(t) \in (BVN)$ cannot be dispensed with,* as the following example shows: $x(t) = 0$ for $0 \leqq t < \frac{1}{2}$, $= 1$ for $t = \frac{1}{2}$, $= \frac{1}{2}$ for $\frac{1}{2} < t \leqq 1$; $x_n(t) = \frac{1}{2}$ for $\frac{1}{2} < t \leqq 1$ and all n; on $(0, \frac{1}{2})$, $x_n(t)$ a sequence of polygonal functions all having the same length $\frac{3}{2}$ and the same total variation $2^{1/2}$, converging uniformly to $x(t) = 0$ and with $x_n(0) = x_n(\frac{1}{2}) = 0$ for all n. We observe that an example of the same type shows that when $x(t)$ has an external saltus at even one point, the condition $x_n(t) \rightarrow x(t)$ on a dense set in (0, 1), plus semi-continuity of $T(x)$ at x, *does not imply* semi-continuity of $L(x)$ at x.

We conclude with the following

COROLLARY 11. *The condition $(x_n, x) \rightarrow 0$ in the metric (3), with $x \in (BVN)$, implies that condition in the metric (2).*

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ABSTRACT DERIVATION AND LIE ALGEBRAS*

BY

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The purpose of this paper is the investigation of the algebraic properties of the set of operations mapping an algebra on itself and having the formal character of derivation in the field of analytic functions. Some of the results obtained are analogous to well-known theorems on automorphisms of algebras.‡ The considerations in I are general and quite elementary. In II and III we restrict ourselves to the derivations of an associative algebra having a finite basis and in the main to semi-simple algebras. A number of results of the theory of algebras are presupposed. These may be found in Deuring's *Algebren*, Springer, 1935.

I. Derivations in an arbitrary algebra

1. Let $\mathfrak{R}$ be an arbitrary algebra (hypercomplex system not necessarily commutative or associative, or of finite order) over a commutative field $\mathfrak{F}$. Then $\mathfrak{R}$ is a vector space (with elements $x, y, \cdots$) over $\mathfrak{F}$ (with elements $\alpha, \beta, \cdots$) in which a composition $xy \in \mathfrak{R}$ is defined such that

$$(1)\quad (x+y)z = xz + yz,\quad z(x+y) = zx + zy,\quad (xy)\alpha = (x\alpha)y = x(y\alpha).$$

A *derivation* D of $\mathfrak{R}$ is a single valued mapping of $\mathfrak{R}$ on itself such that

$$(2)\ \text{(a)}\ (x+y)D = xD + yD,\ \text{(b)}\ (x\alpha)D = (xD)\alpha,\ \text{(c)}\ (xy)D = (xD)y + x(yD).$$

Thus D is a linear transformation in the vector space $\mathfrak{R}$ satisfying the special condition (2c). It is well known that the sum D_1+D_2, difference D_1-D_2, scalar product $D\alpha$ and product D_1D_2 (defined respectively by $x(D_1 \pm D_2) \equiv xD_1 \pm xD_2$, $x(D\alpha) \equiv (xD)\alpha$, $x(D_1D_2) \equiv ((xD_1)D_2)$ of linear transformations are linear transformations. If D, D_1, D_2 are derivations we have besides

$$(3)\quad \begin{aligned}(xy)(D_1 \pm D_2) &= (xy)D_1 \pm (xy)D_2 = (xD_1)y + x(yD_1) \pm (xD_2)y \pm x(yD_2)\\ &= (x(D_1 \pm D_2))y + x(y(D_1 \pm D_2)),\end{aligned}$$

$$(4)\quad (xy)D\alpha = ((xy)D)\alpha = ((xD)y + x(yD))\alpha = (xD\alpha)y + x(yD\alpha),$$

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‡ A direct connection between derivations and automorphisms may sometimes be established. For example if $\mathfrak{R}$ is the ring of polynomials $\mathfrak{F}[x]$ where $\mathfrak{F}$ is a field of characiterstic 0, and D is defined by $f(x)D=f'(x)$ the usual derivative then $\exp D=1+D+D^2/2!+\cdots$ is an automorphism since $f(x)\exp D=f(x+1)$.

$$(5)\qquad \begin{aligned}(xy)D_1D_2 &= ((xy)D_1)D_2 = ((xD_1)y + x(yD_1))D_2\\ &= (xD_1D_2)y + x(yD_1D_2) + (xD_1)(yD_2) + (xD_2)(yD_1).\end{aligned}$$

Thus $D_1 \pm D_2$, $D\alpha$ are derivations, but not in general D_1D_2. However (5) shows that the commutator $[D_1, D_2] = D_1D_2 - D_2D_1$ does satisfy (2c) and so is a derivation. We recall the relations

$$(6)\quad [D_1, D_2] = -[D_2, D_1],\ [[D_1, D_2], D_3] + [[D_2, D_3], D_1] + [[D_3, D_1], D_2] = 0.$$

As a consequence of (2) we have Leibniz's formula:

$$(7)\quad (xy)D^k = (xD^k)y + C_{k,1}(xD^{k-1})(yD) + C_{k,2}(xD^{k-2})(yD^2) + \cdots + x(yD^k).$$

Hence if $\mathfrak{F}$ has characteristic $p \neq 0$ we have

$$(8)\qquad (xy)D^p = (xD^p)y + x(yD^p);$$

i.e., D^p is a derivation.

By *a restricted Lie algebra of linear transformations* we shall mean a system of linear transformations closed relative to the operations of addition, subtraction, scalar multiplication, commutation, and taking pth powers, if p ($=0$ or a prime) is the characteristic of the field over which the vector space is defined.* With this definition we have

THEOREM 1. *The derivations of an algebra $\mathfrak{R}$ over $\mathfrak{F}$ constitute a restricted Lie algebra $\mathfrak{D}$ of linear transformations in $\mathfrak{R}$.*

We call $\mathfrak{D}$ the *derivation algebra* or, more briefly, the *d-algebra* of $\mathfrak{R}$ over $\mathfrak{F}$. It should be noted that we are regarding $\mathfrak{D}$ as an algebra over $\mathfrak{F}$.

2. Suppose $D, E, D_1, D_2, \cdots$ are elements of any associative algebra $\mathfrak{A}$. As a generalization of the multinomial theorem in a commutative algebra we have

$$(9)\qquad (D_1 + D_2 + \cdots + D_r)^k = \sum \begin{Bmatrix} D_1 & D_2 & \cdots & D_r \\ j_1 & j_2 & \cdots & j_r \end{Bmatrix},$$

where the summation is extended over $j_1, \cdots, j_r$ such that $j_\alpha \geqq 0$ and $j_1 + \cdots + j_r = k$ and where $\{D_1 \cdots D_r / j_1 \cdots j_r\}$ denotes the sum of the $(j_1 + \cdots + j_r)!/(j_1! \cdots j_r!)$ terms obtained by multiplying j_1 of the D_1's, j_2 of the D_2's, $\cdots$, j_r of the D_r's together in every possible order. Let $D_{i_1} + D_{i_2} + \cdots + D_{i_s} = D_{i_1 i_2 \cdots i_s}$ where $i_1, i_2, \cdots, i_s$ are distinct and have values in the range $1, 2, \cdots, k$. Consider

$$D^k_{i_1 \cdots i_k} - \sum_C D^k_{i_1 \cdots i_{k-1}} + \cdots + (-1)^{k-1} \sum_C D^k_{i_1} = Q,$$

* We use the convention $D^0 = 0$.

where $\sum_C D^k_{i_1 \cdots i_k}$ denotes the sum of the $C_{k,s}$ terms obtained by letting $i_1, \cdots, i_s$ run through all the combinations of $1, 2, \cdots, k$ taken s at a time. By (9), Q is a sum of terms of the form $\left\{D_{m_1} \cdots D_{m_t}/j_1 \cdots j_t\right\}$ where $j_\alpha > 0$ and $j_1 + j_2 + \cdots + j_t = k$. Since

$$\begin{Bmatrix} D_{m_1} \cdots D_{m_t} \\ j_1 \quad \cdots j_t \end{Bmatrix} = \begin{Bmatrix} D_{m_1} \cdots D_{m_t} & D_{n_1} \cdots D_{n_r} \\ j_1 \quad \cdots j_t & 0 \quad \cdots 0 \end{Bmatrix},$$

where $n_1, n_2, \cdots, n_r$ are distinct indices different from $m_1, m_2, \cdots, m_r$, the term $\left\{D_{m_1} \cdots D_{m_t}/j_1 \cdots j_t\right\}$ has the coefficient $C_{k-t,r}$ in $\sum_C D^k_{i_1 \cdots i_{t+r}}$ and hence the coefficient of this term in Q is

$$C_{k-t,k-t} - C_{k-t,k-t-1} + \cdots + (-1)^{k-t} C_{k-t,0} = \delta_{kt},$$

i.e., $=0$ or 1 according as $k \neq t$ or $k = t$. Hence

$$D^k_{i_1 \cdots i_k} - \sum_C D^k_{i_1 \cdots i_{k-1}} + \cdots + (-1)^k \sum_C D^k_{i_1} = \begin{Bmatrix} D_1 \cdots D_k \\ 1 \quad \cdots 1 \end{Bmatrix}. \tag{10}$$ *

Since

$$\begin{Bmatrix} D_1 + \cdots + D_r & D \\ k & 1 \end{Bmatrix} = \sum \begin{Bmatrix} D_1 \cdots D_r & D \\ j_1 \quad \cdots j_r & 1 \end{Bmatrix},$$

where $j_\alpha \geqq 0$ and $j_1 + \cdots + j_r = k$, we may derive the following formula similar to (10):

$$\begin{Bmatrix} D_{i_1 \cdots i_k} & D \\ k & 1 \end{Bmatrix} - \cdots + (-1)^{k-1} \sum_C \begin{Bmatrix} D_{i_1} & D \\ k & 1 \end{Bmatrix} = \begin{Bmatrix} D_1 \cdots D_k & D \\ 1 \quad \cdots 1 & 1 \end{Bmatrix}. \tag{11}$$

If in (10) and (11) we set j_1 of the D's equal to D_1, j_2 equal to D_2, $\cdots$, j_t equal to D_t then $\left\{D_1 \cdots D_k/1 \cdots 1\right\}$ and $\left\{D_1 \cdots D_k D/1 \cdots 11\right\}$ become respectively $(j_1! \cdots j_t!)\left\{D_1 \cdots D_t/j_1 \cdots j_t\right\}$ and $(j_1! \cdots j_t!)\left\{D_1 \cdots D_t D/j_1 \cdots j_t 1\right\}$ and we obtain expressions for these as sums of kth powers and as sums of terms of the type $\left\{ED/k1\right\}$.

An analogue of (7) is

$$DE^k = E^k D + C_{k,1} E^{k-1} D' + \cdots + D^{(k)} \tag{12}$$

where $D' = [D, E], \cdots, D^{(i)} = [D^{(i-1)}, E]$. Hence

$$E^l D E^{k-l} = E^k D + C_{k-l,1} E^{k-1} D' + \cdots + C_{k-l,j} E^{k-j} D^{(j)} + \cdots + E^l D^{(k-l)},$$

and summing on $l = 0, 1, \cdots, k$ we have

* If we set $D_1 = D_2 = \cdots = D_k = 1$ in (10) we obtain the identity
$$k^k - C_{k,1}(k-1)^k + C_{k,2}(k-2)^k - \cdots + (-1)^{k-1} C_{k,k-1} 1^k = k!.$$

$$(13)\qquad \begin{Bmatrix} E & D \\ k & 1 \end{Bmatrix} = C_{k+1,1}E^kD + \cdots + C_{k+1,j+1}E^{k-j}D^{(j)} + \cdots + D^{(k)},$$

since

$$C_{k,j} + C_{k-1,j} + \cdots + C_{j,j} = C_{k+1,j+1}.$$

If the characteristic of $\mathfrak{A}$ is $p \neq 0$ special cases of (12) and (13) are

$$(14)\qquad \text{(a)}\quad [D, E^p] = D^{(p)}, \qquad \text{(b)}\quad \begin{Bmatrix} E & D \\ p-1 & 1 \end{Bmatrix} = D^{(p-1)}.$$

Equations (11) and (14b) show that $\{D_1 \cdots D_p/1 \cdots 1\}$ is expressible as a linear combination of $(p-1)$-fold commutators, i.e., of the type $D^{(p-1)}$ where $D = D_p$ and E is a sum of the other D_i's. Hence we see also that $(j_1! \cdots j_l!)\ \{D_1 \cdots D_l/j_1 \cdots j_l\}$ where $j_1 + \cdots + j_l = p$ is a linear sum of $(p-1)$-fold commutators. If no $j_i = p$, $(j_1! \cdots j_l!) \not\equiv 0 \pmod{p}$ and so $\{D_1D_2 \cdots D_l/j_1j_2 \cdots j_l\}$ is a linear sum of $(p-1)$-fold commutators and (9) becomes

$$(15)\qquad (D_1 + D_2 + \cdots + D_r)^p = D_1^p + D_2^p + \cdots + D_r^p + S,$$

where S is a linear sum of $(p-1)$-fold commutators.

3. If $\mathfrak{D}$ is any system of linear transformations we define the *enveloping algebra* $\mathfrak{A}$ of $\mathfrak{D}$ to be the totality of linear combinations of products of a finite number of elements of $\mathfrak{D}$. We call k the degree of the monomial $D_1D_2 \cdots D_k$, $D_i \in \mathfrak{D}$. Suppose $\mathfrak{D}$ is a Lie algebra of linear transformations and consider $D_1D_2 \cdots D_k$ where $k < p$ if $p \neq 0$ and arbitrary if $p = 0$. We have

$$D_1 \cdots D_{i-1}D_{i+1}D_iD_{i+2} \cdots D_k = D_1D_2 \cdots D_k + D_1 \cdots D_{i-1}D'D_{i+2} \cdots D_k,$$

where $D' = [D_{i+1}, D_i] \in \mathfrak{D}$. Since any arrangement $i_1i_2 \cdots i_k$ of $1, 2, \cdots, k$ may be obtained from $1, 2, \cdots, k$ by a sequence of transpositions of adjacent indices

$$D_{i_1}D_{i_2} \cdots D_{i_k} = D_1D_2 \cdots D_k + R,$$

where R is a sum of terms of degree $< k$. Hence

$$\begin{Bmatrix} D_1 & D_2 & \cdots & D_k \\ 1 & 1 & \cdots & 1 \end{Bmatrix} = (k!)D_1D_2 \cdots D_k + S,$$

where degree of $S < k$. Since the left-hand side of this equation is expressible by (10) as a sum of kth powers of elements in $\mathfrak{D}$ and $k! \not\equiv 0 \pmod{p}$, we have by induction that $D_1D_2 \cdots D_k$ is a linear combination of lth powers of elements of $\mathfrak{D}$ where $l \leqq k$.

THEOREM 2. *If $\mathfrak{D}$ is a Lie algebra of linear transformations the elements in the enveloping algebra $\mathfrak{A}$ of degree $k<p$ if $p\neq 0$ and of arbitrary degree if $p=0$ are expressible as linear combinations of lth powers $l\leqq k$, of elements of $\mathfrak{D}$.**

If $\mathfrak{D}$ is restricted (10) shows that $\{D_1D_2\cdots D_r/j_1j_2\cdots j_r\}\epsilon\mathfrak{D}$ if $j_1+j_2+\cdots+j_r=p$ and $D_i\epsilon\mathfrak{D}$. This transformation is also expressible as a sum of $(p-1)$-fold commutators of elements of $\mathfrak{D}$. Since $(D_1+D_2+\cdots+D_r)^{p^k}=((D_1+D_2+\cdots+D_r)^{p^{k-1}})^p$ for $j_1, j_2, \cdots, j_r$ such that $j_1+j_2+\cdots+j_r=p^k$, we have

$$\begin{Bmatrix} D_1 & D_2\cdots D_r \\ j_1 & j_2\cdots j_r \end{Bmatrix} = \sum \left\{ \begin{matrix} \begin{Bmatrix} D_1 & D_2\cdots D_r \\ k_{11} & k_{12}\cdots k_{1r} \end{Bmatrix}_{m_1} \begin{Bmatrix} D_1 & D_2\cdots D_r \\ k_{21} & k_{22}\cdots k_{2r} \end{Bmatrix}_{m_2} \cdots \end{matrix} \right\},$$

where the summation is extended over the non-negative integers such that the ordered set $(k_{l1}, k_{l2}, \cdots, k_{lr})\neq(k_{m1}, k_{m2}, \cdots, k_{mr})$ for $l\neq m$ and

$$\begin{aligned} k_{l1}+k_{l2}+\cdots+k_{lr} &= p^{k-1} && (l=1, 2, \cdots), \\ m_1+m_2+\cdots &= p, \\ k_{1i}m_1+k_{2i}m_2+\cdots &= j_i && (i=1, 2, \cdots, r). \end{aligned}$$

Hence we see by induction on k that $\{D_1D_2\cdots D_r/j_1j_2\cdots j_r\}\epsilon\mathfrak{D}$ *for all* $j_1, j_2, \cdots$ *such that* $j_1+j_2+\cdots+j_r=p^k$.

4. Because of (14a) we are led to the definition: A *restricted Lie Algebra* $\mathfrak{R}$ *of characteristic* p ($=0$ or not) is an algebra (i.e., satisfies (1)) in which the composition $[x, y]$ (in place of xy) satisfies

$$[x, y] = -[y, x], \tag{16}$$

$$[[x, y], z]+[[y, z], x]+[[z, x], y]=0, \tag{17}$$

for every y there exists an element denoted as y^p such that

$$[\cdots[[x, \overbrace{y]y]\cdots y}^{p}] = [x, y^p] \tag{18}$$

for all x. A *restricted subalgebra* $\mathfrak{S}$ of $\mathfrak{R}$ is a subalgebra containing y^p for every y in $\mathfrak{S}$. Similarly we define restricted ideal, etc.†

Suppose $\mathfrak{R}$ is an associative algebra. We may define a new composition $[x, y]=xy-yx$ in terms of xy defined in $\mathfrak{R}$. It is readily verified that $\mathfrak{R}$ is a

* This is a slight extension of a result announced recently by M. Zorn (Bulletin of the American Mathematical Society, vol. 42 (1936), p. 485). Cf. H. Poincaré, *Sur les groupes continus*, Cambridge Philosophical Transactions, vol. 18 (1899), pp. 220–255.

† For definitions of the important concepts in the theory of Lie algebras the reader is referred to Jacobson, *Rational methods in the theory of Lie algebras*, Annals of Mathematics, vol. 36 (1935), pp. 875–881.

restricted Lie algebra if y^p is defined as the pth power of y in $\mathfrak{R}$. We shall call this Lie algebra *the restricted Lie algebra determined by the associative* $\mathfrak{R}$.

5. If $\mathfrak{R}$ is any algebra the mapping a_r: $x \to xa$ is a linear transformation and will be called the ***right multiplication*** determined by a. Suppose D is a derivation in $\mathfrak{R}$. Equation (2c) gives the commutation relation

$$[a_r, D] = (aD)_r. \tag{19}$$

Similarly we define a_l as $x \to ax$ and call this mapping the *left multiplication* determined by a. In place of (19) we have $[a_l, D] = (aD)_l$. If $\mathfrak{R}$ is a Lie algebra $a_r = -a_l$ and, by (16) and (17),

$$[x, y]a_r = [xa_r, y] + [x, ya_r].$$

Thus a_r is a derivation which we call *inner*.

THEOREM 3. *The totality of inner derivations of a (restricted) Lie algebra* $\mathfrak{R}$ *is a (restricted) ideal* $\mathfrak{I}$ *in the d-algebra* $\mathfrak{D}$ *of* $\mathfrak{R}$. $\mathfrak{I} \cong \mathfrak{R}/\mathfrak{C}$ *where* $\mathfrak{C}$ *is the centrum of* $\mathfrak{R}$.*

If a_r and b_r are multiplications associated with a and b it follows directly from the definition of $\mathfrak{R}$ that $a_r \pm b_r = (a \pm b)_r$, $a_r\alpha = (a\alpha)_r$, $[a_r, b_r] = [a, b]_r$ and if $\mathfrak{R}$ is restricted $(a_r)^p = (a^p)_r$. Hence $\mathfrak{I}$ is a subalgebra of $\mathfrak{D}$ and is restricted if $\mathfrak{R}$ is. Furthermore the correspondence $a \to a_r$ is a homomorphism between $\mathfrak{R}$ and $\mathfrak{I}$. Since the elements of $\mathfrak{C}$ are the ones corresponding to 0 in this homomorphism $\mathfrak{R}/\mathfrak{C} \cong \mathfrak{I}$. Equation (19) shows that $\mathfrak{I}$ is an ideal.

Suppose $\mathfrak{R}$ is associative and D a derivation. D is also a derivation in the restricted Lie algebra determined by $\mathfrak{R}$. Hence the d-algebra of $\mathfrak{R}$ as an associative algebra is a restricted subalgebra of the d-algebra of $\mathfrak{R}$ as a Lie algebra. Moreover the inner derivations $x \to [x, a]$ are derivations of the associative $\mathfrak{R}$ since

$$[xy, a] = [x, a]y + x[y, a].$$

Thus $\mathfrak{I}$ is a restricted ideal in the d-algebra of the associative $\mathfrak{R}$.

If $\mathfrak{R}$ is associative, $\mathfrak{D}$ its d-algebra, $D \epsilon \mathfrak{D}$ and $c \epsilon \mathfrak{C}$ the centrum of $\mathfrak{R}$ then $c_r = c_l \equiv c$ and it is easily verified that $Dc \epsilon \mathfrak{D}$ also. Hence $\mathfrak{D}$ has $\mathfrak{C}$ as well as $\mathfrak{F}$ as a set of multipliers under which it is invariant. A subalgebra $\mathfrak{E}$ of $\mathfrak{D}$ which contains with every element E also Ec for every c in $\mathfrak{C}$ will be called a $\mathfrak{C}$-*subalgebra* of $\mathfrak{D}$.

If $\mathfrak{R}$ is arbitrary, $D \epsilon \mathfrak{D}$ the elements $k \epsilon \mathfrak{R}$ such that $kD = 0$ are called *D-constants*. Their totality is a subalgebra. If $kD = 0$ for all D then k is a *constant*. If $\mathfrak{R}$ has an identity 1 we have $1^2 = 1$ and hence $1(1D) + (1D)1 = 1D$ or $1D = 0$

* The *centrum* is the set of elements c such that $[c, x] = 0$ for all x in $\mathfrak{R}$.

so that 1 is a constant. More generally if $\mathfrak{D}_1$ is a subalgebra of $\mathfrak{D}$ we denote the set of elements k in $\mathfrak{R}$ such that $kD_1=0$ for all $D_1\epsilon\mathfrak{D}_1$ by $\mathfrak{R}(\mathfrak{D}_1)$. $\mathfrak{R}(\mathfrak{D}_1)$ is a subalgebra. On the other hand if $\mathfrak{R}_1$ is a subalgebra of $\mathfrak{R}$ we define $\mathfrak{D}(\mathfrak{R}_1)$ to be the set of derivations E such that $x_1E=0$ for all $x_1\epsilon\mathfrak{R}_1$. $\mathfrak{D}(\mathfrak{R}_1)$ is a restricted subalgebra of $\mathfrak{D}$. Evidently $\mathfrak{D}(\mathfrak{R}(\mathfrak{D}_1))\supset\mathfrak{D}_1$ and $\mathfrak{R}(\mathfrak{D}(\mathfrak{R}_1))\supset\mathfrak{R}_1$. If $\mathfrak{R}$ is associative with centrum $\mathfrak{C}$, $\mathfrak{D}(\mathfrak{R}_1)$ is a restricted $\mathfrak{C}$-subalgebra of $\mathfrak{D}$.

$\mathfrak{S}$ is a *characteristic subalgebra* of $\mathfrak{R}$ if it is mapped on itself by every element of $\mathfrak{D}$. The subalgebra of constants $\mathfrak{R}_0$, the centrum $\mathfrak{C}$ and the powers of $\mathfrak{R}$ are characteristic. If $\mathfrak{S}$ is characteristic, $\mathfrak{D}(\mathfrak{S})$ is an ideal. In particular $\mathfrak{D}(\mathfrak{C})$ is an ideal containing $\mathfrak{J}$ if $\mathfrak{R}$ is associative or a Lie algebra. The derivations mapping $\mathfrak{R}$ on the characteristic subalgebra $\mathfrak{S}$ also form a restricted ideal $\mathfrak{G}$. In the case of a Lie algebra or an associative algebra the ideal associated in this way with $\mathfrak{C}$ is the *annihilator* of $\mathfrak{J}$, i.e., the set of elements G such that $[a_r, G]=0$ for all a_r. This is an immediate consequence of (19).

II. Derivations in an associative algebra with a finite basis

6. In the remainder of the paper $\mathfrak{R}$ will denote an associative algebra with a finite basis over $\mathfrak{F}$. We propose to study the d-algebra $\mathfrak{D}$ of $\mathfrak{R}$.

Theorem 4. *If* $\mathfrak{R}=\mathfrak{R}_1\oplus\mathfrak{R}_2$ *and* $\mathfrak{R}_1^2=\mathfrak{R}_1$, $\mathfrak{R}_2^2=\mathfrak{R}_2$ *then* $\mathfrak{D}=\mathfrak{D}_1\oplus\mathfrak{D}_2$ *where* $\mathfrak{D}_i$ *is isomorphic to the d-algebra of* $\mathfrak{R}_i$.

$\mathfrak{R}_1$ is characteristic; for $\mathfrak{R}_1^2=\mathfrak{R}_1$ and so the arbitrary element x_1 of $\mathfrak{R}_1$ has the form $\sum y_1z_1$, $y_1, z_1\epsilon\mathfrak{R}_1$. Hence $x_1D=\sum(y_1z_1)D=\sum(y_1D)z_1+\sum y_1(z_1D)\epsilon\mathfrak{R}_1$ since this is an ideal. Similarly $\mathfrak{R}_2$ is characteristic. Let $\mathfrak{D}_i$ be the ideals mapping $\mathfrak{R}$ onto $\mathfrak{R}_i$. Since $\mathfrak{R}_1\cap\mathfrak{R}_2=0$, $\mathfrak{D}_1\cap\mathfrak{D}_2=0$ and hence $[\mathfrak{D}_1, \mathfrak{D}_2]\subset\mathfrak{D}_1\cap\mathfrak{D}_2=0$.† If $x=x_1+x_2$, $x_i\epsilon\mathfrak{R}_i$ and D any derivation, the mappings $x\to x_1D\equiv xD_1$ and $x\to x_2D\equiv xD_2$ are derivations in $\mathfrak{D}_1$ and $\mathfrak{D}_2$ respectively. Since $D=D_1+D_2$, $\mathfrak{D}=\mathfrak{D}_1\oplus\mathfrak{D}_2$. The isomorphism between $\mathfrak{D}_1$ and the d-algebra of $\mathfrak{R}_1$ follows directly from the fact that the transformations of $\mathfrak{D}_1$ induce all the derivations in $\mathfrak{R}_1$ and map $\mathfrak{R}_2$ into 0. Similarly $\mathfrak{D}_2$ is isomorphic to the d-algebra of $\mathfrak{R}_2$.

Let $x_1, x_2, \cdots, x_r$ be a basis for $\mathfrak{R}$ over $\mathfrak{F}$ ($\mathfrak{R}=x_1\mathfrak{F}+x_2\mathfrak{F}+\cdots+x_r\mathfrak{F}$) and suppose $x_ix_j=\sum x_\rho\gamma_{\rho ij}$, $\gamma_{\rho ij}\epsilon\mathfrak{F}$. If D is a derivation in $\mathfrak{R}$ and

$$(x_1D, x_2D, \cdots, x_rD) = (x_1, x_2, \cdots, x_r)\Delta,\ \Delta=(\alpha_{ij}),\ \alpha_{ij}\epsilon\mathfrak{F},$$

then the condition $(x_ix_j)D=(x_iD)x_j=x_i(x_jD)$ gives

$$\sum_\rho \alpha_{k\rho}\gamma_{\rho ij} = \sum_\rho \gamma_{k\rho j}\alpha_{\rho i} + \sum_\rho \gamma_{ki\rho}\alpha_{\rho j} \qquad (i, j, k = 1, 2, \cdots, r), \tag{20}$$

a set of n^3 linear homogeneous equations for the coordinates α_{ij} of Δ. Con-

† $[\mathfrak{A}, \mathfrak{B}]$ denotes the smallest subspace of $\mathfrak{D}$ containing all the elements $[A, B]$, where $A\epsilon\mathfrak{A}$, $B\epsilon\mathfrak{B}$.

versely if Δ is any matrix whose coordinates satisfy (20) the linear transformation D determined by Δ satisfies $(x_ix_j)D=(x_iD)x_j+x_i(x_jD)$ for all i, j and hence $(xy)D=(xD)y+x(yD)$ for all x, y, i.e., D is a derivation. Now suppose $\mathfrak{K}$ is a field containing $\mathfrak{F}$ and let $\mathfrak{R}_{\mathfrak{K}}=x_1\mathfrak{K}+x_2\mathfrak{K}+\cdots+x_r\mathfrak{K}$ and $\mathfrak{D}^*$ be the d-algebra of $\mathfrak{R}_{\mathfrak{K}}$ (over $\mathfrak{K}$). Evidently the matrix Δ also determines a derivation D^* in $\mathfrak{R}_{\mathfrak{K}}$. Furthermore since the maximum number of linearly independent solutions of (20) in $\mathfrak{K}$ is the same as in $\mathfrak{F}$ it follows that if $D_1, D_2, \cdots, D_s$ is a basis for $\mathfrak{D}$ then $D_1^*, D_2^*, \cdots, D_s^*$ is a basis for $\mathfrak{D}^*$, and if $[D_i, D_j]=\sum D_p\mu_{pij}$, $D_i^p=\sum D_p\nu_{pi}$ $(\mu_{pij}, \nu_{pi}\epsilon\mathfrak{F})$, then $[\Delta_j, \Delta_i]=\sum\Delta_p\mu_{pij}$, $\Delta_i^p=\sum\Delta_p\nu_{pi}$ and hence $[D_i^*, D_j^*]=\sum D_p^*\mu_{pij}$, $(D_i^*)^p=\sum D_p^*\nu_{pi}$. Thus we have proved

THEOREM 5. *If $\mathfrak{D}$ is the d-algebra of $\mathfrak{R}$ then $\mathfrak{D}_{\mathfrak{K}}$ is the d-algebra of $\mathfrak{R}_{\mathfrak{K}}$.*

7. We now consider the d-algebra of a semi-simple algebra $\mathfrak{R}$. Since $\mathfrak{R}=\mathfrak{R}_1\oplus\mathfrak{R}_2\oplus\cdots\oplus\mathfrak{R}_t$ where $\mathfrak{R}_i$ are simple and $\mathfrak{R}_i^2=\mathfrak{R}_i$ we have as a consequence of Theorem 4

THEOREM 6. *The d-algebra of a semi-simple algebra is a direct sum of algebras isomorphic to the d-algebras of its simple components.*

We suppose therefore that $\mathfrak{R}$ is simple and let $\mathfrak{C}$ denote its centrum. $\mathfrak{C}$ is an algebraic field over $\mathfrak{F}$ and is characteristic. Let $\mathfrak{C}_0$ be the subfield of constants of $\mathfrak{C}$. Because of (19),

$$[D_1, D_2]c_0 = [D_1c_0, D_2] = [D_1, D_2c_0],$$

where c_0 here denotes the multiplication determined by the element c_0 of $\mathfrak{C}_0$. Thus $\mathfrak{D}$ as well as $\mathfrak{R}$ may be regarded as an algebra over $\mathfrak{C}_0$. We may therefore suppose that $\mathfrak{C}_0=\mathfrak{F}$, i.e., the only constants in $\mathfrak{C}$ are the multiples of 1 by elements of $\mathfrak{F}$. In this case we shall show that $\mathfrak{C}$ is an inseparable field of a simple type over $\mathfrak{F}$.

Let c be any element of $\mathfrak{C}$ not in $\mathfrak{F}$. Since $cD\epsilon\mathfrak{C}$, we have

$$\begin{aligned}\phi(c)D &\equiv (c^r+c^{r-1}\gamma_1+\cdots+\gamma_r)D\\ &= (rc^{r-1}+(r-1)c^{r-2}\gamma_1+\cdots+\gamma_{r-1})(cD)\\ &= \phi'(c)(cD),\end{aligned}\tag{21}$$

where $\phi'(\lambda)$ is the formal derivative of the polynomial $\phi(\lambda)$ in the polynomial ring $\mathfrak{F}[\lambda]$. If $\phi(c)=0$ is the minimum equation of c and D is chosen so that $cD\neq 0$, (21) gives $\phi'(c)=0$ and hence $\phi'(\lambda)=0$. Thus c is inseparable. In particular if the characteristic $p=0$, $\mathfrak{C}=\mathfrak{F}$ and $\mathfrak{R}$ is a normal simple algebra. If $p\neq 0$, $c^p=\gamma\epsilon\mathfrak{F}$ since $c^pD=pc^{p-1}(cD)=0$ for all D.

LEMMA 1. *If $\mathfrak{F}$ is a field of characteristic $p\neq 0$, the polynomial $\lambda^p-\alpha$ is either irreducible or a pth power of a linear factor in $\mathfrak{F}[\lambda]$.*

Suppose $\lambda^p-\alpha$ is reducible and $\phi(\lambda)$ of degree $<p$ is an irreducible factor, say

$$\lambda^p - \alpha = \phi(\lambda)^r\psi(\lambda), \qquad (\phi(\lambda), \psi(\lambda)) = 1.$$

Differentiating we obtain

$$0 = r\phi(\lambda)^{r-1}\phi'(\lambda)\psi(\lambda) + \phi(\lambda)^r\psi'(\lambda).$$

$\psi'(\lambda)\neq 0$ implies that $\phi(\lambda)^r$ divides $r\phi(\lambda)^{r-1}\phi'(\lambda)\psi(\lambda)$ and $\phi(\lambda)$ divides $r\phi'(\lambda)\psi(\lambda)$. Since $\phi'(\lambda)\neq 0$ and $(\phi(\lambda), \psi(\lambda)) = 1$, it follows that $r = p$ and hence $\psi(\lambda)$ has degree 0 contrary to the assumption $\psi'(\lambda)\neq 0$. Hence $\psi'(\lambda) = 0$ or $\psi(\lambda)$ has degree 0 and may be taken to be 1. Then $r\phi(\lambda)^{r-1}\phi'(\lambda) = 0$ and so $r = p$, $\lambda^p-\alpha = \phi(\lambda)^p$.

We return to the consideration of the structure of $\mathfrak{C}$ in the case $p\neq 0$. If $\mathfrak{C}\neq\mathfrak{F}$ choose $c_1\epsilon\mathfrak{C}$, $\epsilon\mathfrak{F}$. $c_1^p = \gamma_1\epsilon\mathfrak{F}$. The polynomial $\lambda^p-\gamma_1$ is irreducible in $\mathfrak{F}[\lambda]$. For otherwise $\lambda^p-\gamma_1 = (\lambda-\delta)^p$, $\delta\epsilon\mathfrak{F}$ and $\lambda^p-\gamma_1 = (\lambda-c_1)^p = (\lambda-\delta)^p$, $c_1 = \delta\epsilon\mathfrak{F}$ contrary to the choice of c_1. The order of $\mathfrak{F}^1 = \mathfrak{F}(c_1)$ over $\mathfrak{F}$ is therefore p. If $\mathfrak{C}\neq\mathfrak{F}^1$ choose $c_2\epsilon\mathfrak{C}$, $\epsilon\mathfrak{F}^1$. $c_2^p = \gamma_2\epsilon\mathfrak{F}$ and the polynomial $\lambda^p-\gamma_2$ is irreducible in $\mathfrak{F}^1$. Hence $\mathfrak{F}^2 = \mathfrak{F}^1(c_2) = \mathfrak{F}(c_1, c_2)$ has order p over $\mathfrak{F}^1$ and consequently p^2 over $\mathfrak{F}$. Continuing in this way we prove that $\mathfrak{C} = \mathfrak{F}(c_1, c_2, \cdots, c_m)$, $c_i^p = \gamma_i$ and $\mathfrak{C}$ has order p^m over $\mathfrak{F}$.

8. We determine first the structure of the d-algebra $\mathfrak{D}$ of a normal simple algebra $\mathfrak{R}$, i.e., $\mathfrak{C} = \mathfrak{F}$. The following theorem is fundamental.

THEOREM 7. *If $\mathfrak{S}$ is a semi-simple subalgebra of $\mathfrak{R}$, any derivation in $\mathfrak{S}$ may be extended to an inner derivation in $\mathfrak{R}$.*†

By Wedderburn's theorem $\mathfrak{R}$ is the totality of $t\times t$ matrices with coordinates in a normal division algebra $\mathfrak{G}$. In particular the elements z of $\mathfrak{S}$ are such matrices and we have a representation $z\to z$ of $\mathfrak{S}$ by matrices in $\mathfrak{G}$. We suppose first that this representation is irreducible. If D is any derivation in $\mathfrak{S}$ it is readily verified that

$$z \to \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}, \qquad z \to \begin{pmatrix} z & 0 \\ zD & z \end{pmatrix} \tag{22}$$

are also representations of $\mathfrak{S}$ by matrices $(2t\times 2t)$ in $\mathfrak{G}$. Since, as E. Noether‡ has shown, every representation of a semi-simple algebra by matrices in a normal division algebra is completely reducible, any two representations with

† This proof is an extension of an argument communicated to me by R. Brauer.

‡ E. Noether, *Nichtkommutative Algebra*, Mathematische Zeitschrift, vol. 37 (1933), pp. 514–541. The theorem is stated here only for simple algebras but the proof given is also valid for semi-simple algebras.

the same irreducible parts are similar. Thus the two representations in (22) are similar, i.e., there exists a fixed non-singular matrix

$$A = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}, \qquad a_{ij} \epsilon \mathfrak{R}$$

such that

$$\begin{pmatrix} z & 0 \\ zD & z \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$$

for all $z \epsilon \mathfrak{S}$. Hence

$$za_{11} = a_{11}z, \qquad za_{12} = a_{12}z, \qquad (zD)a_{11} + za_{21} = a_{21}z,$$
$$(zD)a_{12} + za_{22} = a_{22}z.$$

By Schur's lemma, a_{11} and a_{12} are either 0 or non-singular and both cannot be 0 since A is non-singular. If $a_{11} \neq 0$, we set $a = -a_{21}a_{11}^{-1}$ and if $a_{11} = 0$, we set $a = -a_{22}a_{12}^{-1}$. Then $a \epsilon \mathfrak{R}$ and $zD = [z, a]$ as was to be shown.

If $z \to z$ is not irreducible it is completely reducible and so there exists a fixed matrix b in $\mathfrak{R}$ such that

$$b^{-1}zb = \begin{pmatrix} z_1 & & & \\ & z_2 & & \\ & & \ddots & \\ & & & z_l \end{pmatrix}$$

and $z \to z_i$ are irreducible representations of $\mathfrak{S}$. As before

$$z \to \begin{pmatrix} z_i & 0 \\ (zD)_i & z_i \end{pmatrix}, \qquad z \to \begin{pmatrix} z_i & 0 \\ 0 & z_i \end{pmatrix}$$

are similar representations of $\mathfrak{S}$ and there exists a matrix a_i such that $(zD)_i = [z_i, a_i]$. Then if

$$a = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_l \end{pmatrix},$$

$b^{-1}(zD)b = [b^{-1}zb, a]$ and $zD = [z, bab^{-1}]$, $bab^{-1} \epsilon \mathfrak{R}$.

As a special case we have

THEOREM 8. *The d-algebra of a normal simple algebra contains only inner derivations.*

COROLLARY. *If $\mathfrak{R}$ is simple, $\mathfrak{D}(\mathfrak{C}) = \mathfrak{J}$.*

If $D \in \mathfrak{D}(\mathfrak{C})$, $(xc)D = (xD)c$ for all x and all $c \in \mathfrak{C}$. Thus D is a derivation of $\mathfrak{R}$ considered as an algebra over $\mathfrak{C}$. By Theorem 8, D is inner and so $\mathfrak{D}(\mathfrak{C}) \subset \mathfrak{I}$. Since $\mathfrak{I} \supset \mathfrak{D}(\mathfrak{C})$ we have equality.

Suppose again that $\mathfrak{R}$ is normal simple. Theorem 8 implies that $\mathfrak{D} = \mathfrak{I} \cong \mathfrak{R}/\mathfrak{F}$ where $\mathfrak{R}$ is the restricted Lie algebra determined by the associative $\mathfrak{R}$ and $\mathfrak{F}$ is the centrum consisting of the multiples of 1. We may extend $\mathfrak{F}$ to the field $\mathfrak{K}$ such that $\mathfrak{R}_{\mathfrak{K}} = \mathfrak{K}_n$ is the complete matrix algebra of order n^2 over $\mathfrak{K}$, i.e., $\mathfrak{K}_n$ has a basis e_{ij} $(i, j = 1, 2, \cdots, n)$ such that $e_{ij}e_{kl} = \delta_{jk}e_{il}$. We consider the structure of the Lie algebra $\mathfrak{K}_n$ having basis e_{ij} also and multiplication table

$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{il}e_{kj}. \tag{23}$$

The centrum of $\mathfrak{K}_n$ is $\mathfrak{K}$ the totality of multiples of $1 = e_{11} + e_{22} + \cdots + e_{nn}$. This is an ideal as is $\mathfrak{K}_n' = [\mathfrak{K}_n, \mathfrak{K}_n]$. From (23) follows that e_{ts}, $e_{tt} - e_{ss} \in \mathfrak{K}_n'$ if $t \neq s$. Evidently every element of $\mathfrak{K}_n'$ has trace 0. Conversely if $a = \sum e_{ij}\alpha_{ij}$ and $\operatorname{tr}(a) = \alpha_{11} + \alpha_{22} + \cdots + \alpha_{nn} = 0$,

$$a = (e_{11} - e_{nn})\alpha_{11} + (e_{22} - e_{nn})\alpha_{22} + \cdots + (e_{n-1,n-1} - e_{nn})\alpha_{n-1,n-1} + \sum_{t \neq s} e_{ts}\alpha_{ts} \in \mathfrak{K}_n'$$

and so $\mathfrak{K}_n'$ is the set of matrices of trace 0 and is generated by $e_{11} - e_{nn}$, $e_{22} - e_{nn}, \cdots, e_{n-1,n-1} - e_{nn}, e_{ts}$ $(t \neq s)$. These $n^2 - 1$ elements are evidently linearly independent and hence form a basis for $\mathfrak{K}_n'$. Since $(e_{ss} - e_{tt})^p = e_{ss} - e_{tt}$, $e_{st}^p = 0$ if $p \neq 0$ is the characteristic of $\mathfrak{K}$, $\mathfrak{K}_n'$ by (15) contains the pth power of every element belonging to it, i.e., $\mathfrak{K}_n'$ is a restricted ideal. $\mathfrak{K}_n'$ contains 1 if and only if $\operatorname{tr}(1) = n \equiv 0 \pmod{p}$.

Suppose $\mathfrak{B}$ is an ideal $\neq \mathfrak{K}$ in $\mathfrak{K}_n$ and $b = \sum e_{ij}\beta_{ij} \in \mathfrak{B}$, $\notin \mathfrak{K}$. Suppose first $\beta_{uv} \neq 0$ for some pair u, v, $u \neq v$. If $n > 2$, choose $t \neq v$, $\neq u$ and then $[[[b, e_{vu}], e_{tu}], e_{tt}]\beta_{uv}^{-1} = e_{tu} \in \mathfrak{B}$. If $p \neq 2$, $[[b, e_{vu}], e_{vu}]\ (-2\beta_{uv})^{-1} = e_{vu} \in \mathfrak{B}$. If all $\beta_{uv} = 0$ then $b = e_{11}\beta_{11} + e_{22}\beta_{22} + \cdots + e_{nn}\beta_{nn}$ and since $b \notin \mathfrak{K}$, $\beta_{uu} \neq \beta_{vv}$ for some pair $u \neq v$ and hence $[b, e_{uv}]\ (\beta_{uu} - \beta_{vv})^{-1} = e_{uv} \in \mathfrak{B}$. Thus in any case unless $n = p = 2$ $\mathfrak{B}$ contains an e_{st}, $s \neq t$ and since by (23), $[e_{st}, \mathfrak{K}_n] = \mathfrak{K}_n'$, $\mathfrak{B} \supset \mathfrak{K}_n'$. If $\mathfrak{B} \neq \mathfrak{K}_n$, $\mathfrak{B} = \mathfrak{K}_n'$.

Any ideal of $\mathfrak{K}_n/\mathfrak{K}$ the derivation ring of the associative algebra $\mathfrak{K}_n$ has the form $\mathfrak{B}/\mathfrak{K}$ where $\mathfrak{B}$ is an ideal in the Lie algebra $\mathfrak{K}_n$ containing $\mathfrak{K}$. If $p \nmid n$ the only such ideals are $\mathfrak{K}$ and $\mathfrak{K}_n$. Hence $\mathfrak{K}_n/\mathfrak{K}$ is a simple Lie algebra, i.e., has no proper ideals.

If $p \mid n$ and either $p \neq 2$ or $n > 2$, $\mathfrak{K}_n/\mathfrak{K}$ has one proper ideal $\mathfrak{K}_n'/\mathfrak{K}$ and this is restricted. It may be shown by a direct argument similar to the above that $\mathfrak{K}_n'/\mathfrak{K}$ is simple except when $p = n = 2$ and hence the Lie algebra $\mathfrak{K}_n/\mathfrak{K}$ is semi-

simple. Since $\mathfrak{K}_n'/\mathfrak{K}$ is the only proper ideal in $\mathfrak{K}_n/\mathfrak{K}$ the latter is not a direct sum of simple ideals.†

THEOREM 9. *If $\mathfrak{R}$ is a normal simple algebra of order n^2 and $p \nmid n^2$ then the d-algebra $\mathfrak{D}$ of $\mathfrak{R}$ is simple.*

THEOREM 10. *If $\mathfrak{R}$ is normal simple and $p \mid n^2$ but either $p \neq 2$ or $n > 2$ $\mathfrak{D}$ is semi-simple though not simple.*

To prove these theorems we note that a proper ideal $\mathfrak{B}$ of $\mathfrak{D}$ becomes a proper ideal $\mathfrak{B}_{\mathfrak{K}}$ of $\mathfrak{D}_{\mathfrak{K}}$ the d-algebra of $\mathfrak{R}_{\mathfrak{K}}$ when $\mathfrak{F}$ is extended to $\mathfrak{K}$. By choosing $\mathfrak{K}$ so that $\mathfrak{R}_{\mathfrak{K}} = \mathfrak{K}_n$, $\mathfrak{D}_{\mathfrak{K}} \cong \mathfrak{K}_n/\mathfrak{K}$ it follows that $\mathfrak{D}$ has no such ideals if $p \nmid n^2$. If $p \mid n^2$ and either $p \neq 2$ or $n > 2$, $[\mathfrak{D}, \mathfrak{D}]_{\mathfrak{K}} \cong \mathfrak{K}_n'/\mathfrak{K}$ is a proper restricted ideal of $\mathfrak{D}_{\mathfrak{K}}$ and hence $\mathfrak{D}' = [\mathfrak{D}, \mathfrak{D}]$ is a proper restricted ideal in $\mathfrak{D}$. $\mathfrak{D}'$ is simple since $\mathfrak{D}_{\mathfrak{K}}'$ is.

If $n = p = 2$ it is easily seen that $\mathfrak{K}_2/\mathfrak{K}$ and hence $\mathfrak{D}$ is solvable.

9. We consider next the d-algebra $\mathfrak{D}$ of the other extreme case, namely, $\mathfrak{R} = \mathfrak{C} = \mathfrak{F}(c_1, c_2, \cdots, c_m)$ where $c_i^p = \gamma_i$ and the order of $\mathfrak{R}$ over $\mathfrak{F}$ is p^m, $p \neq 0$. Let D be any element of $\mathfrak{D}$ and consider the correspondence $D \rightarrow (c_1D, c_2D, \cdots, c_mD)$ mapping $\mathfrak{D}$ on the space $\mathfrak{R}^{(m)}$ of ordered m-tuples of elements of $\mathfrak{R}$. This correspondence is linear relative to $\mathfrak{F}$ and since $c_1D = c_2D = \cdots = c_mD = 0$ implies that $D = 0$ it is $(1-1)$. Moreover if $(d_1, d_2, \cdots, d_m)$ is an arbitrary element of $\mathfrak{R}^{(m)}$ there is a $D \epsilon \mathfrak{D}$ such that $c_iD = d_i$. For $\mathfrak{R} \cong \mathfrak{F}[\lambda_1, \lambda_2, \cdots, \lambda_m]/\mathfrak{P}$ where $\mathfrak{P}$ is the ideal having the basis $\lambda_1^p - \gamma_1$, $\lambda_2^p - \gamma_2, \cdots, \lambda_m^p - \gamma_m$. If $d_1(\lambda_1, \cdots, \lambda_m)$, $d_2(\lambda_1, \cdots, \lambda_m), \cdots, d_m(\lambda_1, \cdots, \lambda_m)$ are arbitrary polynomials, then the transformation D defined by

$$c(\lambda_1, \lambda_2, \cdots, \lambda_m)D = \sum_i \frac{\partial c(\lambda_1, \lambda_2, \cdots, \lambda_m)}{\partial \lambda_i} d_i(\lambda_1, \lambda_2, \cdots, \lambda_m)$$

is easily verified to be a derivation in $\mathfrak{F}[\lambda_1, \lambda_2, \cdots, \lambda_m]$. If $z(\lambda_1, \lambda_2, \cdots, \lambda_m) \epsilon \mathfrak{P}$ then $zD \epsilon \mathfrak{P}$ also. It follows that D induces a derivation in $\mathfrak{F}[\lambda_1, \lambda_2, \cdots, \lambda_m]/\mathfrak{P}$, i.e., in $\mathfrak{R}$ and since $d_i(\lambda_1, \cdots, \lambda_m)$ were arbitrary, D may be chosen so that $c_iD = d_i$. We have therefore established an isomorphism between $\mathfrak{D}$ and $\mathfrak{R}^{(m)}$ considered as vector spaces over $\mathfrak{F}$. The order of $\mathfrak{R}^{(m)}$ is mp^m and hence the order of $\mathfrak{D}$ is mp^m also.

LEMMA 2. *If $\mathfrak{R}$ is any commutative field, D a derivation in it, and $\mathfrak{F}$ the subfield of D-constants, a necessary and sufficient condition that the elements $y_1, y_2, \cdots, y_r$ be linearly dependent over $\mathfrak{F}$ is that the Wronskian*

† If $p = 0$ a fundamental theorem due to E. Cartan, *Thèse*, Paris, 1894, states that a semi-simple Lie algebra is a direct sum of simple algebras. The algebras $\mathfrak{K}_n/\mathfrak{K}$ for $p \mid n$ show that this does not hold for $p \neq 0$. A second example of this type will be given below.

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_r \\ y_1D & y_2D & \cdots & y_rD \\ \cdot & \cdot & \cdots & \cdot \\ y_1D^{r-1} & y_2D^{r-1} & \cdots & y_rD^{r-1} \end{vmatrix} = 0.$$

The usual proof of this result for analytic functions is valid here.† As a consequence we have

LEMMA 3. *The differential equation* $y(D^r+D^{r-1}a_1+\cdots+a_r)=0$, $a_i\epsilon\mathfrak{R}$, *has at most* r *solutions* $y_1, y_2, \cdots, y_r$ *in* $\mathfrak{R}$ *linearly independent over* $\mathfrak{F}$.

It has been shown by R. Baer‡ that if $\mathfrak{R}$ is a field of the type $\mathfrak{F}(c_1, c_2, \cdots, c_m)$, $c_i^p=\gamma_i\epsilon\mathfrak{F}$, there exists a derivation D such that the D-constants are precisely the elements of $\mathfrak{F}$. Let D denote a fixed derivation of this type and set $cD=c'$ for any $c\epsilon\mathfrak{R}$. D^p, D^{p^2}, $\cdots$ are derivations and since $\mathfrak{R}$ is commutative the transformation $Da_0+D^pa_1+\cdots+D^{p^{m-1}}a_{m-1}$ is a derivation for arbitrary right multiplication a_i $(=a_{ir})$ in $\mathfrak{R}$. If $Da_0+D^pa_1+\cdots+D^{p^{m-1}}a_{m-1}=0$, i.e., $y(Da_0+D^pa_1+\cdots+D^{p^{m-1}}a_{m-1})=0$ for all y in $\mathfrak{R}$, it follows by Lemma 3 and the fact that $\mathfrak{F}$ is the set of D-constants that all $a_i=0$. Thus as the a_i vary in $\mathfrak{R}$ we obtain in this way mp^m linearly independent (over $\mathfrak{F}$) derivations and hence the complete algebra $\mathfrak{D}$. We shall therefore call D a *generator* of $\mathfrak{D}$. Since D^{p^m} is a derivation we have

$$D^{p^m} = D^{p^{m-1}}b_{m-1} + D^{p^{m-2}}b_{m-2} + \cdots + Db_0.$$

Taking commutators with D we have by (19),

$$0 = D^{p^{m-1}}b'_{m-1} + D^{p^{m-2}}b'_{m-2} + \cdots + Db'_0,$$

and hence $b_i'=0$, i.e., $b_i=\beta_i\epsilon\mathfrak{F}$, and

$$D^{p^m} = D^{p^{m-1}}\beta_{m-1} + D^{p^{m-2}}\beta_{m-2} + \cdots + D\beta_0. \tag{24}$$

As a consequence of (19), we note also

$$[D^{p^k}a, D^{p^j}b] = D^{p^k}a^{(p^j)}b - D^{p^j}b^{(p^k)}a, \tag{25}$$

where $a^{(p^j)}=aD^{p^j}$. If $E=Da\neq 0$ then the E-constants are the same as the D-constants since the multiplication a is non-singular and hence E is a generator of $\mathfrak{D}$ also.

THEOREM 11. *The d-algebra of the field* $\mathfrak{R}=\mathfrak{F}(c_1, c_2, \cdots, c_m)$, $c_i^p=\gamma_i$ *is simple except when* $p=2$, $m=1$.

† See, for example, T. Chaundy, *Differential Calculus*, Oxford, 1935, p. 106.

‡ R. Baer, *Algebraische Theorie der differentiierbaren Funktionenkörper*. I, Sitzungsberichte, Heidelberger Akademie, 1927, pp. 15–32.

Let $\mathfrak{B} \neq 0$ be an ideal in $\mathfrak{D}$ and $E = Db_0 + D^p b_1 + \cdots + D^{p^j} b_j$, $b_j \neq 0$, $j < m$, belong to $\mathfrak{B}$. We call j the *length* of E and suppose E chosen in $\mathfrak{B}$ so that j is minimal. We assert that $j = 0$. For if $j > 0$ we may suppose $b_j = 1$. This is evident if $b_j' = 0$ or $b_j = \beta_j \epsilon \mathfrak{F}$ and if $b_j' \neq 0$, $[E, D(b_j')^{-1}] = Db_0^* + D^p b_1^* + \cdots + D^{p^j} \epsilon \mathfrak{B}$. But if $E = Db_0 + D^p b_1 + \cdots + D^{p^{j-1}} b_{j-1} + D^{p^j}$ then

$$[E, Da] = D(ab_0' - a'b_0 - \cdots - a^{(p^{j-1})} b_{j-1} - a^{(p^j)}) + D^p a b_1' + \cdots + D^{p^{j-1}} a b_{j-1}'.$$

$[E, Da]$ has length $< j$ and may be chosen $\neq 0$ since by Lemma 3 a may be chosen so that $ab_0' - a'b_0 - \cdots - a^{(p^{j-1})} b_{j-1} - a^{(p^j)} \neq 0$. This contradicts the minimality of j in $\mathfrak{B}$ and shows that $E = Db_0$, $b_0 \neq 0$. Since E as well as D is a generator of $\mathfrak{D}$, by changing the notation we may suppose that $\mathfrak{B} \supset D$. Then $\mathfrak{B} \supset [Da, D] = Da'$ also. Since the null space of the linear transformation D in $\mathfrak{R}$ has order 1, the order of $\mathfrak{R}'$ the set of all a' is $p^m - 1$ over $\mathfrak{F}$. If $1 \epsilon \mathfrak{R}'$ the smallest space containing all a' and 1 is $\mathfrak{R}$. Since $\mathfrak{B} \supset D$ and Da', $\mathfrak{B}$ will then contain Da for all a in $\mathfrak{R}$. Also if $p \neq 2$, $\mathfrak{B} \supset \frac{1}{2}[Da', Db] + \frac{1}{2}[Da'b, D] = Da''b$ and since a'' is not identically 0 and b is arbitrary, $\mathfrak{B} \supset Da$ for all a. Suppose finally that $p = 2$ and $\mathfrak{R}' \supset 1$, say $u' = 1$. Here $\mathfrak{B} \supset [[D^2 a, D], Db] = D^2 a''b + Da'b''$. If $m > 1$, a'' is not identically 0 and hence b may be chosen so that $a''b = u$. Set $a'b'' = v$. $\mathfrak{B} \supset [D^2 u + Dv, Da] + D(va + ua' + a)' = D^2 a$ and $[D^2 a, Db] + D^2 a'b = Dab''$. Thus in any case unless $p = 2$, $m = 1$, $\mathfrak{B} \supset Da$ for all a and since $[D^{p^k} b, Da] + Da^{(p^k)} b = D^{p^k} b' a$, $\mathfrak{B} \supset$ all $D^{p^k} a$ so that $\mathfrak{B} = \mathfrak{D}$.

If $p = 2$, $m = 1$, $\mathfrak{D}$ has order 2 and hence is solvable. In all other cases the algebras $\mathfrak{D}$ are simple algebras which, like inseparable fields, have no counterparts for $p = 0$.

If E is any derivation, the totality of expressions $E^{p^k} a_0 + E^{p^{k-1}} a_1 + \cdots + Ea_g$, $a_i \epsilon \mathfrak{R}$ is, by virtue of (15) and (19), a restricted $\mathfrak{R}$-($= \mathfrak{C}$-)subalgebra $\mathfrak{E}$ of $\mathfrak{D}$. Conversely if $\mathfrak{E}$ is any restricted $\mathfrak{R}$-subalgebra of $\mathfrak{D}$, $\mathfrak{E}$ is generated in this fashion. To prove this let $E = D^{p^e} g_0 + D^{p^{e-1}} g_1 + \cdots + Dg_e$, $g_0 \neq 0$, be an element of smallest length in $\mathfrak{E}$. Since $\mathfrak{E}$ is an $\mathfrak{R}$-algebra we may suppose that $g_0 = 1$ and then $E = D^{p^e} + D^{p^{e-1}} g_1 + \cdots + Dg_e$ is unique. If $F = D^{p^f} h_0 + D^{p^{f-1}} h_1 + \cdots + Dh_f \epsilon \mathfrak{E}$, $f \geqq e$ and

$$F_1 = F - E^{p^{f-e}} h_0 = D^{p^{f-1}} k_0 + \cdots + Dk_{f-1}$$

by (15) and (25). F_1 has length $\leqq f - 1$ and belongs to $\mathfrak{E}$. Repeating this process we obtain an expression for F of the form $E^{p^{f-e}} a_0 + \cdots + Ea_{f-e}$.

If as in I we denote the elements of $\mathfrak{R}$ which are constants for all the derivations in $\mathfrak{E}$ by $\mathfrak{R}(\mathfrak{E})$ it is clear that $\mathfrak{R}(\mathfrak{E})$ coincides with the subfield $\mathfrak{S}$ of E-constants. On the other hand if E is any derivation, $\mathfrak{S}$ the set of E-constants, then the argument at the beginning of this section shows that E gen-

erates the d-algebra $\mathfrak{D}(\mathfrak{S})$ of $\mathfrak{R}$ considered as a field over $\mathfrak{S}$. Hence $\mathfrak{D}(\mathfrak{S}) = \mathfrak{E}$, i.e.,

$$\mathfrak{D}(\mathfrak{R}(\mathfrak{E})) = \mathfrak{E}.$$

If $\mathfrak{S}$ is any subfield of $\mathfrak{R}$, $\mathfrak{D}(\mathfrak{S})$ contains an element E such that the E-constants are precisely $\mathfrak{S}$. Hence $\mathfrak{R}(\mathfrak{D}(\mathfrak{S}))$ cannot be larger than $\mathfrak{S}$ and so

$$\mathfrak{R}(\mathfrak{D}(\mathfrak{S})) = \mathfrak{S}.$$

We have therefore proved

THEOREM 12. *There is a* (1–1) *correspondence between the subfields* $\mathfrak{S}$ *of* $\mathfrak{R}$ *containing* $\mathfrak{F}$ *and the restricted* $\mathfrak{R}$*-subalgebras* $\mathfrak{E}$ *of the d-algebra* $\mathfrak{D}$ *of* $\mathfrak{R}$ *over* $\mathfrak{F}$. *The correspondence is given by either* $\mathfrak{E} = \mathfrak{D}(\mathfrak{S})$ *or* $\mathfrak{S} = \mathfrak{R}(\mathfrak{E})$.

10. We now suppose that $\mathfrak{R}$ is simple and that $\mathfrak{R} \supset \mathfrak{C} \supset \mathfrak{F}$ where $\mathfrak{C} = \mathfrak{F}(c_1, c_2, \cdots, c_m)$, $c_i^p = \gamma_i$ and $p \neq 0$. Let $\mathfrak{D}$ denote the d-algebra of $\mathfrak{R}$ over $\mathfrak{F}$ and $\mathfrak{E}$ that of $\mathfrak{C}$ over $\mathfrak{F}$. If $D \in \mathfrak{D}$, D induces a derivation in $\mathfrak{C}$ and hence $\mathfrak{D}$ is homomorphic with a subalgebra $\mathfrak{E}_1$ of $\mathfrak{E}$. Since $\mathfrak{D}(\mathfrak{C})$ is the set of elements corresponding to 0 in this homomorphism, we have $\mathfrak{E}_1 \cong \mathfrak{D}/\mathfrak{D}(\mathfrak{C})$. But by the corollary to Theorem 8, $\mathfrak{D}(\mathfrak{C}) = \mathfrak{J}$ and hence $\mathfrak{E}_1 \cong \mathfrak{D}/\mathfrak{J}$. We wish to show that $\mathfrak{E}_1 = \mathfrak{E}$.

$\mathfrak{R}$ may be regarded as a normal simple algebra over $\mathfrak{C}$ and there exists a separable field $\mathfrak{C}(s)$ over $\mathfrak{C}$ such that $\mathfrak{R} \times \mathfrak{C}(s) = \mathfrak{C}(s)_n$ the matrix algebra of order n^2 with elements in $\mathfrak{C}(s)$. As has been shown by Albert† the separable extension $\mathfrak{C}(s)$ of the inseparable field $\mathfrak{C}$ has the form $\mathfrak{K}(c_1, \cdots, c_m) = \mathfrak{C}_{\mathfrak{K}}$ where $\mathfrak{K}$ is a separable field over $\mathfrak{F}$. Now consider $\mathfrak{R}_{\mathfrak{K}}$. The centrum of this algebra is $\mathfrak{C}_{\mathfrak{K}} = \mathfrak{C}(s)$ and if $x_1, x_2, \cdots, x_{n^2}$ form a basis of $\mathfrak{R}$ over $\mathfrak{C}$ they are also a basis for $\mathfrak{R}_{\mathfrak{K}}$ over $\mathfrak{C}_{\mathfrak{K}} = \mathfrak{C}(s)$. It follows that $\mathfrak{R}_{\mathfrak{K}} = \mathfrak{R} \times \mathfrak{C}(s) = \mathfrak{C}(s)_n = (\mathfrak{C}_{\mathfrak{K}})_n$.

The d-algebra of $\mathfrak{R}_{\mathfrak{K}}$ is $\mathfrak{D}_{\mathfrak{K}}$ and the ideal of inner derivations of $\mathfrak{D}_{\mathfrak{K}}$ is $\mathfrak{J}_{\mathfrak{K}}$. If E^* is any derivation in $\mathfrak{C}_{\mathfrak{K}}$ over $\mathfrak{K}$, the correspondence $\sum e_{ij} c_{ij}^* \to \sum e_{ij}(c_{ij}^* E^*)$, $c_{ij}^* \in \mathfrak{C}_{\mathfrak{K}}$ is readily verified to be a derivation in $\mathfrak{R}_{\mathfrak{K}}$ inducing E^* in $\mathfrak{C}_{\mathfrak{K}}$. Hence $\mathfrak{D}_{\mathfrak{K}}/\mathfrak{J}_{\mathfrak{K}}$ is isomorphic to the complete d-algebra of $\mathfrak{C}_{\mathfrak{K}}$ and so has order mp^m over $\mathfrak{K}$. Since $\mathfrak{D}_{\mathfrak{K}}/\mathfrak{J}_{\mathfrak{K}} \cong (\mathfrak{D}/\mathfrak{J})_{\mathfrak{K}}$, $\mathfrak{D}/\mathfrak{J}$ has order mp^m over $\mathfrak{F}$. Comparing orders we have $\mathfrak{E} \cong \mathfrak{D}/\mathfrak{J}$.

THEOREM 13. *Suppose* $\mathfrak{R}$ *is a simple algebra of order* n^2 *over its centrum* $\mathfrak{C} = \mathfrak{F}(c_1, c_2, \cdots, c_m)$, $c_i^p = \gamma_i$, $p \neq 0$. *Then the d-algebra of* $\mathfrak{D}$ *over* $\mathfrak{R}$ *is semi-simple unless* $p = 2$ *and either* $n \leqq 2$ *or* $m = 1$.

Let $\mathfrak{B}$ be a solvable ideal in $\mathfrak{D}$. $\mathfrak{B} + \mathfrak{J}$‡ is an ideal and $(\mathfrak{B} + \mathfrak{J})/\mathfrak{J}$ is a solva-

† A. A. Albert, *Simple algebras of degree* p^e *over a centrum of characteristic* p, these Transactions, vol. 40 (1936), p. 113.

‡ $\mathfrak{B} + \mathfrak{J}$ denotes the smallest space containing $\mathfrak{B}$ and $\mathfrak{J}$.

ble ideal in $\mathfrak{D}/\mathfrak{J}\cong\mathfrak{E}$. But by Theorem 11, $\mathfrak{E}$ is simple. Hence $(\mathfrak{B}+\mathfrak{J})/\mathfrak{J}=0$ or $\mathfrak{B}+\mathfrak{J}=\mathfrak{J}$ and $\mathfrak{B}\subset\mathfrak{J}$. However, by Theorem 10, $\mathfrak{J}$ is semi-simple and so $\mathfrak{B}=0$.

$\mathfrak{D}$ is not a direct sum of $\mathfrak{J}$ and a second ideal. For we have seen (§5) that the elements commutative with all elements in $\mathfrak{J}$ are those mapping $\mathfrak{R}$ into $\mathfrak{C}$. If F is such an element, then F^* the extension of F maps $\mathfrak{R}_\mathfrak{g}$ into $\mathfrak{C}_\mathfrak{g}$ (cf. Theorem 5). If $e_{ij}F^*=c_{ij}^*\epsilon\mathfrak{C}_\mathfrak{g}$, it follows from $e_{ij}e_{kl}=\delta_{jk}e_{il}$ that $c_{ij}^*=0$. Hence $(e_{ij}c^*)F^*=e_{ij}(c^*F^*)$ for $c^*\epsilon\mathfrak{C}_\mathfrak{g}$. If this belongs to $\mathfrak{C}_\mathfrak{g}$ we must have $c^*F^*=0$. Thus $F^*=0$, $F=0$, and $\mathfrak{D}$ is not a direct sum.

III. Theory of D-fields

11. In this part we propose to study $\mathfrak{R}=\mathfrak{F}(c_1, c_2, \cdots, c_m)$, $c_i^p=\gamma_i$, $p\neq 0$ relative to the fixed derivation D and shall obtain several analogues of theorems on automorphisms of cyclic fields. Without loss of generality we may assume that $\mathfrak{F}$ is the field of D-constants and hence D is a generator of the d-algebra of $\mathfrak{R}$. We have seen that D satisfies (24),

$$D^{p^m}=D^{p^{m-1}}\beta_1+D^{p^{m-2}}\beta_2+\cdots+D\beta_m,$$

and no equation of lower degree of the form $D^r+D^{r-1}a_1+\cdots+a_r$, $a_i\epsilon\mathfrak{R}$.

Suppose $y_1, y_2, \cdots, y_{p^m}$ is a basis for $\mathfrak{R}$ and

$$(y_1D, y_2D, \cdots, y_{p^m}D)=(y_1, y_2, \cdots, y_{p^m})\Delta \qquad \Delta=(\alpha_{ij}).$$

If $f(\lambda)$ is the characteristic function $|\lambda 1-\Delta|$, then by the Hamilton-Cayley theorem, $f(D)=0$. Since the degree of $f(\lambda)$ is p^m we have

$$f(\lambda)=|\lambda 1-\Delta|=\lambda^{p^m}-\lambda^{p^{m-1}}\beta_1-\cdots-\lambda\beta_m. \tag{26}$$

Since the characteristic and minimum equations of A are identical, A is similar to

$$B=\begin{pmatrix} 0 & & & 0 \\ 1 & & & \beta_m \\ & 1 & & 0 \\ & & \ddots & \vdots \\ & & & \beta_1 \\ & & & 0 \\ & & & \vdots \\ & & 1 & 0 \end{pmatrix}.$$

It follows that $\mathfrak{R}$ has a basis of the form $z, zD, zD^2, \cdots, zD^{p^m-1}$, i.e., $\mathfrak{R}$ is a

cyclic space relative to the linear transformation D.†

A polynomial of the form $\lambda^{p^e}+\lambda^{p^{e-1}}\rho_1+\cdots+\lambda\rho_e$ will be called a *p-polynomial*.‡ A subfield $\mathfrak{S}$ of $\mathfrak{R}$ containing $\mathfrak{F}$ and vD for every v in $\mathfrak{S}$ will be called a *D-subfield* of $\mathfrak{R}$. Thus $\mathfrak{S}$ is a space invariant under the transformation D.

THEOREM 14. *There is a* (1–1) *correspondence between the D-subfields of* $\mathfrak{R}$ *and the p-polynomial factors of* $f(\lambda)$.

Any subspace $\mathfrak{S}$ of $\mathfrak{R}$ is cyclic with generator w. If $g(\lambda)$ is a polynomial of least degree such that $wg(D)=0$ then $g(\lambda)$ is the minimum function of D acting in $\mathfrak{S}$ and the order of $\mathfrak{S}$ = degree of $g(\lambda)$. $g(\lambda)$ is therefore uniquely determined by $\mathfrak{S}$ and is a factor of $f(\lambda)$. For if $h(\lambda)=f(\lambda)q(\lambda)+g(\lambda)r(\lambda)$ $=(g(\lambda), f(\lambda))$ then $wh(D)=0$ and since $g(\lambda)$ is minimal, $g(\lambda)=h(\lambda)$. Conversely if $g(\lambda)$ is a factor of $f(\lambda)$, $f(\lambda)=g(\lambda)k(\lambda)$, the vectors v such that $vg(D)=0$ form an invariant subspace $\mathfrak{S}$. $\mathfrak{S}\supset zk(D)$, $zd(D)D, \cdots$ if z is a generator of $\mathfrak{R}$, and if the degree of $g(\lambda)$ is r, $zk(D)$, $zk(D)D, \cdots, zk(D)D^{r-1}$ are linearly independent. Hence the order of $\mathfrak{S}$ is $\geqq r$. On the other hand the minimum function of D in $\mathfrak{S}$ is $g(\lambda)$ so that order of $\mathfrak{S}$ is r, $\mathfrak{S}=(zk(D),$ $zk(D)D, \cdots)$. Thus we have a (1–1) correspondence between the invariant subspaces $\mathfrak{S}$ of $\mathfrak{R}$ and the factors $g(\lambda)$ of $f(\lambda)$. If $\mathfrak{S}$ is a field, D is a generator of the d-algebra of $\mathfrak{S}$ over $\mathfrak{F}$ and hence $g(\lambda)$ is a p-polynomial. Conversely if $g(\lambda)$ is a p-polynomial and $v_1, v_2\epsilon\mathfrak{S}$, i.e., $v_1g(D)=v_2g(D)=0$, then since $g(D)$ is a derivation, $v_1v_2g(D)=(v_1g(D))v_2+v_1(v_2g(D))=0$ so that $\mathfrak{S}$ is closed under multiplication and hence is a D-subfield of $\mathfrak{R}$.

Suppose $g(\lambda)=\lambda^{p^e}+\lambda^{p^{e-1}}\rho_1+\cdots+\lambda\rho_e$, $h(\lambda)=\lambda^{p^f}+\lambda^{p^{f-1}}\sigma_1+\cdots+\lambda\sigma_f$ and $e\leqq f$. Then $g(\lambda)-h(\lambda)^{p^{e-f}}=\lambda^{p^{e-1}}\tau_1+\cdots+\lambda\tau_{e-1}$. By repeating this process we may express $g(\lambda)$ in the form

$$g(\lambda) = h(\lambda)^{p^{e-f}} + h(\lambda)^{p^{e-f-1}}\omega_1 + \cdots + h(\lambda)\omega_{e-f} + r(\lambda) \qquad (\omega_1 = \tau_1),$$

where $r(\lambda)$ is a p-polynomial of degree $<p^f$. Since $r(\lambda)$ is the remainder obtained by dividing $g(\lambda)$ by $h(\lambda)$, by continuing the euclid algorithm we find that $(g(\lambda), h(\lambda))$ *is a p-polynomial.*

If $k(\lambda)$ is any polynomial (coefficients in $\mathfrak{F}$), then

$$\lambda^{p^j} = k(\lambda)q_j(\lambda) + s_j(\lambda) \qquad (j = 0, 1, 2, \cdots),$$

where degree $s_j(\lambda)<$degree $k(\lambda)=r$. Since there are at most r independent polynomials of degree $<r$ there exist elements, $\alpha_0, \alpha_1, \cdots, \alpha_r$ not all 0 such that $s_r(\lambda)\alpha_0+s_{r-1}(\lambda)\alpha_1+\cdots+s_0(\lambda)\alpha_r=0$ and hence

† For a discussion of cyclic spaces see Jacobson, *Pseudo-linear transformations*, Annals of Mathematics, vol. 38 (1937), p. 496.

‡ This term is due to O. Ore, *On a special class of polynomials*, these Transactions, vol. 35 (1933), p. 560.

$$h(\lambda) = \lambda^{p^r}\alpha_0 + \lambda^{p^{r-1}}\alpha_1 + \cdots + \lambda\alpha_r = k(\lambda)\sum q_j(\lambda)\alpha_j,$$

i.e., *any polynomial is a factor of some p-polynomial.*† Since the h.c.f. of p-polynomials is a p-polynomial, the p-polynomial of least degree divisible by $k(\lambda)$ is unique. We denote it by $\{k(\lambda)\}$.

Now suppose $\mathfrak{S}$ is a subspace of $\mathfrak{R}$ invariant under D and $k(\lambda)$ is the minimum function of D in $\mathfrak{S}$. Let $\{\mathfrak{S}\}$ denote the enveloping field of $\mathfrak{S}$. $\{\mathfrak{S}\}$ is a D-field and D has minimum function $\{k(\lambda)\}$ in $\{\mathfrak{S}\}$. If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are invariant subspaces, $k_1(\lambda)$, $k_2(\lambda)$ the corresponding minimum functions, then $\mathfrak{S}_1+\mathfrak{S}_2$ and $\mathfrak{S}_1 \cap \mathfrak{S}_2$ are invariant and the associated functions are respectively $[k_1(\lambda), k_2(\lambda)]$ and $(k_1(\lambda), k_2(\lambda))$.

12. Let $\mathfrak{M}$ denote the algebra of linear transformations generated by D and the multiplications of $\mathfrak{R}$. Since $D^{p^m-1}a_1+D^{p^m-2}a_2+\cdots+a_{p^m}=0$ implies all $a_i=0$, $\mathfrak{M}$ has order p^{2m} over $\mathfrak{F}$ and hence is isomorphic to $\mathfrak{F}_{p^m}$ the algebra of all $p^m \times p^m$ matrices in $\mathfrak{F}$. The multiplication of the elements of $\mathfrak{M}$ may be ascertained from the multiplications of the elements of $\mathfrak{R}$ and the rules

$$(27) \quad \text{(a)}\ aD = Da + a', \qquad \text{(b)}\ f(D) = D^{p^m} - D^{p^{m-1}}\beta_1 - \cdots - D\beta_m = 0.$$

Let c be an arbitrary element of $\mathfrak{R}$ and consider the powers of $D_1 = D+c$. From (27a) we obtain by induction

$$(28) \quad D_1^k = (D+c)^k = D^k + C_{k,1}D^{k-1}V_1(c) + C_{k,2}D^{k-2}V_2(c) + \cdots + V_k(c),$$

where

$$(29) \qquad V_1(c) = c, \qquad V_j(c) = V_{j-1}(c)' + V_{j-1}(c)c.$$

For $k=p^j$, (28) specializes to

$$(30) \qquad D_1^{p^j} = (D+c)^{p^j} = D^{p^j} + V_{p^j}(c).$$

D_1 evidently satisfies (27a), and from (30) and (27b) we have as the condition that D_1 also satisfies (27b),

$$(31) \qquad V(c) = V_{p^m}(c) - V_{p^{m-1}}(c)\beta_1 - \cdots - V_1(c)\beta_m = 0.$$

On the other hand if D_1 satisfies (27) the correspondence $D \to D_1$, $a \to a$ defines an automorphism of $\mathfrak{M}$ and conversely. Since every automorphism of $\mathfrak{M} \cong \mathfrak{F}_{p^m}$ is inner there exists an element $B \in \mathfrak{M}$ such that

$$B^{-1}aB = a, \qquad B^{-1}D_1B = D$$

for all a in $\mathfrak{R}$. Since $\mathfrak{R}$ has maximum order for a commutative subfield of $\mathfrak{M}$, $B = b \in \mathfrak{R}$ and hence the second condition gives

† This result is due to Ore, loc. cit., p. 581.

$$c = b^{-1}b', \tag{32}$$

i.e., c is a *logarithmic derivative*. We have therefore proved

THEOREM 15. *A necessary and sufficient condition that* $c \in \mathfrak{R}$ *be a logarithmic derivative is that* (31) *hold.*

This is an analogue of Hilbert's theorem on the elements of norm 1 in a cyclic field. $V(c)$ takes the part of the norm and derivation that of the generating automorphism of the cyclic field.

We denote the set of logarithmic derivatives by $\mathfrak{L}$. Since $-b'/b = (b^{-1})'/(b^{-1})$and $b'/b+c'/c=(bc)'/bc$, $\mathfrak{L}$ is a group under addition and the correspondence $b \to b'/b$ establishes a homomorphism between the multiplicative group of $\mathfrak{R}$ and $\mathfrak{L}$. The elements corresponding to 0 here are those of $\mathfrak{F}$. Hence $\mathfrak{L} \cong \mathfrak{R}/\mathfrak{F}$.

By means of the recursion formula (29) we may prove by induction

$$V_j(c) = \sum_{i=1}^{j} P_{ij}, \qquad P_{ij} = \sum \frac{j!}{\alpha!\beta!\cdots}\left(\frac{c}{1!}\right)^{\alpha}\left(\frac{c'}{2!}\right)^{\beta}\cdots, \tag{33}$$

where the summation in P_{ij} is extended over all non-negative integers such that

$$\alpha + \beta + \gamma + \cdots = i, \qquad \alpha + 2\beta + 3\gamma + \cdots = j.$$

(The coefficients in P_{ij} are understood to be the integers obtained by cancelling the common factors in $j!/(\alpha!\beta!\cdots)(1!)^{\alpha}(2!)^{\beta}\cdots$.) By (33) it is easily seen that $V_p(c) = c^p + c^{(p-1)}$. Since

$$D_1^{p^j} = (D_1^{p^{j-1}})^p = (D^{p^{j-1}} + V_{p^{j-1}}(c))^p = D^{p^j} + V_{p^j}(c),$$

we have

$$V_{p^j}(c) = (V_{p^{j-1}}(c))^p + (V_{p^{j-1}}(c))^{(p^j - p^{j-1})}, \tag{34}$$

and hence

$$V_{p^j}(c) = c^{p^j} + (c^{(p-1)})^{p^{j-1}} + (c^{(p^2-1)})^{p^{j-2}} + \cdots + c^{(p^j-1)}. \tag{35}$$

Then $V_{p^j}(c)' = c^{(p^j)}$ and so by (27b), $(V(c))' = 0$, i.e., $V(c) \in \mathfrak{F}$ *for any* c *in* $\mathfrak{R}$. Also by (35), or more directly by (30),

$$V(b + c) = V(b) + V(c). \tag{36}$$

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THEORY OF NON-LINEAR SINGULAR DIFFERENTIAL SYSTEMS*

BY

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1. **Introduction.** The most important published paper dealing with this subject is a recent work by the present author† in which further references to the literature of the subject will be found. A method, introduced for the first time by Trjitzinsky in (T_1), which is specifically applicable to non-linear problems lies at the basis of the developments in the present work.‡

In this paper we consider the system of non-linear differential equations

$$(A) \qquad t^{-p}y_j^{(1)}(t) = a_j(t, y_1, \cdots, y_n) \qquad (j = 1, \cdots, n;\ p \text{ an integer} \geqq 0),\ \S$$

where

$$(1.1) \qquad a_j(t, y_1, \cdots, y_n) = l_j(t, y_1, \cdots, y_n) + q_j(t, y_1, \cdots, y_n),$$

$$(1.1a) \qquad l_j(t, y_1, \cdots, y_n) = l_{1,j}(t)y_1 + \cdots + l_{n,j}(t)y_n,$$

$$(1.1b) \qquad q_j(t, y_1, \cdots, y_n) = \sum {}_j a_{i_1, i_2, \cdots, i_n}(t) y_1^{i_1} y_2^{i_2} \cdots y_n^{i_n}$$
$$(i_1, i_2, \cdots, i_n \geqq 0;\ i_1 + \cdots + i_n \geqq 2;\ j = 1, \cdots, n).$$

The coefficients $l_{i,j}(t)$, ${}_j a_{i_1, \cdots, i_n}(t)$ are assumed to be analytic at $t = \infty$ for $|t| \geqq r$ (>0), while the series of the second member of (1.1b) are supposed to be convergent for‖

$$(1.2) \qquad |t| \geqq r;\ |y_1|, |y_2|, \cdots, |y_n| \leqq \rho.$$

Moreover, *it will be assumed that the linear system obtained by letting* $q_j(t, y_1, \cdots, y_n) \equiv 0$ $(j = 1, \cdots, n)$ *is actually of order n, and that it is not of Fuchsian type* (cf. §2, italics preceding (2.6)).

The analytic theory will be developed for the complex neighborhood of the singular point $t = \infty$. The main results of this work are embodied in the theorems at the end of §§7, 9, 10, and 15.

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† W. J. Trjitzinsky, *Analytic theory of non-linear singular differential equations*, Mémorial des Sciences Mathématiques, Paris. In the sequel this paper will be referred to as (T_1).

‡ This method has been also applied in the paper by W. J. Trjitzinsky, *Non-linear difference equations*, Compositio Mathematica, vol. 5 (1937), pp. 1–66.

§ p is taken as small as is compatible with the stated hypotheses.

‖ ρ is taken sufficiently small so that the function represented by the series is analytic at every point of the specified region.

The system (A) contains as special cases some of the important instances of dynamical differential systems. From the point of view of dynamical applications the real variable theory is important. However, in so far as the problems of the present paper are concerned our interest lies in the complex plane. A considerable variety of new situations of mathematical interest arises in the field now under consideration.

We shall also consider *systems* (B), *obtained from* (A) *by assuming that the*

$$l_{i,j}(t), \qquad {}_{j}a_{i_1,i_2,\cdots,i_n}(t)$$

are all independent of t *and that* $p=0$. These systems are of importance in dynamics. Our concern here, as well as throughout the paper, is with the analytic character of solutions "in the small" and, as stated before, with the complex neighborhood of the singular point.

Finally, in §§11–15 an investigation is given *for systems* (C) *of the form*

$$\text{(C)} \qquad \lambda^{-p}y_j^{(1)}(x) = a_j(\lambda, x, y_1, \cdots, y_n) \qquad (j = 1, \cdots, n;\ \textit{integer}\ p \geqq 0)$$

where λ *is a parameter and where the second members are analytic in* $\lambda, y_1, \cdots, y_n$ *at* $\lambda = \infty, y_1 = \cdots = y_n = 0$, *for* x *on a real interval, and continuous in* x *on this interval.* The theory of such systems will be given for the neighborhood of the singular point $\lambda = \infty$. The precise formulation of this problem, together with some references, will be found in §11.

2. **Semi-formal aspects.** Formal solutions of the system (A) will now be constructed. These will appear as series, in general divergent, whose coefficients are functions determined by a sequence of well-defined analytic processes. For these reasons the construction and consideration of these solutions can be referred to as the domain of "semi-formal aspects" of the theory of systems (A).

Consider now the linear system associated with (A),

$$\text{(LA)} \qquad t^{-p}y_i^{(1)}(t) = l_i(t, y_1, \cdots, y_n) \qquad (i = 1, \cdots, n)$$

(cf. (1.1a)). By hypothesis

$$(2.1) \qquad |(l_{i,j}(t))| \not\equiv 0.\dagger$$

The general asymptotic theory of such systems has been given by Trjitzinsky.‡ It will be assumed that the reader is acquainted with the main results

† $A = (a_{i,j})$ is to denote a matrix of n^2 elements $(i, j = 1, \cdots, n)$ with the displayed element in the ith row and jth column. The symbol $|(a_{i,j})| = |A|$ is to stand for the determinant of A. The inverse of the matrix A will be denoted by $A^{-1}(=(a_{i,j})^{-1})$.

‡ W. J. Trjitzinsky, *Analytic theory of linear differential equations*, Acta Mathematica, vol. 62 (1934), pp. 167–226. In the sequel this paper will be referred to as (T_2).

and methods contained in (T_2).

The system (LA) can be written in matrix form as

$$(LA_1) \qquad Y^{(1)}(t) = Y(t)A(t), \qquad Y(t) = (y_{i,j}(t)),$$

where

$$(2.2) \qquad A(t) = (t^{p_{i,j}}(t)) \qquad (i, j = 1, \cdots, n).$$

The elements of a row in $Y(t)$ will constitute a solution of (LA). The system (LA_1) has a singularity of finite rank at $t=\infty$.

The following definitions will now be introduced.

DEFINITION 1. *Generically* $\{t_q\}$ (q *an integer* $\geqq 0$) *is to denote an expression*

$$\rho_0(t) + \rho_1(t) \log t + \cdots + \rho_q(t) \log^q t,$$

where $\rho_j(t)$ ($j=0, \cdots, q$) *is a series, possibly divergent, of the form*

$$(2.3) \qquad \rho_{j,0} + \rho_{j,1}t^{-1/k} + \rho_{j,2}t^{-2/k} + \cdots$$

(k *a positive integer*). $\{t\}^*$ *is to stand for an expression* $\{t\}_q$.

DEFINITION 2. *A curve B will be said to be regular if it is analytic in every finite part and extends to infinity where it possesses a unique limiting direction.*†

DEFINITION 3. *A region R will be said to be regular if it is closed, extends to infinity, and is such that for t in R* $|t| \geqq r_1 (>0)$. *The boundary of R is to be simple and is to consist of an arc* γ *of the circle* $|t| = r_1$ *and of two regular* (cf. Definition 2) *curves extending from different extremities of* γ. *Generically* $R(\theta_1, \theta_2)$ *is to denote a regular region for which the two regular curves (making part of the boundary) have limiting directions* θ_1 *and* θ_2, *respectively. The number* $|\theta_1 - \theta_2|$ *will be termed the opening of R.*

DEFINITION 4. *Generically* $[t]_q$ *is to denote a function of the form*

$$p_0(t) + p_1(t) \log t + \cdots + p_q(t) \log^q t,$$

where $p_j(t)$ ($j=0, \cdots, q$) *is a function, analytic for* $t \neq \infty$ *in a regular* (cf. Definition 3) *region R, such that*

$$(2.4) \qquad p_j(t) \sim \rho_j(t) \qquad (t \text{ in } R),$$

where $\rho_j(t)$ *is of the form* (2.3).‡ $[t]^*$ *is to stand for* $[t]_q$.

† The implication is that a regular curve ($|t| \geqq r'$) is representable by an equation $Rf(t)=0$, where $f(t)$ is analytic for $|t| \geqq r'$, but may possess a singularity at $t=\infty$.

‡ Unless otherwise stated asymptotic relations are asymptotic in the ordinary sense, i.e., to infinitely many terms. Thus, if $a(t) \sim a_0 + a_1 t^{-1/k} + \cdots$ (t in a regular region R) to infinitely many terms, one has $a(t) = a_0 + \cdots + a_{n-1}t^{-(n-1)/k} + a_n(t)t^{-n/k}$ ($|a_n(t)| < b_n$; t in R) for $n=1, 2, \cdots$. If n cannot be increased beyond n_0 the above relation is to n_0 terms.

One may write $[t]_q \sim \{t\}_q$ and $[t]^* \sim \{t\}^*$ (t in R).

The system (LA_1) possesses a formal matrix solution†

$$S(t) = (s_{i,j}(t)) = (e^{Q_i(t)} t^{r_{i,j}} \sigma_{i,j}(t)), \tag{2.5}$$

where

$$Q_i(t) = \sum_{\nu=0}^{l_i-1} q_{\nu}^{i} t^{(l_i-\nu)/k_i} \qquad \text{(integer } k_i \geqq 1;\ i = 1, \cdots, n;\ l_i \geqq 1), \tag{2.5a}$$

the $r_{i,j}$ for a fixed i may differ only by rational numbers, and

$$\sigma_{i,j}(t) = \{t\}_q \qquad (i, j = 1, \cdots, n). \ddagger \tag{2.5b}$$

Moreover, formally $|S(t)|$ does not vanish.§

In consequence of a hypothesis made in §1 the system (LA) is not of Fuchsian type; that is, *not all of the polynomials* $Q_i(t)$ $(i = t, \cdots, n)$ *involved in* (2.5) *are identically zero.* We let $B_{i,j}$ denote a regular curve along which

$$R(Q_i(t) - Q_j(t)) = 0. \tag{2.6}$$

It will be understood that there are no $B_{i,j}$ curves corresponding to any pair of values i, j for which $Q_i(t) \equiv Q_j(t)$. Let

$$R_1', R_2', \cdots, R_N' \tag{2.6a}$$

be regular regions (cf. Definition 3) separated by the $B_{i,j}$ curves (formed for all admissible pairs of values i, j) in such a way that interior to any such region there is no $B_{i,j}$ curve. Consider a particular region R_k' of the set (2.6a). It is of the form $R(\theta_{k,1}, \theta_{k,2})$ (cf. Definition 3) where, let us say, $\theta_{k,1} \leqq \theta_{k,2}$. Let $B_{k,l}'$ and $B_{k,r}'$ denote the regular curves forming part of the boundary of R_k' and possessing at infinity the limiting directions $\theta_{k,1}$ and $\theta_{k,2}$, respectively.

In view of the Fundamental Existence Theorem given in (T_2) and in consequence of the connection between single nth order linear differential equations with systems, the following can be asserted for any fixed k $(1 \leqq k \leqq N')$. If $\theta_{k,1} = \theta_{k,2}$, the matrix equation (LA_1) has a matrix solution

$$Y(t) = (y_{i,j}(t)) \qquad (i, j = 1, \cdots, n),$$

whose elements $y_{i,j}(t)$ are analytic in R_k' for $t \neq \infty$, while

$$Y(t) \sim S(t) \qquad (t \text{ in } R_k'). \tag{2.7}$$

† (T_2), p. 171.

‡ It is seen that the $\sigma_{i,j}(t)$ can be so selected that the $r_{i,j}$ are independent of j.

§ This implies that when $|S(t)|$ is formally computed as an expression of the type (exp. [polynomial in t^{α}]) $t^{\beta} \{t\}_0$ (cf. Definition 1), not all the coefficients in $\{t\}_0$ are zero.

The implication of (2.7) is

$$y_{i,j}(t) = e^{Q_i(t)}t^{r_{i,j}}[t]_q \qquad (i, j = 1, \cdots, n; t \text{ in } R_k') \tag{2.7a}$$

(cf. Definition 4). If $\theta_{k,1}<\theta_{k,2}$ there exist regular overlapping subregions of R_k',

$$ {}_rR_k' = R(\theta_{k,1}, \theta_{k,2}), \qquad {}_lR_k' = R(\theta_{k,1}, \theta_{k,2}), \tag{2.8}$$

whose boundaries contain $B_{k,l}'$ and $B_{k,r}'$, respectively.† The regions (2.8) can be so selected (depending on the polynomials (2.5a); for details cf. (T_2)) that (LA_1) possesses two matrix solutions,

$$ {}_rY(t) = ({}_ry_{i,j}(t)), \qquad {}_lY(t) = ({}_ly_{i,j}(t)), \tag{2.8a}$$

such that

$$ {}_rY(t) \sim S(t) \qquad ({}_rt \text{ in } {}_rR_k'), \tag{2.8b}$$

$$ {}_lY(t) \sim S(t) \qquad ({}_lt \text{ in } {}_lR_k'). \tag{2.8c}$$

Let $P(t)$ stand for a polynomial (2.5a) which is not identically zero. It is clear that one may find a regular region $R' = R(\theta', \theta'')(\theta'<\theta'')$ such that

$$e^{P(t)} \sim 0 \qquad (t \text{ in } R').$$

Since not all of the $Q_i(t)$ are identically zero it is observed that *there exist regions*

$$R_1, R_2, \cdots, R_N \tag{2.9}$$

such that, if R stands for a particular one of them, the following statements are true:

(i) *R is a regular region* (cf. Definition 3) *which is a subset of a region referred to in* (2.7), (2.8b), (2.8c).

(ii) *There exist polynomials of the set* (2.5a), $Q_{i_1}(t), Q_{i_2}(t), \cdots, Q_{i_m}(t)$ $(1 \leqq m \leqq n)$ *such that*

$$e^{Q_{i_\nu}(t)} \sim 0 \qquad (\nu = 1, 2, \cdots, m; t \text{ in } R). \tag{2.9a}$$

As a matter of notation, which does not entail any loss of generality, the polynomials referred to in (2.9a) *will be designated as*

$$Q_1(t), Q_2(t), \cdots, Q_m(t). \tag{2.9b}$$

Moreover, without any loss of generality one may write

$$RQ_1(t) \leqq RQ_2(t) \leqq \cdots \leqq RQ_n(t) \qquad (t \text{ in } R). \tag{2.9c}$$

† The other regular curve, which forms part of the boundary of ${}_rR_k'$ (or ${}_lR_k'$), is interior to R_k' and has at infinity the limiting direction of $B_{k,r}'$ (or $B_{k,l}'$).

In the sequel, unless stated otherwise, we confine ourselves to a particular region R referred to in connection with (i), (ii), and (2.9b). In consequence of (i) there exists a matrix solution of (LA_1), ${}_0Y(t)=({}_0y_{i,j}(t))$, such that

$$ {}_0Y(t) \sim S(t) \qquad (t \text{ in } R). \tag{2.10} $$

We consider now a solution of (LA) of the form

$$ {}_1y_j(t) = \sum_{\lambda=1}^{m} c_\lambda \, {}_0y_{\lambda,j}(t) \qquad (j = 1, \cdots, n), \tag{2.11} $$

where $c_1, c_2, \cdots, c_m$ are arbitrary constants. The non-linear system (A) of §1 will be formally satisfied by the series (in general divergent) of the form

$$ y_j(t) = {}_1y_j(t) + {}_2y_j(t) + \cdots + {}_\nu y_j(t) + \cdots \qquad (j = 1, \cdots, n). \tag{2.12} $$

Here the ${}_\nu y_j(t)$ $(\nu=2, 3, \cdots)$ will be determined in the sequel as certain functions analytic in $R(t \neq \infty)$ and of the form

$$ {}_\nu y_j(t) = \sum_{k_1, \cdots, k_m} c_1^{k_1} c_2^{k_2} \cdots c_m^{k_m} \, {}_\nu\eta_{k_1, k_2, \cdots, k_m; j}(t) \tag{2.12a} $$

$$ (k_1, k_2, \cdots, k_m \geqq 0; \; k_1 + k_2 + \cdots + k_m = \nu). $$

The functions ${}_1y_j(t)$ will be representable by (2.12a) with $\nu=1$ if we let

$$ {}_0y_{\lambda,j}(t) = {}_1\eta_{k_1, k_2, \cdots, k_m; j}(t) \qquad (\lambda = 1, \cdots, m; \; j = 1, \cdots, n), \tag{2.12b} $$

where $k_\lambda=1$ and $k_i=0$ for $i \neq \lambda$.

To determine the terms of the series (2.12) these series are substituted in (A). If we take account of (1.1), (1.1a), (1.1b), it follows that, since ${}_1y_j(t)$ $(j=1, \cdots, n)$ is a solution of (LA), we have

$$ \sum_{\nu=2}^{\infty} {}_\nu y_j^{(1)}(t) - t^p \sum_{\nu=2}^{\infty} l_j(t, {}_\nu y_1, \cdots, {}_\nu y_n) = t^p q_j \qquad (j = 1, \cdots, n), \tag{2.13} $$

where

$$ q_j = \sum_{i_1, \cdots, i_n} {}_j a_{i_1, \cdots, i_n}(t) \prod_{\alpha=1}^{n} \Big[\sum_{\nu=1}^{\infty} {}_\nu y_\alpha(t) \Big]^{i_\alpha} \tag{2.13a} $$

$$ (i_1, \cdots, i_n \geqq 0; \; i_1 + \cdots + i_n \geqq 2). $$

The first member in (2.13) *may be written in the form*

$$ \sum_{\nu=2}^{\infty} \sum_{k_1 + \cdots + k_m = \nu} c_1^{k_1} \cdots c_m^{k_m} L_j(t, {}_\nu\eta_{k_1, \cdots, k_m; 1}, \cdots, {}_\nu\eta_{k_1, \cdots, k_m; n}), \tag{2.14} $$

where

$$L_j \equiv {}_{\nu}\eta_{k_1,\cdots,k_m:j}(t) - t^{\nu} l_j(t, {}_{\nu}\eta_{k_1,\cdots,k_m:1}, \cdots, {}_{\nu}\eta_{k_1,\cdots,k_m:n}). \tag{2.14a}$$

We now proceed to derive a formal expansion of q_j according to powers of $c_1, \cdots, c_m$. First it is noted that, with $i_\alpha > 0$,

$$\begin{aligned}\left[\sum_{\nu=1}^{\infty} {}_{\nu}y_\alpha(t)\right]^{i_\alpha} &= \sum_{\nu_1,\cdots,\nu_{i_\alpha}=1} {}_{\nu_1}y_\alpha\, {}_{\nu_2}y_\alpha \cdots {}_{\nu_{i_\alpha}}y_\alpha \\ &= \sum_{\gamma=i_\alpha}^{\infty} \sum_{\nu_1+\cdots+\nu_{i_\alpha}=\gamma} {}_{\nu_1}y_\alpha\, {}_{\nu_2}y_\alpha \cdots {}_{\nu_{i_\alpha}}y_\alpha \quad (1 \leqq \nu_1, \nu_2, \cdots).\end{aligned} \tag{2.15}$$

In view of (2.12a) it is observed that†

$$\begin{aligned}\prod_{\beta=1}^{i_\alpha} {}_{\nu_\beta}y_\alpha &= \prod_{\beta=1}^{i_\alpha} \sum_{k_1^\beta+\cdots+k_m^\beta=\nu_\beta} c_1^{k_1^\beta} c_2^{k_2^\beta} \cdots c_m^{k_m^\beta}\, {}_{\nu_\beta}\eta_{k_1,\cdots,k_m:\alpha}^{\beta} \\ &= \sum_{k_1^1,\cdots,k_m^{i_\alpha}}^{(1)} c_1^{k_1^1+\cdots+k_1^{i_\alpha}} c_2^{k_2^1+\cdots+k_2^{i_\alpha}} \cdots c_m^{k_m^1+\cdots+k_m^{i_\alpha}}\, {}_{\nu_1}\eta_{k_1^1,\cdots,k_m^1:\alpha} \\ &\qquad \cdot {}_{\nu_2}\eta_{k_1^2,\cdots,k_m^2:\alpha} \cdots {}_{\nu_{i_\alpha}}\eta_{k_1^{i_\alpha},\cdots,k_m^{i_\alpha}:\alpha},\end{aligned} \tag{2.15a}$$

the summation symbol of the last member above being specified as follows:

$$\begin{aligned}\sum^{(1)} = \sum (k_i^\beta \geqq 0;\ & k_1^1 + k_2^1 + \cdots + k_m^1 = \nu_1; \\ & k_1^2 + k_2^2 + \cdots + k_m^2 = \nu_2; \cdots; k_1^{i_\alpha} + k_2^{i_\alpha} + \cdots + k_m^{i_\alpha} = \nu_{i_\alpha}).\end{aligned} \tag{2.15b}$$

It is next inferred that

$$\prod_{\beta=1}^{i_\alpha} {}_{\nu_\beta}y_\alpha = \sum_{\delta_1+\cdots+\delta_m=\gamma} c_1^{\delta_1} c_2^{\delta_2} \cdots c_m^{\delta_m}\, {}_{\gamma}\phi_{\nu_1,\nu_2,\cdots,\nu_{i_\alpha}:\alpha}^{\delta_1,\delta_2,\cdots,\delta_m:i_\alpha}, \tag{2.16}$$

where

$${}_{\gamma}\phi_{\nu_1,\cdots,\nu_{i_\alpha}:\alpha}^{\delta_1,\cdots,\delta_m:i_\alpha} = \sum_{k_1^1,\cdots,k_m^{i_\alpha}}^{(2)} \left[{}_{\nu_1}\eta_{k_1^1,\cdots,k_m^1:\alpha} \cdots {}_{\nu_{i_\alpha}}\eta_{k_1^{i_\alpha},\cdots,k_m^{i_\alpha}:\alpha}\right], \tag{2.16a}$$

$$\begin{aligned}\sum^{(2)} = \sum [& k_1^1 + \cdots + k_1^{i_\alpha} = \delta_1;\ k_2^1 + \cdots + k_2^{i_\alpha} = \delta_2; \cdots; \\ & k_m^1 + \cdots + k_m^{i_\alpha} = \delta_m;\ k_1^1 + \cdots + k_m^1 = \nu_1;\ k_1^2 + \cdots + k_m^2 = \nu_2; \cdots; \\ & k_1^{i_\alpha} + \cdots + k_m^{i_\alpha} = \nu_{i_\alpha}].\end{aligned} \tag{2.16b}$$

Substitution of (2.16) into (2.15) will yield

$$\left[\sum_{\nu=1}^{\infty} {}_{\nu}y_\alpha(t)\right]^{i_\alpha} = \sum_{\gamma=i_\alpha}^{\infty} \sum_{\delta_1+\cdots+\delta_m=\gamma} c_1^{\delta_1} c_2^{\delta_2} \cdots c_m^{\delta_m}\, {}_{\gamma}\psi_\alpha^{\delta_1,\delta_2,\cdots,\delta_m:i_\alpha} \tag{2.17}$$

† Here the superscripts attached to the k_i do not indicate powers.

with

$$(2.17a)\qquad {}_{\gamma}\psi_{\alpha}^{\delta_1,\cdots,\delta_m:i_\alpha} = \sum_{\nu_1+\cdots+\nu_{i_\alpha}=\gamma} {}_{\gamma}\phi_{\nu_1,\cdots,\nu_{i_\alpha}:\alpha}^{\delta_1,\cdots,\delta_m:i_\alpha}.$$

It will be convenient to introduce the notation

$$(2.17b)\qquad \eta_{\gamma}^{\alpha:i_\alpha} = \sum_{\delta_1+\cdots+\delta_m=\gamma} C_1^{\delta_1}\cdots C_m^{\delta_m}\,{}_{\gamma}\psi_{\alpha}^{\delta_1,\cdots,\delta_m:i_\alpha}.$$

We may then write

$$(2.18)\qquad \prod_{\alpha=1}^{n}\Big[\sum_{\nu=1}^{\infty} {}_{\nu}y_{\alpha}(t)\Big]^{i_\alpha} = \prod_{\alpha=1}^{n}\sum_{\gamma_\alpha=i_\alpha}^{\infty}\eta_{\gamma_\alpha}^{\alpha:i_\alpha} = \sum_{\gamma_1,\cdots,\gamma_n}\prod_{q=1}^{n}\eta_{\gamma_q}^{q:i_q}$$
$$(\gamma_1 \geqq i_1,\ \gamma_2 \geqq i_2,\ \cdots,\ \gamma_n \geqq i_n).$$

In (2.18) the terms can be grouped so that

$$(2.18a)\qquad \prod_{\alpha=1}^{n}\Big[\sum_{\nu=1}^{\infty} {}_{\nu}y_{\alpha}(t)\Big]^{i_\alpha} = \sum_{H=i_1+\cdots+i_n}^{\infty}\ \sum_{\gamma_1+\cdots+\gamma_n=H}\prod_{q=1}^{n}\eta_{\gamma_q}^{q:i_q}$$
$$(\gamma_1 \geqq i_1,\ \cdots,\ \gamma_n \geqq i_n).$$

Before we proceed further, the product involved in the second member of (2.18a) will be expanded. In view of (2.17b)

$$(2.19)\qquad \begin{aligned}\prod_{q=1}^{n}\eta_{\gamma_q}^{q:i_q} &= \prod_{q=1}^{n}\sum_{{}_q\delta_1+\cdots+{}_q\delta_m=\gamma_q} C_1^{{}_q\delta_1}\cdots C_m^{{}_q\delta_m}\,{}_{\gamma_q}\psi_q^{{}_q\delta_1,\cdots\,{}_q\delta_m:i_q}\\ &= \sum_{{}_1\delta_1,\cdots,{}_n\delta_m} C_1^{{}_1\delta_1+\cdots+{}_n\delta_1}\,C_2^{{}_1\delta_2+\cdots+{}_n\delta_2}\cdots C_m^{{}_1\delta_m+\cdots+{}_n\delta_m}\\ &\quad\cdot {}_{\gamma_1}\psi_1^{{}_1\delta_1,\cdots,{}_1\delta_m:i_1}\,{}_{\gamma_2}\psi_2^{{}_2\delta_1,\cdots,{}_2\delta_m:i_2}\cdots {}_{\gamma_n}\psi_n^{{}_n\delta_1,\cdots,{}_n\delta_m:i_n}\end{aligned}$$
$$({}_1\delta_1+\cdots+{}_1\delta_m=\gamma_1;\ {}_2\delta_1+\cdots+{}_2\delta_m=\gamma_2;\ \cdots;\ {}_n\delta_1+\cdots+{}_n\delta_m=\gamma_n).$$

Substitute ${}_1\delta_q+{}_2\delta_q+\cdots+{}_n\delta_q=h_q$ $(q=1,2,\cdots,m)$ in (2.19). Then we have

$$(2.19a)\qquad \prod_{q=1}^{n}\eta_{\gamma_q}^{q:i_q} = \sum_{h_1,\cdots,h_m} C_1^{h_1}\cdots C_m^{h_m}\sum_{{}_1\delta_1,\cdots,{}_n\delta_m}^{(3)} {}_{\gamma_1}\psi_1^{{}_1\delta_1,\cdots,{}_1\delta_m:i_1}\cdots {}_{\gamma_n}\psi_n^{{}_n\delta_1,\cdots,{}_n\delta_m:i_n},$$

where

$$(2.19b)\qquad \sum^{(3)} = \sum\big[{}_1\delta_q+\cdots+{}_n\delta_q = h_q(q=1,\cdots,m);\ {}_1\delta_1+\cdots+{}_1\delta_m=\gamma_1;\ \cdots;\ {}_n\delta_1+\cdots+{}_n\delta_m=\gamma_n\big].$$

Now, as indicated in (2.18a), $\gamma_1+\cdots+\gamma_n=H$. Hence in (2.19a)

$$\sum_{q=1}^{m}h_q = \sum_{q=1}^{m}\sum_{i=1}^{n}{}_i\delta_q = \sum_{i=1}^{n}\gamma_i = H.$$

We shall write

$$\sum_{1^{\delta_1},\cdots,n^{\delta_m}}^{(3)} \cdots = {}_H\Delta_{\gamma_1,\cdots,\gamma_n}^{h_1,\cdots,h_m;i_1,\cdots,i_n}. \tag{2.19c}$$

Then (2.19a) will assume the form

$$\prod_{q=1}^{n} \eta_{\gamma_q}^{q;i_q} = \sum_{h_1+\cdots+h_m=H} c_1^{h_1} \cdots c_m^{h_m}\, {}_H\Delta_{\gamma_1,\cdots,\gamma_n}^{h_1,\cdots,h_m;i_1,\cdots,i_n}. \tag{2.20}$$

Substitution of (2.20) into (2.18a) will yield

$$\prod_{\alpha=1}^{n}\left[\sum_{\nu=1}^{\infty} {}_\nu y_\alpha(t)\right]^{i_\alpha} = \sum_{H=i_1+\cdots+i_n}^{\infty} \sum_{h_1+\cdots+h_m=H} c_1^{h_1} \cdots c_m^{h_m}\, {}_H\Delta_{i_1,\cdots,i_n}^{h_1,\cdots,h_m}, \tag{2.21}$$

where

$${}_H\Delta_{i_1,\cdots,i_n}^{h_1,\cdots,h_m} = \sum_{\gamma_1+\cdots+\gamma_n=H} {}_H\Delta_{\gamma_1,\cdots,\gamma_n}^{h_1,\cdots,h_m;i_1,\cdots,i_n} \qquad (\gamma_1 \geqq i_1, \cdots, \gamma_n \geqq i_n). \tag{2.21a}$$

Substitution of (2.21) in (2.13a) will result in

$$\begin{aligned} q_j &= \sum_{s=2}^{\infty} \sum_{i_1+\cdots+i_n=s} {}_j a_{i_1,\cdots,i_n}(t) \sum_{H=s}^{\infty} \sum_{h_1+\cdots+h_m=H} c_1^{h_1} \cdots c_m^{h_m}\, {}_H\Delta_{i_1,\cdots,i_n}^{h_1,\cdots,h_m} \\ &= \sum_{s=2}^{\infty} \sum_{H=s}^{\infty} \sum_{h_1+\cdots+h_m=H} c_1^{h_1} \cdots c_m^{h_m}\, {}_H\Delta_{s;j}^{h_1,\cdots,h_m}, \end{aligned} \tag{2.22}$$

where

$${}_H\Delta_{s;j}^{h_1,\cdots,h_m} = \sum_{i_1+\cdots+i_n=s} {}_j a_{i_1,\cdots,i_n}\, {}_H\Delta_{i_1,\cdots,i_n}^{h_1,\cdots,h_m}. \tag{2.22a}$$

If we denote the expression following the second summation symbol of the last member of (2.22) by $o_{s,H;j}$, it is inferred that

$$q_j = \sum_{s=2}^{\infty} \sum_{H=s}^{\infty} o_{s,H;j} = \sum_{H=2}^{\infty} \sum_{s=2}^{H} o_{s,H;j}. \tag{2.23}$$

By substituting the expression for $o_{s,H;j}$ in (2.23) one obtains

$$q_j = \sum_{H=2}^{\infty} \sum_{h_1+\cdots+h_m=H} c_1^{h_1} \cdots c_m^{h_m}\, {}_H\Gamma_j^{h_1,\cdots,h_m} \tag{2.23a}$$

with

$${}_H\Gamma_j^{h_1,\cdots,h_m} = \sum_{s=2}^{H} {}_H\Delta_{s;j}^{h_1,\cdots,h_m}. \tag{2.23b}$$

In consequence of the italicized statement following (2.13a) and in view of (2.23a) it is inferred from (2.13) that

$$\sum_{H=2}^{\infty} \sum_{h_1+\cdots+h_m=H} c_1^{h_1} \cdots c_m^{h_m} L_j(t, {}_H\eta_{h_1,\cdots,h_m;1}, \cdots, {}_H\eta_{h_1,\cdots,h_m;n})$$

$$= \sum_{H=2}^{\infty} \sum_{h_1+\cdots+h_m=H} c_1^{h_1} \cdots c_m^{h_m} t^p {}_H\Gamma_j^{h_1,\cdots,h_m}.$$

Thus the following set of differential equations is obtained:

$$(S_H) \qquad L_j(t, {}_H\eta_{h_1,\cdots,h_m;1}, \cdots, {}_H\eta_{h_1,\cdots,h_m;n}) = t^p {}_H\Gamma_j^{h_1,\cdots,h_m}$$

$$(j = 1, 2, \cdots, n; H = h_1 + \cdots + h_m; h_1 \geqq 0, \cdots, h_m \geqq 0)$$

(cf. (2.14a)). There exist systems (S_H) for $H = 2, 3, \cdots$. It will be shown that the systems (S_2), (S_3), $\cdots$ can be solved in succession in order to determine all the ${}_\nu\eta_{k_1,\cdots,k_m;j}$ involved in the formal solutions (2.12) of the system (A), §1 (cf. (2.12a)).

We shall now establish in detail the nature of the dependence of the second members of (S_H) on the ${}_\nu\eta_{k_1,\cdots,k_m;j}$. Let

$$(2.24) \qquad \sum{}^{(4)} = \sum_{s=2}^{H} \sum_{i_1+\cdots+i_n=s} \sum_{\gamma_1+\cdots+\gamma_n=H} \sum{}^{(3)}_{1\delta_1,\cdots,n\delta_m} \qquad (\gamma_1 \geqq i_1, \cdots, \gamma_n \geqq i_n)$$

(cf. (2.19b)). Here $s \leqq H$. For if $s > H$ it would follow that $\gamma_1 + \cdots + \gamma_n \geqq i_1 + \cdots + i_n = s > H$, which would be contrary to the equality $\gamma_1 + \cdots + \gamma_n = H$.

Consecutive application of (2.23b), (2.22a), (2.21a), and (2.19c) will yield

$$(2.25) \qquad \begin{aligned} {}_H\Gamma_j^{h_1,\cdots,h_m} &= \sum_{s=2}^{H} {}_H\Lambda_{s;j}^{h_1,\cdots,h_m} \\ &= \sum_{s=2}^{H} \sum_{i_1+\cdots+i_n=s} {}_j a_{i_1,\cdots,i_n} {}_H\Lambda_{i_1,\cdots,i_n}^{h_1,\cdots,h_m} \\ &= \sum_{s=2}^{H} \sum_{i_1+\cdots+i_n=s} {}_j a_{i_1,\cdots,i_n} \sum_{\gamma_1+\cdots+\gamma_n=H} {}_H\Lambda_{\gamma_1,\cdots,\gamma_n}^{h_1,\cdots,h_m;i_1,\cdots,i_n} \\ &= \sum{}^{(4)} {}_j a_{i_1,\cdots,i_n} \prod_{\alpha=1}^{n} {}_\gamma\psi_\alpha^{\alpha\delta_1,\cdots,\alpha\delta_m;i_\alpha}. \end{aligned}$$

In consequence of (2.17a) and (2.16a) it is finally deduced that

$$(2.26) \qquad \begin{aligned} {}_H\Gamma_j^{h_1,\cdots,h_m} &= \sum{}^{(4)} {}_j a_{i_1,\cdots,i_n} \prod_{\alpha=1}^{n} \sum_{\nu_1+\cdots+\nu_{i_\alpha}=\gamma_\alpha} {}_\gamma\phi_{\nu_1,\cdots,\nu_{i_\alpha}}^{\alpha\delta_1,\cdots,\alpha\delta_m;i_\alpha} \\ &= \sum{}^{(4)} {}_j a_{i_1,\cdots,i_n} \prod_{\alpha=1}^{n} \sum_{\nu_1+\cdots+\nu_{i_\alpha}=\gamma_\alpha} \sum{}^{(2)}_{k_1^1,\cdots,k_m^{i_\alpha}} \prod_{r=1}^{i_\alpha} {}_\nu\eta_{k_1^r,\cdots,k_m^r;\alpha} \end{aligned}$$

$$(H = h_1 + \cdots + h_m)$$

(cf. (2.24), (2.16b)).† *In* (2.26) *the expression displayed after the product symbol with respect to* α ($\alpha=1, \cdots, n$) *is to be replaced by unity for every* α *for which* $i_\alpha=0$; *moreover, for every such* α *we have* $\gamma_\alpha=0$.

DEFINITION 5. *In* ${}_\nu\eta_{k_1,\cdots,k_m;\alpha}$ *the subscript* ν *will be called rank*.

It will be now established that ${}_H\Gamma_j^{h_1,\cdots,h_m}$ *is independent of the* ${}_\nu\eta_{k_1,\cdots,k_m;\alpha}$ *of rank* $\geqq H$. To prove this one notes that, in view of (2.26), the ranks of the ${}_\nu\eta_{k_1,\cdots,k_m;\alpha}$ involved in ${}_H\Gamma_j^{h_1,\cdots,h_m}$ satisfy the conditions

$$(2.27)\quad \nu_1+\cdots+\nu_{i_\alpha}=\gamma_\alpha\geqq i_\alpha\ (\alpha=1, 2, \cdots, n);\quad \nu_k\geqq 1\ (k=1, 2, \cdots);$$

$$(2.27a)\quad \gamma_1+\cdots+\gamma_n=H;\quad i_1+\cdots+i_n\geqq 2;\quad i_k\geqq 0\quad (k=1, \cdots, n).$$

Thus

$$(2.27b)\quad (\nu_1+\cdots+\nu_{i_1})+(\nu_1+\cdots+\nu_{i_2})+\cdots+(\nu_1+\cdots+\nu_{i_n})=H.$$

The number of terms in the first member here is $i_1+\cdots+i_n$. If we had a $\nu_\rho\geqq H$ the left member in (2.27b) would certainly be $\geqq H+1$ since there are at least two terms (we have $i_1+\cdots+i_n\geqq 2$) in this member, while all the ν_k are $\geqq 1$. This makes the truth of the italicized statement subsequent to Definition 5 evident.

The η's of rank one are known; they are the functions defined by (2.12b) where the ${}_0y_{i,j}(t)$ are elements of the matrix solution ${}_0Y(t)=({}_0y_{i,j}(t))$ of (LA). It is to be noted that in R, ${}_0Y(t)$ satisfies (2.10). The ${}_2\Gamma_j^{h_1,\cdots,h_m}$ ($h_1+\cdots+h_m=2$; $h_1\geqq 0, \cdots, h_m\geqq 0$; $j=1, \cdots, n$) can accordingly be determined as functions of t. One may then determine all the η's of rank two with the aid of the system (S_2). In general, if the η's of ranks $1, 2, \cdots, H-1$ are known, the ${}_H\Gamma_j^{h_1,\cdots,h_m}$ may be computed as functions of t and the η's of rank H may be then constructed as solutions of the system (S_H). Thus a mechanism has been set up for determining the functions ${}_\nu y_j(t)$ of (2.12a); i.e., a device for computation of consecutive terms of the formal solution $(y_1(t), \cdots, y_n(t))$ (cf. (2.12)) of the system (A) of §1.

A matrix equation

$$(2.28)\quad Y^{(1)}(t)=Y(t)A(t)+G(t)\quad (Y(t)=(y_{i,j}(t));\quad G(t)=(g_{i,j}(t)))$$

is satisfied by

$$(2.28a)\qquad Y(t)=C(t){}_0Y(t);\quad C(t)=\int^t G(\tau){}_0Y^{-1}(\tau)d\tau=(c_{i,j}(\tau)),$$

where ${}_0Y(t)=({}_0y_{i,j}(t)$ is a matrix solution of

$$(2.28b)\qquad {}_0Y^{(1)}(t)={}_0Y(t)A(t).$$

† In (2.16b) let $\delta_i={}_\alpha\delta_i$ and $\gamma=\gamma_\alpha$.

If the $g_{i,j}(t)=g_j(t)$ are independent of i, it is not difficult to verify that the $c_{i,j}(t)=c_j(t)$ are independent of i. The same will be true of the

$$y_{i,j}(t) = y_j(t) = \sum_{\lambda=1}^{n} c_\lambda(t)\,{}_0y_{\lambda,j}(t). \tag{2.29}$$

By writing

$$A(t) = (t^p l_{i,j}(t));\ g_{i,j}(t) = g_j(t) = t^p\,{}_H\Gamma_j^{h_1,\cdots,h_m}(t),$$

we accordingly infer that the system (S_H) is satisfied with the aid of the formulas

$$ {}_H\eta_{h_1,\cdots,h_m;j}(t) = \sum_{\lambda=1}^{n} c_\lambda(t)\,{}_0y_{\lambda,j}(t), \tag{2.30}$$

where

$$\begin{aligned} c_\lambda(t) &= c_\lambda(t;\ {}_H\Gamma_1^{h_1,\cdots,h_m},\ \cdots,\ {}_H\Gamma_n^{h_1,\cdots,h_m}) \\ &= \sum_{\lambda_1=1}^{n}\int^t \tau^p\,{}_H\Gamma_{\lambda_1}^{h_1,\cdots,h_m}(\tau)\,{}_0\bar{y}_{\lambda_1,\lambda}(\tau)\,d\tau \end{aligned} \tag{2.30a}$$

$$(j=1,\cdots,n;\ h_1+\cdots+h_m=H;\ h_1\geqq 0,\cdots,h_m\geqq 0).$$

In (2.30a) the ${}_0\bar{y}_{i,j}(\tau)$ are the elements of the inverse of the matrix $({}_0y_{i,j}(\tau))$; that is,

$$({}_0\bar{y}_{i,j}(\tau)) = ({}_0y_{i,j}(\tau))^{-1}. \tag{2.30b}$$

The above may be summarized as follows.

LEMMA 1. *Let R be a region referred to in connection with* (i), (ii) (*subsequent to* (2.9)) *and* (2.9b). *Let ${}_0Y(t)$ be the matrix involved in* (2.10). *The system* (A), §1 *is formally satisfied by series* (2.12) *for $j=1,\cdots,n$, in general divergent. The first term, ${}_1y_j(t)$, in such a series is defined by* (2.11). *The subsequent terms ${}_\nu y_j(t)$ are of the form* (2.12a) *with $\nu=2,3,\cdots$. The coefficients ${}_\nu\eta_{k_1,\cdots,k_m;j}(t)$ involved in ${}_\nu y_j(t)$ can be determined in succession with the aid of* (2.30), (2.30a), (2.30b), *and* (2.26). (In connection with (2.26) cf. italics subsequent to (2.26) and Definition 5.)

Note. R can be replaced by a more general region. However, we restrict ourselves to regions of the stated kind, inasmuch as only corresponding to such regions will "actual" solutions of (A), §1 be obtained.

3. **Preliminaries to integration.** It is to be noted that

$${}_0y_{\lambda,\alpha}(t) = e^{Q_\lambda(t)}t^{r_\lambda}[t]_q \qquad (t \text{ in } R;\ \lambda,\alpha=1,\cdots,n). \tag{3.1}$$

Thus the elements of the matrix (2.30b) are of the form

(3.1a) $\quad {}_0\bar{y}_{\lambda_1,\lambda}(\tau) = e^{-Q_\lambda(\tau)\tau - r_\lambda + \omega}[\tau]^* \qquad (\omega \text{ rational}; \tau \text{ in } R; \lambda_1, \lambda = 1, \cdots, n)$

(cf. Definition 4, §2)). In view of the notation (2.12a), we have

$$(3.2)\qquad {}_1\eta_{k_1^r,\cdots,k_m^r:\alpha}(\tau) = \exp\left[k_1^r Q_1(\tau) + \cdots + k_m^r Q_m(\tau)\right] t^{k_1^r r_1 + \cdots + k_m^r r_m}[t]_q$$
$$(k_1^r + \cdots + k_m^r = 1;\ k_i^r \geqq 0;\ \alpha = 1, \cdots, n;\ \tau \text{ in } R).$$

In consequence of (2.26),

$$(3.3)\qquad {}_2\Gamma_j^{h_1,\cdots,h_m}(\tau) = \sum\nolimits^{(4)} {}_j a_{i_1,\cdots,i_n}(\tau) \prod_{\alpha=1}^{n} \sum_{\nu_1+\cdots+\nu_{i_\alpha}=i_\alpha} \sum\nolimits^{(2)} \prod_{r=1}^{i_\alpha} {}_{\nu_r}\eta_{k_1^r,\cdots,k_m^r:\alpha}(\tau).$$

With the aid of (2.16b) and (3.2) one obtains

$$(3.3a)\qquad \begin{aligned} \prod_{r=1}^{i_\alpha} {}_{\nu_r}\eta_{k_1^r,\cdots,k_m^r:\alpha}(\tau) &= \exp\left[(k_1^1 + \cdots + k_1^{i_\alpha})Q_1 + \cdots \right. \\ &\quad \left. + (k_m^1 + \cdots + k_m^{i_\alpha})Q_m\right] t^{(k_1^1+\cdots+k_1^{i_\alpha})r_1+\cdots+(k_m^1+\cdots+k_m^{i_\alpha})r_m} \cdot [\tau]^* \\ &= \exp\left[\delta_1 Q_1 + \cdots + \delta_m Q_m\right] \tau^{\delta_1 r_1 + \cdots + \delta_m r_m}[\tau]^*. \end{aligned}$$

Application of the summation

$$\sum_{\nu_1+\cdots+\nu_{i_\alpha}=i_\alpha} \sum\nolimits^{(2)}$$

(cf. (2.16b) with $\delta_i = {}_\alpha\delta_i$) to the left member of (3.3a) will yield the function

(3.3b) $\quad F_\alpha^1 =$ generic form of the last member of (3.3a).

With the aid of (2.24) and (2.19b) it is inferred that

$$(3.3c)\qquad \begin{aligned} \prod_{\alpha=1}^{n} F_\alpha^1 &= \exp\left[({}_1\delta_1 + \cdots + {}_n\delta_1)Q_1 + \cdots + ({}_1\delta_m + \cdots + {}_n\delta_m)Q_m\right] \\ &\quad \cdot \tau^{({}_1\delta_1+\cdots+{}_n\delta_1)r_1+\cdots+({}_1\delta_m+\cdots+{}_n\delta_m)r_m}[\tau]^* \\ &= \exp\left[h_1 Q_1(\tau) + \cdots + h_m Q_m(\tau)\right] \tau^{h_1 r_1 + \cdots + h_m r_m}[\tau]^*. \end{aligned}$$

Since ${}_j a_{i_1,\cdots,i_n}(\tau) = [\tau]_0$, in view of (2.24) it is concluded that

$$(3.4)\qquad {}_2\Gamma_j^{h_1,\cdots,h_m}(\tau) = \sum_{i_1+\cdots+i_n=2} \sum_{\gamma_1+\cdots+\gamma_n=2} \sum\nolimits^{(3)} F^{h_1,\cdots,h_m}_{i_1,\cdots,i_n;\gamma_1,\cdots,\gamma_n;{}_1\delta_1,\cdots,{}_n\delta_m}$$

where $\gamma_1 = i_1, \cdots, \gamma_n = i_n$ and

(3.4a) $\quad F^{h_1,\cdots}_{i_1;\cdots;\cdots;\cdots} =$ generic form of the last member of (3.3c).

The combined summation represented by the three displayed summation symbols involved in the second member of (3.4) is not with respect to $(h_1, \cdots, h_m)$. Hence, in view of (3.4a),

$$(3.5)\quad {}_2\Gamma_j^{h_1,\cdots,h_m}(\tau) = \exp\,[h_1Q_1(\tau) + \cdots + h_mQ_m(\tau)]\tau^{h_1r_1+\cdots+h_mr_m}[\tau]^* \quad (\tau \text{ in } R),$$

$$(3.5a)\qquad h_1 + \cdots + h_m = 2, \quad h_1 \geqq 0, \cdots, h_m \geqq 0; \quad j = 1, \cdots, n.$$

If we make use of (3.5) and (3.1a), we infer that the integrand displayed in (2.30a) (for $n=2$) is of the form

$$(3.6)\qquad \exp\,[Q(\tau)]\tau^r[\tau]^*,$$

where

$$(3.6a)\qquad \begin{aligned} Q(\tau) &= h_1Q_1(\tau) + \cdots + h_mQ_m(\tau) - Q_\lambda(\tau);\\ r &= h_1r_1 + \cdots + h_mr_m - r_\lambda + \omega + p. \end{aligned}$$

In view of our present purposes it will be essential to integrate expressions of the form (3.6) ($Q(\tau)$ a polynomial in $\tau^{1/k}$; k an integer $\geqq 1$) in such a manner that the integral is of the same generic form as the integrand, with r possibly changed. This can certainly be achieved by the methods of Trjitzinsky† if $RQ(\tau)$ does not change sign in R. Now, when one examines the particular integrand (3.6) for which (3.6a) holds, the following is noted regarding $Q(\tau)$. If $\lambda > m$ we pick out an $h_\alpha(>0)$‡ and write

$$(3.7)\qquad \begin{aligned} Q(\tau) = {}& [h_1Q_1(\tau) + \cdots + (h_\alpha - 1)Q_\alpha(\tau) + \cdots + h_mQ_m(\tau)]\\ & + Q_\alpha(\tau) - Q_\lambda(\tau). \end{aligned}$$

Here $h_1 \geqq 0, \cdots, h_\alpha - 1 \geqq 0, \cdots, h_m \geqq 0$; thus, since one may write in consequence of (2.9a) and (2.9b)

$$(3.8)\qquad RQ_1(\tau) \leqq \cdots \leqq RQ_m(\tau) < 0 \qquad (\tau \text{ in } R),$$

it is inferred that the real part of the expression within the brackets of the second member of (3.7) is certainly negative in R. In this connection we make use of the fact that

$$h_1 + \cdots + h_{\alpha-1} + (h_\alpha - 1) + h_\alpha + \cdots + h_m = 1,$$

as can be seen from (3.5a). On the other hand, by (2.9c)

$$R(Q_\alpha(\tau) - Q_\lambda(\tau)) \leqq 0 \qquad (\tau \text{ in } R).$$

Whence it is concluded that, for $\lambda > m$,

† Cf. (T_2).

‡ This is possible in view of (3.5a).

$$RQ(\tau) < 0 \qquad (\tau \text{ in } R).$$

Moreover, by (2.9a), (2.9b), $\exp [Q(\tau)] \sim 0$ in R. Suppose now that $\lambda \leqq m$. If $h_\lambda > 0$ we have

$$Q(\tau) = h_1 Q_1(\tau) + \cdots + (h_\lambda - 1)Q_\lambda(\tau) + \cdots + h_m Q_m(\tau)$$

with $h_1 \geqq 0, \cdots, h_{\lambda-1} - 1 \geqq 0, \cdots, h_m \geqq 0$ and $h_1 + \cdots + (h_\lambda - 1) + \cdots + h_m = 1$. Hence $\exp [Q(\tau)] \sim 0$ (τ in R) in this case as well. If $\lambda \leqq m$ and $h_\lambda = 0$, the following is noted. Some h_α $(1 \leqq \alpha < \lambda)$† may be positive; by (3.8) one then has $R(Q_\alpha(\tau) - Q_\lambda(\tau)) \leqq 0$. Since

$$Q(\tau) = [h_1 Q_1(\tau) + \cdots + (h_\alpha - 1)Q_\alpha(\tau) + \cdots + h_m Q_m(\tau)] + Q_\alpha(\tau) - Q_\lambda(\tau)$$
$$(h_1 \geqq 0, \cdots, h_\alpha - 1 \geqq 0, \cdots, h_m \geqq 0; h_1 + \cdots + (h_\alpha - 1) + \cdots + h_m = 1),$$

it is inferred that, for the case under consideration, $RQ(\tau) < 0$ and $\exp [Q(\tau)] \sim 0$ (τ in R). In the remaining case, when $\lambda \leqq m$ and $h_\lambda = 0$ one has $h_1 = h_2 = \cdots = h_\lambda = 0$. This necessitates that some h_α $(\lambda < \alpha \leqq m)$‡ should be positive, and we have

$$Q(\tau) = -Q_\lambda(\tau) + h_{\lambda+1}Q_{\lambda+1}(\tau) + \cdots + h_m Q_m(\tau) \qquad (3.9)$$
$$(h_{\lambda+1} \geqq 0, \cdots, h_m \geqq 0; h_{\lambda+1} + \cdots + h_m = 2).$$

Now $RQ(\tau)$, where $Q(\tau)$ is given by (3.9), *will not necessarily maintain its sign*§ *in R*, unless $\lambda = m$, in which case some h_α $(1 \leqq \alpha < \lambda)$ would have to be positive. If, more generally, functions are considered of the form

$$Q(\tau) = g_1 Q_1(\tau) + g_2 Q_2(\tau) + \cdots + g_m Q_m(\tau) = \sum_{\nu=0}^{l-1} q_\nu \tau^{(l-\nu)/k}, \qquad (3.10)$$

where the g_i $(i = 1, \cdots, n)$ are real, and where, unless $Q(\tau) \equiv 0$, the leading term actually present is

$$q_{\nu_1}\tau^{(l-\nu_1)/k} \qquad (0 \leqq \nu_1 \leqq l-1; q_{\nu_1} \neq 0), \qquad (3.10a)$$

it is observed that, *with*

$$q_0{}^1\tau^{l_1/k_1},\ q_0{}^2\tau^{l_2/k_2},\ \cdots,\ q_0{}^m\tau^{l_m/k_m} \qquad (3.10b)$$

respectively denoting the leading terms in $Q_1(\tau), Q_2(\tau), \cdots, Q_m(\tau)$, we necessarily have

$$\frac{l - \nu_1}{k} \leqq l', \qquad (3.10c)$$

† This may arise only if $\lambda > 1$.

‡ This may occur only if $\lambda < m$.

§ Examples can be given when $RQ(\tau)$ changes sign in R.

where l' is the greater one of the numbers l_i/k_i $(i=1, \cdots, m)$. The regular curves along which $RQ(\tau)=0$ will possess at infinity limiting directions $\bar{\tau}_0$ satisfying the equation

$$R(q_{\nu}\tau^{(l-\nu_1)/k}) = |q_{\nu_1}||\tau|^{(l-\nu_1)/k}\cos\left[\bar{q}_{\nu_1}+\left(\frac{l-\nu_1}{k}\right)\bar{\tau}_0\right]=0 \tag{3.10d}$$

$$(\bar{q}_{\nu_1} = \angle\, q_{\nu_1};\ \bar{\tau} = \angle\, \tau).$$

These directions have accordingly the values

$$\bar{\tau}_0 = \bar{\tau}_0^{(m)} = \frac{k}{l-\nu_1}\left[\frac{\pi}{2}+m\pi-\bar{q}_{\nu_1}\right] \qquad (m=0, \pm 1, \pm 2, \cdots). \tag{3.10e}$$

Thus they differ by non-zero integral multiples of $k\pi/(l-\nu_1)$. By (3.10c)

$$\frac{k\pi}{l-\nu_1} \geqq \frac{\pi}{l'}. \tag{3.11}$$

At this stage it will be convenient to introduce a definition.

DEFINITION 6. *Let R denote a region satisfying conditions* (i), (ii) *of* §2. *Let $Q(\tau)$ be a polynomial of the form* (3.10). *A region W will be said to be of the order $\nu(\geqq 1)$ with respect to $Q(\tau)$ if the following conditions are satisfied.*

(I) *W is coincident with or is a regular* (cf. Definition 3, §2) *subregion of R.*

(II) *$W=W_1+W_2+\cdots+W_\nu$, where $W_1, \cdots, W_\nu$ are regular non-overlapping regions such that along $W_{i,i+1}$, the common boundary of W_i and W_{i+1} $(i=1, \cdots, \nu-1)$, $RQ(\tau)=0$, while interior to W_i $(i=1, \cdots, \nu)$ $RQ(\tau)$ does not change sign.*

When $\nu=1$ Condition (II) is to be replaced by a statement to the effect that $RQ(\tau)$ does not change sign in W.

With $\nu>1$ and $\theta_{i,i+1}$ denoting the limiting direction of $W_{i,i+1}$ $(i=1, \cdots, \nu-1)$ at infinity, it may be supposed without any loss of generality that

$$\theta_{1,2} < \theta_{2,3} < \cdots < \theta_{\nu-1,\nu}.$$

One may then write

$$\theta_{i,i+1} = \theta_{1,2} + (i-1)\frac{k\pi}{l-\nu_1} \qquad (i=1, 2, \cdots, \nu-1).$$

LEMMA 2. *With R denoting a region satisfying* (i) *and* (ii) *of* §2, *let W be a regular subregion of R of opening* (cf. Definition 3, §2) *less than π/l', where l' is the number referred to in the italicized statement in connection with* (3.10b),

(3.10c). *The region W will then be of order* $\leqq 2$ (cf. Definition 6) *with respect to every* $Q(\tau)$ *of the form* (3.10).†

In consequence of this lemma, R can be covered with a finite number of regions of order $\leqq 2$ with respect to every $Q(\tau)$ of the form (3.10). The inequality "$\leqq 2$" cannot be replaced by the equality "$=1$" because, generally, it is impossible to find a regular subregion W of R (of opening however small) such that for no $Q(\tau)$ of the form (3.10) is there a curve $RQ(\tau)=0$ interior to W.

The truth of this lemma can be inferred as follows. Consider a region $W=R(\alpha_1, \alpha_2)$ (cf. Definition 3) for which, as required by the lemma, $0\leqq\alpha_2-\alpha_1<\pi/l_1$. Unless implied otherwise we take a fixed set of numbers $g_1, \cdots, g_m$. If $Q(\tau)\equiv 0$, W will be of order unity with respect to this $Q(\tau)$. If $Q(\tau)\not\equiv 0$ let the regular curves (Definition 2) be designated by

$$B_1, B_2, \cdots,$$

and let the limiting directions at infinity be the numbers

$$\beta_1, \beta_2, \cdots,$$

respectively. It is a matter of notation to arrange these numbers so that

$$\beta_1 < \beta_2 < \cdots ; \beta_1 \leqq \alpha_1$$

and so that B_1, if not coincident with the boundary of W for which limiting direction is α_1, lies exterior to W in such a way that between B_1 and the mentioned boundary of W there is no curve B_i $(i=2, 3, \cdots)$ for which

$$\beta_1 < \beta_i \leqq \alpha_1. \tag{3.11a}$$

The implication of the above is that, if (3.11a) holds for $i=2$, the curve B_2 must extend interior to W (in which case necessarily $\beta_2=\alpha_1$).

For some subscript ν (>1) B_ν will be either coincident with the boundary of W, for which the limiting direction is α_2, or it will lie exterior to W so that between the boundary of W, just referred to, and B_ν there is no curve B_i $(i<\nu)$ such that

$$\alpha_2 \leqq \beta_i < \beta_\nu. \tag{3.11b}$$

If, however, (3.11b) holds for $i=\nu-1$, the above necessitates that we should have $\alpha_2=\beta_{\nu-1}$, while $B_{\nu-1}$ extends interior to W.

Corresponding to a particular $Q(\tau)$ $(\not\equiv 0)$ of the form (3.10), W has been covered by a succession of $\nu-1$ adjacent regions separated by the curves $B_2, B_3, \cdots, B_{\nu-1}$, all interior to W. Designate these regions by

† l' is independent of the choice of $Q(\tau)$; i.e., l' is independent of $g_1, \cdots, g_m$.

$$R^{1,2},\ R^{2,3},\ \cdots,\ R^{\nu-1,\nu} \qquad (R^{i,i+1} = R(\beta_i, \beta_{i+1})). \tag{3.12}$$

With the leading term in $Q(\tau)$ given by (3.10a) one has, in view of the statements preceding Definition 6,

$$\beta_{\nu-1} - \beta_2 = (\nu - 3)\frac{k\pi}{l - \nu_1} \geqq (\nu - 3)\frac{\pi}{l'}, \tag{3.12a}$$

whenever $\nu > 3$.

When $\nu \leqq 3$ there are at most two regions (3.12), and W will be of order $\leqq 2$. If $\nu > 3$ consider the region

$$R^{2,3} + \cdots + R^{\nu-2,\nu-1} = R(\beta_2, \beta_{\nu-1}).$$

It will be interior to W.† Now $W = R(\alpha_1, \alpha_2)$. Thus

$$\beta_{\nu-1} - \beta_2 \leqq \alpha_2 - \alpha_1 < \frac{\pi}{l_1}. \tag{3.12b}$$

This, in view of (3.12a) implies that

$$(\nu - 3)\frac{\pi}{l'} < \frac{\pi}{l'}, \quad \nu > 3 \qquad (\nu \text{ an integer}).$$

These inequalities are, however, incompatible. Hence $\nu \leqq 3$ necessarily and W is at most of the second order.

In the sequel, unless stated otherwise, W is to denote a particular regular subregion of R with opening $< \pi/l'$.

4. **Integrations.** *Let $W = R(\alpha_1, \alpha_2)$, $\alpha_1 < \alpha_2$* (cf. Definition 3, §2) *be a region such that $\alpha_2 - \alpha_1 < \pi/l'$. Consider the problem*

$$y^{(1)}(t) = e^{Q(t)}t^r a(t) \tag{4.1}$$

where, unless $Q(t) \equiv 0$,

$$Q(t) = \sum_{\nu=\nu_1}^{l-1} q_\nu t^{(l-\nu)/k} \qquad (0 \leqq \nu_1 \leqq l - 1;\ q_{\nu_1} \neq 0) \tag{4.1a}$$

and $(l - \nu_1)/k \leqq l'$. Assume $a(t)$ to be of the form

$$a(t) = [t]^* \qquad (t \text{ in } W) \tag{4.1b}$$

(cf. Definition 4, §2).

It is not difficult to see that Lemma 2 holds when $Q(t)$, instead of being assumed to be of the form (3.10) is given by (4.1a). The essential fact is that the inequality subsequent to (4.1a) holds. *Thus W is of order $\leqq 2$ (with respect to all the $Q(t)$ of stated type—provided, of course, that l' is fixed).* In this

† That is, all of its points, with the possible exception of those on the circular part of its boundary, are interior to W.

connection "order" is specified according to Definition 6, §3. When $Q(t) \not\equiv 0$ we have $W = W_1 + W_2$, where

$$W_1 = R(\alpha_1, \beta_2), \qquad W_2 = R(\beta_2, \alpha_2) \qquad (\alpha_1 \leqq \beta_2 \leqq \alpha_2), \tag{4.2}$$

unless

$$W = W_1 = R(\alpha_1, \alpha_2). \tag{4.2a}$$

The symbols involved here have the same significance as in §3. With B_i denoting the regular curves satisfying the equation $RQ(t) = 0$ and β_i denoting the limiting direction of B_i we have B_2 interior to W (in case (4.2)) or there are no curves B_i interior to W. The curve B_1 is exterior to W or is coincident with the right boundary† of W and the curve B_3 is exterior to W or is coincident with the left boundary of W. However, in view of our present purposes, it will be necessary to examine this case in greater detail, taking advantage of the conditions satisfied by W.

When $W = W_1$ the integration methods given in (T_2) are applicable. In the case of $W = W_1 + W_2$ an extension of these methods will be necessary. The latter case will now be considered.

By hypothesis $a(t) \sim \alpha(t)$ (t in W), where

$$\alpha(t) = \{t\}^* \tag{4.3}$$

(cf. Definition 1, §2). As demonstrated in (T_2) the formal equation

$$s^{(1)}(t) = e^{Q(t)} t^r \alpha(t), \tag{4.3a}$$

associated with (4.1), possesses a formal solution

$$s(t) = e^{Q(t)} t^{r+w} \sigma(t), \tag{4.4}$$

where

$$\sigma(t) = \{t\}^*, \qquad w = 1 - \left(\frac{l - \nu_1}{k}\right) \qquad (1 - l' \leqq w < 1). \tag{4.4a}$$

Retaining in the power series involved in $\sigma(t)$ the first β_1 terms only, we obtain a certain function $\sigma_{\beta_1}(t)$. Apply now the transformation

$$y(t) = e^{Q(t)} t^{r+w} \sigma_{\beta_1}(t) + z(t) \tag{4.5}$$

to the given equation (4.1). It is observed that $z(t)$ will satisfy

$$z^{(1)}(t) = \zeta(t), \tag{4.6}$$

where

† The "right" boundary of W is that one of the two regular curves, forming part of the boundary of W, which appears on the right when the region W is faced from the origin.

$$\zeta(t) = e^{Q(t)}t^{r}a(t) - \frac{d}{dt}\left[e^{Q(t)}t^{r+w}\sigma_{\beta_1'}(t)\right]$$

$$= e^{Q(t)}t^{-\beta}b(t) \qquad (|b(t)| \leqq b;\ t \text{ in } W), \tag{4.6a}$$

as can be seen from the developments given in (T_2). The essential fact to be noted is that in (4.6a) β $(=\beta(\beta_1))$ *can be made as large as desired by choosing* β_1 *sufficiently great.* The function $b(t)$ will, of course, be analytic in W $(t \neq \infty)$.

Suppose, for instance, *that* $RQ(t) > 0$ *interior to* W_1. Let t_0 denote the point where B_2 meets the circular part of the boundary of W. A solution of (4.6) will be defined as follows:

$$z(t) = \zeta_0 + \int_{t_0}^{t} \zeta(\tau)d\tau, \tag{4.7}$$

$$\zeta_0 = \int_{\infty}^{t_0} \zeta(\tau)d\tau. \tag{4.7a}$$

In (4.7) the path of integration is interior to W_2 when t is interior to W_2, and it is interior to W_1 when t is interior to W_1. This path is along B_2 when t is on B_2. In (4.7a) the path is along B_2. We deliberately avoid using the notation

$$z(t) = \int_{\infty}^{t} \zeta(\tau)d\tau \qquad (t \text{ in } W) \tag{4.7b}$$

unless further qualifications are introduced. This is because, for t in W_1, there may exist paths (t, ∞) extending interior to W_1 for which the displayed integral (4.7b) diverges. Formulas (4.7) and (4.7a) define a solution of (4.6) analytic throughout W $(t \neq \infty)$.

If t is in W_1 and there are no curves†

$$\frac{\partial RQ(t)}{\partial \theta} = 0 \qquad (t = |t| \exp\left[(-1)^{1/2}\theta\right])‡ \tag{4.8}$$

between t and B_2, we shall write

$$z(t) = \left(\int_{\infty}^{t'} + \int_{t'}^{t}\right)\zeta(\tau)d\tau \qquad (|t'| = |t|). \tag{4.9}$$

The first integration displayed here is along B_2; the second is along an arc of the circle $|\tau| = |t|$. With the aid of (4.6a) it follows that

† The whole discussion here and in the sequel is given sufficiently far away from the origin so that no two distinct regular curves, under consideration, intersect.

‡ All such regular curves have limiting directions distinct from those of the regular curves $RQ(t) = 0$ (cf. (T_2)).

$$|z(t)| \leqq b\left(\int_{\infty}^{t'} + \int_{t'}^{t}\right) e^{RQ(\tau)} |\tau|^{-\beta+2} \left|\frac{d\tau}{\tau^2}\right|. \tag{4.9a}$$

By taking β_1 sufficiently great the inequality $-\beta+2 \geqq 0$ will be secured. Since $RQ(\tau)$ increases monotonically as τ describes the circular arc $|\tau| = |t|$ from t' to t, it is inferred that

$$e^{RQ(\tau)} |\tau|^{-\beta+2}$$

attains its upper bound, for the combined path under consideration, at t. Thus

$$|z(t)| \leqq b\, |e^{Q(t)t^{-\beta+2}}| \left(\int_{\infty}^{t'} + \int_{t'}^{t}\right) \left|\frac{d\tau}{\tau^2}\right|. \tag{4.9b}$$

As established in (T_2) we have, when $\tau = r \exp[(-1)^{1/2}\theta]$, is on B_2,

$$\theta - \beta_2 = f_0 r^{-p/k} + f_1 r^{-(p+1)/k} + \cdots \qquad (p \text{ an integer} \geqq 1),$$

the series here involved being convergent for $r \geqq r_1$ (r_1 sufficiently great). Hence along B_2

$$|d\tau| = [r^2(d\theta)^2 + (dr^2)]^{1/2} = (1 + g_1 r^{-1/k} + g_2 r^{-2/k} + \cdots)\,|dr|, \tag{4.9c}$$

so that along B_2

$$|d\tau| < g\,|dr|, \tag{4.9d}$$

where g (>0) is independent of τ and depends only on the curve B_2. Thus

$$\int_{\infty}^{t'} \left|\frac{d\tau}{\tau^2}\right| < \frac{g}{|t|} \qquad (|t'| = |t|). \tag{4.10}$$

On the other hand,

$$\int_{t'}^{t} \left|\frac{d\tau}{\tau^2}\right| = \int_{t'}^{t} \left|\frac{d\theta}{\tau}\right| = \frac{1}{|t|}(\angle t' - \angle t) < \frac{g'}{|t|}, \tag{4.10a}$$

since $\angle t' - \angle t$ is obviously bounded when t' is on B_2 and t ($|t| = |t'|$) is any other point in W. By (4.10), (4.10a) it is inferred from (4.9a) that

$$|z(t)| < (g + g')b\,|e^{Q(t)t^{-\beta+1}}| \tag{4.11}$$

when t is in W_1 while no regular curves (4.8) are between t and B_2. In other words, *if no curve* (4.8) *extends interior to* W_1 *the inequality* (4.11) *will certainly hold throughout* W_1.

Suppose now there are regular curves (4.8) *interior to* W_1. As indicated in (T_2) there is just one curve (4.8) between B_1 and B_2. Hence there will be only

one curve (4.8) interior to W_1. Let λ_0 be its limiting direction. Then †

$$\alpha_1 \leqq \lambda_0 < \beta_2 \tag{4.12}$$

and

$$\lambda_0 = \tfrac{1}{2}(\beta_1 + \beta_2). \tag{4.12a}$$

We need to obtain an inequality for $|z(t)|$ when t is in W_1 between the curve (4.8) (with the limiting direction λ_0) and the right boundary of W_1 (i.e., of W). (4.7), (4.7a) will now be written in the form

$$z(t) = \zeta_0 + \zeta_1 + \left(\int_{t_0'}^{t'} + \int_{t'}^{t}\right)\zeta(\tau)d\tau \qquad (|t'| = |t|). \tag{4.13}$$

Here

$$\zeta_1 = \int_{t_0}^{t_0'} \zeta(\tau)d\tau \tag{4.13a}$$

with the integration extended over the circular part of the boundary of W_1 (t_0 and t_0' representing the points where B_2 and the right boundary of W meet the circular arc). The first integral displayed in (4.13) is along the right boundary of W. The second is along an arc of the circle $|\tau| = |t|$. If B_1 had the same limiting direction at infinity as the right boundary of W, that is, if $\beta_1 = \alpha_1$, a contradiction would necessarily follow, since it has been previously assumed that $\alpha_1 < \beta_2 \leqq \alpha_2$. In fact, one would have $\beta_2 - \beta_1$ $(= \beta_2 - \alpha_1) \leqq \alpha_2 - \alpha_1 < \pi/l'$; on the other hand,‡

$$\beta_2 - \beta_1 = \frac{k\pi}{l - \nu_1} \geqq \frac{\pi}{l'}.$$

Hence, whenever there exists a curve $RQ(t) = 0$ interior to W, $\alpha_1 > \beta_1$. We will have necessarily

$$a = \cos\left(\bar{q}_{\nu_1} + \frac{l - \nu_1}{k}\alpha_1\right) > 0.$$

---

† $$RQ(t) = |q_{\nu_1}||t|^{(l-\nu_1)/k}\cos\left(\bar{q}_{\nu_1} + \frac{l-\nu_1}{k}\theta\right) + \cdots \qquad (\theta = \angle t)$$

and

$$\frac{\partial}{\partial\theta}RQ(t) = -|q_{\nu_1}||t|^{(l-\nu_1)/k}\left(\frac{l-\nu_1}{k}\right)\sin\left(\bar{q}_{\nu_1} + \frac{l-\nu_1}{k}\theta\right) + \cdots.$$

Thus the limiting directions of the curves $RQ(t) = 0$ are the values of θ for which $\cos(\bar{q}_{\nu_1} + ((l-\nu_1)/k)\theta) = 0$, while those of the curves (4.8) are the values of θ for which $\sin(\bar{q}_{\nu_1} + ((l-\nu_1)/k)\theta) = 0$.

‡ This is a consequence of an inequality stated in connection with (4.1a).

Accordingly along the right boundary† of W

$$RQ(\tau) = a\,|\,q_{\nu_1}\,|\,|\,\tau\,|^{(l-\nu_1)/k}(1+g_\tau) \qquad (a\,|\,q_{\nu_1}\,|>0), \tag{4.14}$$

where $g_\tau \to 0$ as $|\tau| \to \infty$. In consequence of (4.14) it is inferred that there exists a number $r(\beta)$ so that, for all t' on the right boundary of W and such that $|t'| \geqq r(\beta)$, we have

$$|\,e^{Q(\tau)}\tau^{-\beta}\,| \leqq |\,e^{Q(t')}(t')^{-\beta}\,|, \tag{4.14a}$$

when τ is restricted to the part of the mentioned boundary between t_0' and t'. Whence, in view of (4.6a),

$$\begin{aligned}\left|\int_{t_0'}^{t'}\zeta(\tau)d\tau\right| &\leqq b\int_{t_0'}^{t'}|\,e^{Q(\tau)}\tau^{-\beta}\,|\,|\,d\tau\,| \leqq b\,|\,e^{Q(t')}(t')^{-\beta}\,|\int_{t_0'}^{t'}|\,d\tau\,| \\ &< b_1\,|\,e^{Q(t')}(t')^{-\beta+1}\,| \qquad (|\,t'\,| \geqq r(\beta)).\end{aligned} \tag{4.15}$$

Take b' ($\geqq b_1$), depending on β, sufficiently great so that

$$\left|\int_{t_0'}^{t'}\right| < b'\,|\,e^{Q(t')}(t')^{-\beta+1}\,| \qquad (|\,t'\,| < r(\beta)).$$

We then have

$$\left|\int_{t_0'}^{t'}\right| < b'\,|\,e^{Q(t')}(t')^{-\beta+1}\,| \tag{4.15a}$$

for all t' on the right boundary of W.

On the circular arc (t', t) constituting the path of integration in the second integral displayed in (4.13), $RQ(\tau)$ attains its upper bound at t (t between the right boundary of W_1 and the curve (4.8) extending interior to W_1). Thus by the same reasoning as before one obtains

$$\left|\int_{t'}^{t}\zeta(\tau)d\tau\right| \leqq bg'\,|\,e^{Q(t)}t^{-\beta+1}\,|. \tag{4.16}$$

If we apply the relations

$$|\,t'\,| = |\,t\,|, \qquad |\,e^{Q(t')}\,| \leqq |\,e^{Q(t)}\,|$$

to (4.15a), in consequence of (4.16) we infer that

$$\left|\left(\int_{t_0'}^{t'}+\int_{t'}^{t}\right)\zeta(\tau)d\tau\right| < b''\,|\,e^{Q(t)}t^{-\beta+1}\,|, \tag{4.17}$$

where b'' may depend on β. When t is between the right boundary of W and the curve (4.8) (extending in W_1), we have

† Use is made of the fact that, if $\theta = f(r)$ ($\tau = r\exp[(-1)^{1/2}\theta]$) is the equation of this boundary, one has $\lim f(r) = \alpha_1$ as $r \to \infty$.

$$(4.17a) \qquad |e^{-Q(t)t^{\beta+1}}| \leqq |e^{-Q(t')(t')^{\beta+1}}| \leqq b_1'' \qquad (|t'| = |t|),$$

where t' has the significance indicated before and b_1'' may depend on β. With the aid of (4.17) and (4.17a) from (4.13) *we obtain the inequality*

$$(4.18) \qquad |z(t)| < b_2 |e^{Q(t)t^{-\beta+1}}| \qquad (b_2 \text{ dependent on } \beta)$$

for t in the region specified in connection with (4.17a). In view of (4.11) it is seen that (4.18) *is valid throughout W_1 in any case.*

When t is in W_2 there are the following four possibilities.

Case 1. The regular curve (4.8) whose limiting direction is $(\beta_1+\beta_2)/2$ is coincident with the left boundary of W_2 or extends interior to W_2.

Case 2. The curve (4.8) mentioned in Case 1 extends exterior to W_2, while $\alpha_2 > \beta_2$.

Case 3. $\alpha_2 = \beta_2$ and there exists a regular curve

$$(4.19) \qquad RQ(\tau) = -h(<0),\dagger$$

with limiting direction β_2, which is coincident with the left boundary of W_2 or extends exterior to W_2.

Case 4. $\alpha_2 = \beta_2$ and for no h (>0) is there a curve (4.19) satisfying the conditions laid down in the formulation of Case 3.

In Case 1 one has

$$(4.20) \qquad RQ(\tau) = -a' |\tau|^{(l-\nu)/k}(1+g_\tau) \qquad (a' > 0)$$

where $g_\tau \to 0$ as $|\tau| \to \infty$ along the curve (4.8) under consideration. The function $z(t)$, as defined by (4.7), can be expressed as follows:

$$(4.21) \qquad z(t) = \left(\int_\infty^{t_1} + \int_{t_1}^{t}\right) \zeta(\tau)d\tau \qquad (|t_1| = |t|).$$

Here the first displayed integration is along the curve (4.8); the second is along an arc of the circle $|\tau| = |t|$. *With the radius of the circular part of the boundary of W sufficiently great* (*but independent of* β), the upper bound of

$$|e^{Q(\tau)\tau^{-\beta+2}}|$$

is attained at t_1 when τ is the variable of integration in the first displayed integral of (4.21). This follows from (4.20). On the other hand, along the arc (t_1, t), $|\exp[Q(\tau)]|$ increases monotonically, attaining the upper bound at t. Thus an inequality like (4.9b) is obtained. With the aid of a reasoning closely analogous to that employed in deriving (4.11) it is inferred that in Case 1

$$(4.21a) \qquad |z(t)| < a_1 |e^{Q(t)t^{-\beta+1}}| \qquad (t \text{ in } W_2).$$

† Regarding such curves cf. (T_2).

In Case 2, a relation like (4.20) will hold along the left boundary of W_2. This follows from the inequality $\beta_3 > \alpha_2 > \beta_2$. We express $z(t)$ as in (4.21) with the first integration involved there along the left boundary of W_2. Making an assumption as stated in italics subsequent to (4.21) we conclude, as in Case 1, that (4.21a) holds (for t in W_2) in Case 2 as well.

In Case 3, $-h \leqq RQ(\tau) \leqq 0$ for τ in W_2. Thus

$$e^{-h} \leqq \left| e^{Q(\tau)} \right| \leqq 1 \qquad (\tau \text{ in } W_2). \tag{4.22}$$

On writing (4.21) with the path of integration as in Case 2, we note that

$$\left| e^{Q(\tau)} \tau^{-\beta+2} \right| \leqq |t|^{-\beta+2}$$

when τ is on the combined path of integration. In view of (4.21) one accordingly has (cf. (4.6a))

$$|z(t)| \leqq b|t|^{-\beta+2}\left(\int_{\infty}^{t_1} + \int_{t_1}^{t}\right)\left|\frac{d\tau}{\tau^2}\right|. \tag{4.23}$$

Inequalities (4.10), (4.10a) will hold with g and g' having a meaning analogous to that previously assigned. Whence it is concluded that

$$|z(t)| < (g + g')b|t|^{-\beta+1} \qquad (t \text{ in } W_2). \tag{4.23a}$$

In view of (4.22) it is inferred that (4.21a) holds in Case 3, provided that we take

$$a_1 \geqq e^h(g + g')b.$$

In the remaining Case 4, $RQ(\tau)$ will be monotonically decreasing along the left boundary of W_2 (as $|\tau| \to \infty$ along this boundary), provided that the radius of the circular part of the boundary is taken sufficiently great. Along any circular arc

$$(t_1, t) \qquad (|t_1| = |t|),$$

where t_1 is on the left boundary of W_2 and t is in W_2, $RQ(\tau)$ will attain its upper bound at t. The repetition, then, of the argument of Case 2 will yield an analogous inequality for $|z(t)|$.

In view of the statement previously made in connection with (4.18) *it is now seen that* (4.18) (*with a suitable* b_2, *dependent on* β) *will hold throughout* $W = W_1 + W_2$. If we write

$$z(t) = e^{Q(t)}t^{r+w}z_1(t) \tag{4.24}$$

it is accordingly observed that

$$|z_1(t)| < b_2'|t|^{-\beta'} \qquad (\beta' = \beta - 1 + r' + w)\dagger \tag{4.24a}$$

† r' = real part of r.

for t in W_1+W_2. Moreover, $\beta'\to\infty$ with $\beta_1\to\infty$. By virtue of (4.24) and (4.5) it is concluded that the following is true for every positive integer α.

The problem formulated at the beginning of this section possesses a solution $y_\alpha(t)$, analytic in W_1+W_2 $(t\neq\infty)$, of the form

$$y_\alpha(t) = e^{Q(t)}t^{r+w}\eta_\alpha(t), \tag{4.25}$$

where

$$\eta_\alpha(t)\sim\sigma(t) \qquad (t \text{ in } W_1+W_2) \tag{4.25a}$$

to α terms (cf. (4.4a)). Let α_1 be any integer $>\alpha$. One has

$$y_{\alpha_1}(t) - y_\alpha(t) \equiv c_{\alpha_1,\alpha} = e^{Q(t)}t^{r+w}(\eta_{\alpha_1}(t) - \eta_\alpha(t)). \tag{4.25b}$$

With the aid of (4.25a) (stated for α and for α_1) it is concluded that the constant $c_{\alpha_1,\alpha}$ must be zero. To demonstrate this it is sufficient to let t recede to infinity along the curve $RQ(t)=0$, separating W_1 and W_2, and to note that the last member in (4.25b) will then manifestly approach zero. Thus all the solutions $y_\alpha(t)$ are identical. Whence it can be asserted that (4.1) *has a solution $y(t)$, analytic in W_1+W_2 $(t\neq\infty)$, of the form*

$$y(t) = e^{Q(t)}t^{r+w}\eta(t), \tag{4.26}$$

$$\eta(t)\sim\sigma(t) \qquad (t \text{ in } W_1+W_2) \tag{4.26a}$$

(cf. (4.4a)), *the asymptotic relations being valid in the ordinary sense.* The above will hold also when $RQ(t)>0$ interior to W_1 and $RQ(t)<0$ in W_2.

We shall now consider briefly the case of (4.2a). The curves B_1, B_2 will both be exterior to $W=W_1$, or one of them will be coincident with a boundary of W_1, while the other one is exterior to W_1. One now has

$$\beta_1 \leqq \alpha_1 < \alpha_2 \leqq \beta_2 \qquad \left(\alpha_2-\alpha_1<\frac{\pi}{l'}\leqq\beta_2-\beta_1\right). \tag{4.27}$$

Suppose first that $RQ(t)<0$ (interior to W_1). If the regular curve (4.8) with the limiting direction $(\beta_1+\beta_2)/2$ is not exterior to W_1 the function $z(t)$ (solution of (4.6)) will be expressed as an integral extending from infinity along the curve (4.8) to a point t_1 ($|t_1|=|t|$) and from t_1 to t along an arc of the circle $|\tau|=|t|$. If the mentioned curve (4.8) is exterior to W_1, an inequality for $|z(t)|$ is obtained with the aid of an integration, extending from infinity to $t_1(|t_1|=|t|)$ along that one of the two regular boundaries of W which is nearer to the curve (4.8); from t_1 the path of integration is continued (away from the curve (4.8) towards t) along an arc of the circle $|\tau|=|t|$. With the radius of the circular part of the boundary of W_1 sufficiently great (this choice can be made independently of β, β_1) it is observed that

$$| e^{Q(\tau)} |, \qquad | e^{Q(\tau)\tau^{-\beta}} |$$

will be monotonically increasing along the specified paths, from infinity to t. With the aid of the reasoning previously employed in proving (4.26), (4.26a) for t in W_1+W_2 the truth of (4.26), (4.26a) is again made evident in the new case under consideration as well; the remarks regarding the choice of path are to be noted.

Suppose now that interior to W_1, $RQ(t)>0$. The following is observed. If, for instance, the right boundary of W_1 is considered, either

$$| e^{Q(\tau)\tau^{-\beta}} | \to \infty \qquad \text{(for every } \beta > 0) \tag{4.28}$$

as $|\tau| \to \infty$ along this boundary, or

$$| e^{Q(\tau)\tau^{-\beta}} | \to 0 \qquad \text{(for some } \beta) \tag{4.28a}$$

as $|\tau| \to \infty$ along the mentioned boundary. We have $\beta_1 \leqq \alpha_1 < \alpha_2 \leqq \beta_2$. It is observed that (4.28) will hold† when $\alpha_1 > \beta_1$; (4.28) may hold even when $\alpha_1 = \beta_1$. However, with $\alpha_1 = \beta_1$, the case (4.28a) will sometimes occur. Since the equalities

$$\alpha_1 = \beta_1, \qquad \alpha_2 = \beta_2$$

cannot be satisfied at the same time, only one of the following two cases may present itself (when $RQ(t)>0$ interior $W = W_1$).

Case (a_1). Along both regular boundaries of W_1 (4.28) is satisfied.

Case (a_2). Along one of the boundaries of W_1 (4.28) holds, while along the other we have (4.28a).

In the Case (a_1) if the curve (4.8) (between B_1 and B_2) is exterior to W_1 or is coincident with one of the regular boundaries of W_1, the path of integration is to be taken from a point on the circular part of the boundary of W_1 along that one of the regular boundaries of W_1 which is further removed from the mentioned curve (4.8). Along this boundary of W_1 integration is to be extended to t_1 ($|t_1| = |t|$). The path is further extended along an arc of the circle $|\tau| = |t|$ from t_1 to t (t in W_1). If, in the Case (a_1), a curve (4.8) extends interior to W_1, the path is taken from t_0, the intersection of (4.8) with the circular part of the boundary of W_1, to t. When t is, for instance, between the curve (4.8) and the right boundary of W_1, the path is deformed so as to extend from t_0 along the circular part of the boundary of W_1 to the intersection with the right boundary of W_1; the path is further continued along the latter boundary to t_1 ($|t_1| = |t|$) and from t_1 it is finally extended along an arc of the circle $|\tau| = |t|$ to t. An analogous deformation is made when t is in W_1 to the left of the curve (4.8) under consideration. Given α,

† Along the left boundary of W_1 (4.28) will certainly hold if $\alpha_2 < \beta_2$.

however large, a solution (4.25), (4.25a) (t in W_1) can be obtained. The various solutions $y_\alpha(t)$ will in general differ by constants distinct from zero. However, any particular one of the $y_\alpha(t)$ will maintain in the ordinary sense and throughout W_1 the asymptotic form of the formal solution. The latter fact is due to the relation

$$e^{-Q(t)} \sim 0 \qquad (t \text{ in } W_1)$$

which certainly holds in Case (a_1). The above choice of the path having been made, the following property will hold. If $|t| \geqq r_\beta$ (r_β sufficiently great), the upper bound of $|\tau^{-\beta} \exp [Q(\tau)]|$, for τ on the path under consideration, will be attained at $\tau = t$.

Consider now Case (a_2). Suppose, for instance, that (4.28a) holds along the left boundary of W_1; let β (>0) be sufficiently great so that we have (4.28a). In this case integration will be precisely as in the previously discussed case when $W = W_1 + W_2$, where W_1 and W_2 are separated by the curve B_2 along which $RQ(t) = 0$, $RQ(t) > 0$ in W_1, and t is in W_1 (cf. the developments stated in connection with (4.8)–(4.18)). One needs only to replace B_2 by the left boundary of W_1. The result will be precisely analogous to the one obtained in the previous case.

Consider now the following modified problem:

$$y^{(1)}(t) = e^{Q(t)} t^r a(t), \tag{4.29}$$

$$Q(t) = \textit{polynomial in } t^{1/k}, \tag{4.29a}$$

$$a(t) = [t]^* \qquad (t \textit{ in } R'), \tag{4.29b}$$

$$e^{Q(t)} \sim 0 \qquad (t \textit{ in } R'). \tag{4.29c}$$

Here $R' = R(\alpha_1, \alpha_2)$ ($\alpha_1 < \alpha_2$) *is a regular region* (cf. Definition 3, §2). In view of (4.29c) it is not difficult to infer that not more than one curve (4.8) may extend interior to R'. Hence the reasoning of the type employed in the text between (4.27) and (4.28) is again applicable leading to an analogous result.

An examination of the preceding developments of this section enables us to assert the following.

Let the region $W = R(\alpha_1, \alpha_2)$ *be subject to the condition*

$$0 < \alpha_1 - \alpha_2 < \frac{\pi}{2l'}, \tag{4.30}$$

and let the radius r' of the circular part of the boundary of W be suitably great. With t in W and t_1 ($|t_1| = |t|$) on a regular boundary of W or on a regular curve $RQ(\tau) = 0$ or on a regular curve (4.8), as the case may be, the path of integration indicated in the previous discussion extends towards t_1 along one of the mentioned curves either from infinity or from a point

on the circular part of the boundary of W (as the case may be);† from t_1 the path extends towards t along an arc of the circle $|\tau| = |t|$. *The upper bound of* $|\exp [Q(\tau)]|\tau|^{-\gamma}|$ $(\gamma > 0)$, *when* τ *is on such a path, is attained at* $\tau = t$ (at least for $|t| \geqq r'' > r'$). *A similar statement* (with $\gamma = 0$ and W replaced by R) *will hold, with the condition* (4.30) *deleted, when the problem* (4.29)–(4.29c) *is under consideration.*

LEMMA 3. *Consider the problem represented by* (4.1), (4.1a), (4.1b) *with* l' *fixed. A solution of this problem can be evaluated as follows:*

$$y(t) = \int^{t} e^{Q(\tau)} \tau^{r} [\tau]^{*} d\tau = e^{Q(t)} t^{r+w} [t]^{*} \qquad (t \text{ in } W) \tag{4.31}$$

(cf. Definition 4, §2), *where*

$$1 - l' \leqq w = 1 - \frac{l - \nu_1}{k} < 1, \tag{4.31a}$$

unless $Q(t) \equiv 0$, *when* $w = 1$. *For the problem* (4.29)–(4.29c), *with the leading term in* $Q(t)$ *given by* $qt^{\nu'/k}$ $(q \neq 0;\ \nu' > 0)$, *a solution of the form* (4.31) *will exist for* t *in* R'; *in this case*

$$w = 1 - \frac{\nu'}{k} < 1. \tag{4.31b}$$

5. **Formal solutions.** Let R be a region of positive opening, referred to in §2 in connection with (i), (ii), (9a), and (9b). Thus, when τ is interior to R, we have

$$RQ_1(\tau) \leqq \cdots \leqq RQ_m(\tau) < RQ_{m+1}(\tau) \leqq \cdots \leqq RQ_n(\tau). \tag{5.1}$$

Also

$$e^{Q_i(\tau)} \sim 0 \qquad (i = 1, \cdots, m;\ \tau \text{ in } R). \tag{5.1a}$$

Let l' be the number defined in the italicized statement made in connection with (3.10b), (3.10c). Finally, let $W = R(\alpha_1, \alpha_2)$ $(0 < \alpha_1 < \alpha_2;\ \alpha_2 - \alpha_1 < \pi/l')$ denote a subregion of R.

Our purpose will now be to determine the coefficients of the series (2.12) (cf. (2.12a)), that is, the

$$H\eta_{h_1, \cdots, h_m; j}(t) \tag{5.2}$$

$$(h_1, \cdots, h_m \geqq 0;\ h_1 + \cdots + h_m = H;\ j = 1, \cdots, n;\ H = 1, 2, \cdots).$$

† In a certain case the path extends from a fixed point on the circular part of the boundary of W, along this part to the intersection with one of the regular boundaries of W and from there on along the latter boundary towards t_1; from t_1 it is continued to t along a circular arc.

For $H=1$ the functions (5.2) are seen to be of the form (3.2). With the aid of (3.2) the relations (3.5), (3.5a) were obtained; that is, the form of the

$$ {}_2\Gamma_j^{h_1,\cdots,h_m}(\tau) \qquad (\tau \text{ in } R) \tag{5.3} $$

was determined.

Before proceeding further we note that, in consequence of (3.1), (3.1a), (2.30), and (2.30a), one may write

$$ {}_H\eta_{h_1,\cdots,h_m;j}(t) = \sum_{\lambda=1}^{n} c_\lambda(t) e^{Q_\lambda(t)} t^{r_\lambda} \gamma_{\lambda,j}(t), \tag{5.4} $$

where

$$ c_\lambda(t) = \sum_{\lambda_1=1}^{n} \int^t {}_H\Gamma_{\lambda_1}^{h_1,\cdots,h_m}(\tau) e^{-Q_\lambda(\tau)} \tau^{-r_\lambda+\omega+p} \bar{\gamma}_{\lambda_1,\lambda}(\tau) d\tau, \tag{5.4a} $$

$$ \gamma_{\lambda,j}(\tau) = [\tau]^*, \qquad \bar{\gamma}_{\lambda_1,\lambda}(\tau) = [\tau]^* \qquad (\tau \text{ in } R). \tag{5.4b} $$

Here the $\gamma_{\lambda,j}(\tau)$ and $\bar{\gamma}_{\lambda_1,\lambda}(\tau)$ are independent of H, $h_1, \cdots, h_m$. For $H=2$ the integrand displayed in (5.4a) has the form (3.6), (3.6a). Lemma 3, §4 is seen to be applicable for evaluation of all of the integrals involved in (5.4a) for $H=2$ when t is in W. The integral displayed in (5.4a) for $H=2$ will be of the form

$$ \exp\left[h_1Q_1(t) + \cdots + h_mQ_m(t) - Q_\lambda(t)\right] t^{h_1r_1+\cdots+h_mr_m-r_\lambda+\omega+p+w_2} [t]^* \tag{5.5} $$

when t is in W. Here w_2 is taken the same for all of the involved terms in (5.4a) for $H=2$; w_2 is rational and $w_2 \leqq 1$. It can be asserted that $c_\lambda(t)$ has the form (5.5) when t is in W (and $H=2$). Substitution of these forms in (5.4) (for $H=2$) will yield, in view of (5.4b),

$$ \begin{aligned} {}_2\eta_{h_1,\cdots,h_m;j}(t) &= \exp\left[h_1Q_1(t) + \cdots + h_mQ_m(t)\right] t^{h_1r_1+\cdots+h_mr_m+w_2'} [t]^* \\ &(h_1, \cdots, h_m \geqq 0;\ h_1 + \cdots + h_m = 2;\ j = 1, \cdots, n;\ t \text{ in } W). \end{aligned} \tag{5.6} $$

Here $w_2' = \omega + p + w_2$. On taking account of (3.2) in view of (5.6) one may write

$$ \begin{aligned} {}_{\nu_r}\eta_{k_1^r,\cdots,k_m^r;\alpha}(\tau) &= \exp\left[k_1^rQ_1(\tau) + \cdots + k_m^rQ_m(\tau)\right] \tau^{k_1^rr_1+\cdots+k_m^rr_m+(\nu_r-1)w_2'} [\tau]^* \\ &(\tau \text{ in } W;\ k_1^r, \cdots, k_m^r \geqq 0;\ k_1^r + \cdots + k_m^r = \nu_r;\ \alpha = 1, \cdots, n;\ \nu_r = 1, 2). \end{aligned} \tag{5.7} $$

We shall now substitute (5.7) in (2.26) (for $H=3$). If the equalities of (2.16b) and (5.7) are used it is inferred that

$$ \prod_{r=1}^{i_\alpha} \cdots = \exp\left[\delta_1Q_1(\tau) + \cdots + \delta_mQ_m(\tau)\right] \tau^{\delta_1r_1+\cdots+\delta_mr_m+\nu(\gamma_\alpha,i_\alpha)w_2'} [\tau]^*, \tag{5.8} $$

where

$$\nu(\gamma_\alpha, i_\alpha) = \sum_{r=1}^{i_\alpha} (\nu_r - 1) = \gamma_\alpha - i_\alpha. \tag{5.8a}$$

The symbol

$$\sum_{\nu_1+\cdots+\nu_{i_\alpha}=\gamma_\alpha} \sum_{k_1',\cdots,k_m'}^{(2)} \qquad (\delta_i = {}_\alpha\delta_i)$$

(cf. (2.16b)) applied to the product (5.8) will yield a function

$$F_\alpha' = \text{generic form of the last member of (5.8)} \qquad (\tau \text{ in } W). \tag{5.8b}$$

In view of (2.24) and (2.19b) (where $H=3$)

$$\begin{gathered} {}_1\delta_q + \cdots + {}_n\delta_q = h_q \ (q = 1, \cdots, m); \\ \gamma_1 + \cdots + \gamma_n = 3;\ i_1 + \cdots + i_n = 2, 3. \end{gathered} \tag{5.8c}$$

Accordingly, in consequence of (5.8b) (with $\delta_i = {}_\alpha\delta_i$) and (5.8a), one has

$$\prod_{\alpha=1}^{n} F_\alpha' = \exp\left[h_1Q_1(\tau) + \cdots + h_mQ_m(\tau)\right]\tau^{h_1r_1+\cdots+h_mr_m}\tau^{(3-s)w_2'}[\tau]^* \tag{5.8d}$$
$$(s = i_1 + \cdots + i_n;\ \tau \text{ in } W).$$

Since without any loss of generality one may consider that $w_2' \geqq 0$ and since the only values that s may assume (when $H=3$) are 2 and 3, it is concluded that†

$$\prod_{\alpha=1}^{n} F_\alpha' = \exp\left[h_1Q_1(\tau) + \cdots + h_mQ_m(\tau)\right]\tau^{h_1r_1+\cdots+h_mr_m}\tau^{w_2'}[\tau]^*. \tag{5.9}$$

We next write

$$F^{h_1,\cdots,h_m}_{i_1,\cdots,i_n;\gamma_1,\cdots,\gamma_n;{}_1\delta_1,\cdots,{}_n\delta_m} = {}_j a_{i_1,\cdots,i_n} \prod_{\alpha=1}^{n} F_\alpha' \tag{5.10}$$
$$= \text{generic form of last member of (5.9)}.$$

Next it is noted that the first summation symbol of the last member of (2.26) is defined by (2.24), (2.19b); this summation is extended only over the subscripts of the left member of (5.10). The generic form displayed in (5.9) does not contain these subscripts. Hence, for τ in W,

$${}_3\Gamma_j^{h_1,\cdots,h_m} = \sum^{(4)} F^{h_1,\cdots,h_m}_{i_1,\cdots;\cdots;\cdots} = \text{generic form of last member of (5.9)}. \tag{5.11}$$

Assume now that, with a rational $w_{H-1}' \geqq w_2'$ $(w_{H-1}' \leqq \omega+p+1)$ and for τ in W, one has

† $\tau^{(3-s)w_2'} = \tau^{w_2'} \cdot \tau^{(2-s)w_2'} = \tau^{w_2'}[\tau]_0$ since $(2-s)w_2'$ is rational and $\leqq 0$.

$$(5.12)\qquad {}_{q+1}\Gamma_j^{h_1,\cdots,h_m} = \exp\left[h_1Q_1(\tau)+\cdots+h_mQ_m(\tau)\right]\tau^{h_1r_1+\cdots+h_mr_m-(q-1)w_{H-1}}[\tau]^*$$
$$(h_1,\cdots,h_m\geqq 0;\ h_1+\cdots+h_m=q+1;\ j=1,\cdots,n),$$

$$(5.13)\qquad {}_q\eta_{k_1,\cdots,k_m;\alpha}(\tau) = \exp\left[k_1Q_1(\tau)+\cdots+k_mQ_m(\tau)\right]\tau^{k_1r_1+\cdots+k_mr_m-(q-1)w_{H-1}}[\tau]^*$$
$$(k_1,\cdots,k_m\geqq 0;\ k_1+\cdots+k_m=q;\ \alpha=1,\cdots,n)$$

for $q=1, 2, \cdots, H-1$ $(H\geqq 3)$. For $H=3$, that is for $q=1, 2$, relations (5.12) and (5.13) have been established previously in (3.2), (3.5), (5.6), and (5.11).

Substitute (5.12) with $q=H-1$ in (5.4a). The integrand there displayed will be of the form

$$(5.14)\qquad e^{Q(\tau)}\tau^r[\tau]^* \qquad (\tau \text{ in } W),$$

where

$$(5.14a)\qquad Q(\tau) = h_1Q_1(\tau)+\cdots+h_mQ_m(\tau)-Q_\lambda(\tau),$$

$$(5.14b)\qquad r = h_1r_1+\cdots+h_mr_m-r_\lambda+\omega+p+(H-2)w'_{H-1}$$

with

$$(5.14c)\qquad h_1,\cdots,h_m\geqq 0;\quad h_1+\cdots+h_m=H \qquad (\geqq 3).$$

In view of (5.14c) and (5.1) it is inferred that, for $\lambda>m$, one certainly has

$$e^{Q(\tau)}\sim 0 \qquad (\tau \text{ in } W).$$

When $\lambda\leqq m$, $Q(\tau)$ will be a linear combination (with constant coefficients) of

$$(5.15)\qquad Q_1(\tau),\cdots,Q_m(\tau)$$

only. With (3.10b) denoting the leading terms of the polynomials (5.15) it is recalled that l' was defined as the greater one of the numbers l_i/k_i $(i=1,\cdots,m)$. Thus, when $\lambda\leqq m$, $Q(\tau)$ is of the form (3.10) and, unless $Q(\tau)\equiv 0$, there will be a leading term,

$$q_{\nu_1}\tau^{(l-\nu_1)/k} \qquad (q_{\nu_1}\neq 0;\ l-\nu_1>0),$$

present in $Q(\tau)$; moreover, we shall have $(l-\nu_1)/k\leqq l'$. It is accordingly seen that, when $\lambda\leqq m$, $Q(\tau)$ will be a polynomial of the variety of polynomials involved in the integration problem (4.1), (4.1a), (4.1b). On the other hand, when $\lambda>m$, since $\exp[Q(\tau)]\sim 0$ (τ in W), it is observed that $Q(t)$ will be a polynomial of the type which could occur in the problem (4.29)–(4.29c), provided one lets $R'=W$. Thus *Lemma* 3, §4 *is applicable for evaluation of the integrals involved in* (5.4a). With the aid of this lemma and in view of (5.14) we conclude that we may write

$$(5.16)\qquad \int^t (\text{integrand displayed in (5.4a)}) = e^{Q(t)}t^{r+w_H}[t]^* \qquad (t \text{ in } w),$$

where w_H is rational and $w_H \leqq 1$. The functions $c_\lambda(t)$ (cf. (5.4a)) will be all of the generic form of the last member of (5.16). Hence, in view of (5.14a), (5.14b), one obtains from (5.4) and (5.4b)

$$
\begin{aligned}
&{}_H\eta_{h_1,\cdots,h_m;j}(t) \\
(5.16a)\qquad &= \sum_{\lambda=1}^{n} \exp\left[h_1Q_1(t) + \cdots + h_mQ_m(t)\right] t^{h_1r_1+\cdots+h_mr_m+(H-2)w_{H-1}+w_H''}[t]^* \\
&\qquad (t \text{ in } W;\ h_1 + \cdots + h_m = H;\ j = 1, \cdots, n),
\end{aligned}
$$

where $w_H'' = \omega + p + w_H \leqq \omega + p + 1$. *Let w_H' be the greater of the numbers w_H'' and w_{H-1}'. Necessarily w_H' will be rational and $w_H' \leqq \omega + p + 1$.* On noting that

$$
t^{(H-2)(w_{H-1}-w_H')+w_H''-w_H'} = [t]_0,
$$

in view of (5.16a) it is inferred that, for t in W

$$
(5.17)\qquad {}_H\eta_{h_1,\cdots,h_m;j}(t) = \exp\left[h_1Q_1(t) + \cdots + h_mQ_m(t)\right] t^{h_1r_1+\cdots+h_mr_m+(H-1)w_H'}[t]^*.
$$

This, however, implies that (5.13) holds also when H is increased by unity; that is (5.13) holds for $q = 1, 2, \cdots, H$, provided that in (5.13) we write w_H' in place of w_{H-1}'. This is possible for $q = 1, 2, \cdots, H-1$ because $w_{H-1}' - w_H'$ is rational and non-positive.

To determine the form of ${}_{H+1}\Gamma_j^{h_1,\cdots,h_m}$ use will be made of (5.13), with w_{H-1}' replaced by w_H', and of (5.17). Substituting the known forms of the ${}_q\eta_{\cdots;j}(t)$ $(q = 1, 2, \cdots,)$ in (2.26), where H is replaced by $H+1$, one obtains

$$
(5.18)\qquad \prod_{r=1}^{i_\alpha} \cdots = \text{generic form of the last member of (5.8)} \quad (w_2' \text{ replaced by } w_H').
$$

Here (5.8a) will hold. The summation symbol, displayed subsequent to (5.8a), applied to the product (5.18) will yield

$$
(5.18a)\qquad F_\alpha' = \text{generic form of the last member of (5.18)}.
$$

By virtue of the relations (2.24), (2.19b), where H is replaced by $H+1$, one has

$$
\begin{aligned}
&{}_1\delta_q + \cdots + {}_n\delta_q = h_q \qquad (q = 1, \cdots, m); \\
(5.18b)\qquad &\gamma_1 + \cdots + \gamma_n = H + 1;\ i_1 + \cdots + i_n = 2, 3, \cdots, H + 1.
\end{aligned}
$$

Thus from (5.18a), where we put $\delta_i = {}_\alpha\delta_i$, and in view of (5.8a) it is inferred that

$$
\begin{aligned}
\prod_{\alpha=1}^{n} F_\alpha' &= \exp\left[h_1Q_1(\tau) + \cdots + h_mQ_m(\tau)\right] \tau^{h_1r_1+\cdots+h_mr_m\tau^{(H+1-s)w_H'}}[\tau]^* \\
(5.19)\qquad &= \exp\left[h_1Q_1(\tau) + \cdots + h_mQ_m(\tau)\right] \tau^{h_1r_1+\cdots+h_mr_m\tau^{(H-1)w'}}[\tau]^* \\
&\qquad (s = i_1 + \cdots + i_n = 2, 3, \cdots, H + 1;\ \tau \text{ in } W),
\end{aligned}
$$

since $w_H' \geqq 0$ and w_H' is rational. The functions (5.10) will have the generic form of the last member of (5.19). In view of this fact

$$
\begin{aligned}
{}_{H+1}\Gamma_j^{h_1,\cdots,h_m} &= \sum {}^{(4)}F_{i_1,\cdots,i_n;\gamma_1,\cdots,\gamma_n;j_1,\cdots,n\delta_n}^{h_1,\cdots,h_m} \\
&= \exp\left[h_1Q_1(\tau) + \cdots + h_mQ_m(\tau)\right]\tau^{h_1r_1+\cdots+h_mr_m\tau^{(H-1)w_H'}}[\tau]^* \qquad (\tau \text{ in } W),
\end{aligned} \tag{5.20}
$$

since the summation here involved is only with respect to the subscripts of F.

We accordingly conclude that relations (5.12) and (5.13) will be certainly valid for $H = 3, 4, \cdots$ when τ is in W. *The numbers w_H' are rational and*

$$0 \leqq w_2' \leqq w_3' \leqq \cdots \leqq \omega + p + 1. \tag{5.21}$$

LEMMA 4. *Let R be a region of positive opening* (cf. Definition 3, §2), *referred to in* §2 *in connection with* (i), (ii), (5.9a), (5.9b). *As a matter of notation, within R we have* (5.1) *and* (5.1a), *the $Q_i(\tau)$ $(i = 1, \cdots, n)$ being the polynomials involved in the formal matrix solution* (2.5) *of the linear differential system* (LA_1), §2.† *The highest power of τ actually present in $Q_i(\tau)$ being designated as τ^{l_i/k_i} we let l' denote the greatest of the numbers l_i/k_i $(i = 1, \cdots, m)$. Let W be of the form $R(\alpha_1, \alpha_2)$* (cf. Definition 3), *where*

$$0 < \alpha_2 - \alpha_1 < \pi/l', \tag{5.22}$$

and be a subregion of R. The system (A), §1 *will possess a formal solution*

$$s_j(t) = {}_1y_j(t) + {}_2y_j(t) + \cdots + {}_\nu y_j(t) + \cdots \qquad (j = 1, 2, \cdots, n), \tag{5.23}$$

where, with $c_1, c_2, \cdots, c_m$ denoting arbitrary constants,

$$
\begin{aligned}
{}_\nu y_j(t) &= \sum_{k_1,\cdots,k_m} c_1^{k_1}c_2^{k_2}\cdots c_m^{k_m}\,{}_\nu\eta_{k_1,k_2,\cdots,k_m;j}(t) \\
&(k_1, \cdots, k_m \geqq 0;\ k_1 + k_2 + \cdots + k_m = \nu;\ j = 1, \cdots, n).
\end{aligned} \tag{5.23a}
$$

In (5.23a) *the coefficients are functions which for t in W, have the form*

$${}_\nu\eta_{k_1,\cdots,k_m;j}(t) = \exp\left[k_1Q_1(t) + \cdots + k_mQ_m(t)\right]t^{k_1r_1+\cdots+k_mr_mt^{(\nu-1)w_j'}}[t]_{q(\nu)} \tag{5.24}$$

(cf. Definition 4, §2).‡ *The w_j' satisfy the conditions stated in connection with* (5.21).

Note. A function $[t]_{q(\nu)}$ involved in the second member of (5.24) satisfies an asymptotic relation

$$[t]_{q(\nu)} \sim \{t\}_{q(\nu)} \qquad (t \text{ in } W)$$

† This system is identical with (LA), §2 and is the linear system associated with the non-linear problem (A), §1.

‡ It might happen that $q(\nu) \to \infty$ as $\nu \to \infty$.

(cf. Definition 1, §2). The formal expression $\{t\}_{q(\nu)}$ contains power series (generally divergent) of the form

$$\rho_0 + \rho_1 t^{-1/k_\nu} + \rho_2 t^{-2/k_\nu} + \cdots \qquad (k_\nu \text{ an integer} > 0).$$

The k_ν $(\nu = 1, 2, \cdots)$ may approach infinity with ν.

In general the series (5.23) will diverge.

6. **Transformations.** With Lemma 4, §5 in view, consider the transformation

$$y_j(t) = Y_j(t) + \rho_j(t) \qquad (j = 1, \cdots, n), \tag{6.1}$$

where

$$Y_j(t) = {}_1y_j(t) + {}_2y_j(t) + \cdots + {}_{N-1}y_j(t). \tag{6.1a}$$

Here the ${}_\nu y_j(t)$ $(\nu = 1, \cdots, N-1; j = 1, \cdots, n)$ *are the functions* (5.23a) *and* (5.24). *N is a fixed positive integer* (which may be taken as large as desired). This transformation is to be applied to (A), §1.

We let

$$|c_i| < c' \qquad (c' > 0;\ i = 1, \cdots, m). \tag{6.2}$$

Denoting the part of a region G for which $|t| \geqq r'$ by G $(|t| \geqq r')$ we observe that

$$|Y_j(t)| \leqq \rho' \qquad [j = 1, \cdots, n;\ t \text{ in } W(|t| \geqq r')], \tag{6.3}$$

where

$$0 < \rho' < \rho, \tag{6.3a}$$

where ρ is the ρ of (1.2), if r' (depending on N and c') in (6.3) is taken sufficiently great. In order that (6.3) hold r' may be taken independent of N, provided c' (depending on N) of (6.2) has been taken sufficiently small.

Substitution of (6.1) in (A), §1 will yield

$$\begin{aligned} t^{-p}\rho_j^{(1)}(t) - l_j(t, \rho_1, \cdots, \rho_n) &= -t^{-p}Y_j^{(1)}(t) + l_j(t, Y_1, \cdots, Y_n) \\ &\quad + q_j(t, Y_1 + \rho_1, \cdots, Y_n + \rho_n) \qquad (j = 1, \cdots, n), \end{aligned} \tag{6.4}$$

where

$$\begin{aligned} q_j(t, Y_1 + \rho_1, \cdots) &= q_j(t, Y_1, \cdots, Y_n) \\ &\quad + \sum_{\nu_1, \cdots, \nu_n} \alpha_{\nu_1, \nu_2, \cdots, \nu_n; j}(t)\rho_1^{\nu_1}(t) \cdots \rho_n^{\nu_n}(t) \end{aligned} \tag{6.4a}$$

$$(\nu_1 \geqq 0, \cdots, \nu_n \geqq 0;\ \nu_1 + \nu_2 + \cdots + \nu_n = 1, 2, \cdots),$$

$$\nu_1! \cdots \nu_n! \alpha_{\nu_1, \cdots, \nu_n; j}(t) = \frac{\partial^{\nu_1 + \cdots + \nu_n}}{\partial^{\nu_1} y_1 \cdots \partial^{\nu_n} y_n} q_j(t, y_1, \cdots, y_n)\Big] \tag{6.4b}$$

$$(y_1 = Y_1(t), \cdots, y_n = Y_n(t))$$

(cf. (1.1b)). In view of (6.3) and the convergence conditions satisfied by the series (1.1b), *the series* (6.4a) *with* $j=1, \cdots, n$ *are absolutely and uniformly convergent for* t *in* W, *provided*

$$|\rho_1(t)|, \cdots, |\rho_n(t)| \leqq \rho'' \tag{6.5}$$

where

$$0 < \rho'' < \rho - \rho', \tag{6.5a}$$

with ρ' defined as in (6.3a).

We may write (6.4) in the form

$$t^{-p}\rho_j^{(1)}(t) - l_j(t, \rho_1, \cdots, \rho_n) = \bar{q}_j(t, \rho_1, \cdots, \rho_n) - F_j(t) \quad (j = 1, \cdots n), \tag{6.6}$$

where

$$\bar{q}_j = \sum_{\nu_1+\cdots+\nu_n \geqq 1} \alpha_{\nu_1,\cdots,\nu_n;j}(t)\rho_1^{\nu_1} \cdots \rho_n^{\nu_n} \tag{6.6a}$$

(cf. (6.4b)) and

$$F_j(t) = t^{-p}Y_j^{(1)}(t) - l_j(t, Y_1, \cdots, Y_n) - q_j(t, Y_1, \cdots, Y_n). \tag{6.7}$$

To determine the form of $F_j(t)$ we first note that, by (2.23a),

$$q_j(t, Y_1, \cdots, Y_n) = \sum_{H=2}^{\infty} \sum_{h_1+\cdots+h_m=H} c_1^{h_1} \cdots c_m^{h_m} {}_H\Gamma_j'^{h_1,\cdots,h_m}(t). \tag{6.8}$$

We now recall that *the functions*

$${}_H\Gamma_j^{h_1,\cdots,h_m}(t), \tag{6.9}$$

referred to in §5, *depend only on the functions*

$${}_\nu\eta_{k_1,\cdots,k_m;j}(t) \tag{6.9a}$$

of rank less than n (cf. Definition 5 and (2.26)); the functions (6.9a) have in W the form (5.24). It is clear that the function (6.8) can be given by the expression $q_j(t, y_1, \cdots, y_n)$ $(y_1(t) = s_1(t), \cdots, y_n(t) = s_n(t)$ the formal series (5.23)) provided that in the latter expression all the functions (6.9a) with rank equal to or greater than N have been replaced by zeros. Hence, in (6.8), one has

$${}_H\Gamma_j'^{h_1,\cdots,h_m}(t) = {}_H\Gamma_j^{h_1,\cdots,h_m}(t); \tag{6.10}$$

in the second member of (6.10) *the functions* (6.9a), *for which* $\nu \geqq N$, *are replaced by* 0. In view of the statements in connection with (6.9), (6.9a), and (6.10) we have

$$(6.10a)\qquad {}_H\Gamma_j'^{h_1,\cdots,h_m}(t) = {}_H\Gamma_j^{h_1,\cdots,h_m}(t) \qquad (H = 2, 3, \cdots, N).$$

For $H > N$ (6.10) will, in general, not hold. However, in view of (2.26) and in consequence of the indicated connection between the coefficients in (6.8) and the functions (6.9), it is inferred that, for t in W,

$$(6.10b)\qquad \begin{aligned} {}_H\Gamma_j'^{h_1,\cdots,h_m}(t) &= \text{generic form of } {}_H\Gamma_j^{h_1,\cdots,h_m}(t) \\ &= \exp\left[h_1Q_1(t) + \cdots + h_mQ_m(t)\right] t^{h_1r_1+\cdots+h_mr_m} L^{(H-2)w_{H-1}'}[t]^* \end{aligned}$$

for all values of the subscripts and superscripts (cf. (5.20) and (5.21)).

In view of the developments of §2 and in particular in consequence of (2.13) and (2.14) we see that if in the second member of (6.7) one replaces the Y_i by the $y_i(t)$ (formal solutions referred to in Lemma 4, §5), respectively, this member can be formally expanded as follows:

$$(6.11)\qquad \sum_{\nu=2}^{\infty} \sum_{k_1+\cdots+k_m=\nu} c_1^{k_1} \cdots c_m^{k_m} {}_\nu\lambda_{k_1,\cdots,k_m;j}(t),$$

where

$$(6.11a)\qquad {}_\nu\lambda_{k_1,\cdots,k_m;j}(t) = t^{-p}\, {}_\nu\eta^{(1)}_{k_1,\cdots,k_m;j}(t) - l_j(t, \cdots) - {}_\nu\Gamma_j^{k_1,\cdots,k_m}(t).$$

From the definition of the functions (6.9a) it follows that

$$(6.11b)\qquad {}_\nu\lambda_{k_1,\cdots,k_m;j}(t) = 0$$

for all values of the subscripts.

If in (6.11) the functions (6.9a) of rank $\geqq N$ are all replaced by zeros we obtain an expansion of (6.7),

$$(6.12)\qquad F_j(t) = \sum_{\nu=2}^{\infty} \sum_{k_1+\cdots+k_m=\nu} c_1^{k_1} \cdots c_m^{k_m} {}_\nu\lambda'_{k_1,\cdots,k_m;j}(t),$$

$$(6.12a)\qquad {}_\nu\lambda'_{k_1,\cdots,k_m;j}(t) = t^{-p}\, {}_\nu\eta^{(1)}_{k_1,\cdots,k_m;j}(t) - l_j(t, \cdots) - {}_\nu\Gamma_j'^{k_1,\cdots,k_m}(t),$$

where the functions (6.9a) are replaced by zeros for $\nu \geqq N$. Thus, by (6.10a), (6.11a), and (6.11b),

$$(6.13)\qquad {}_\nu\lambda'_{k_1,\cdots,k_m;j}(t) = {}_\nu\lambda_{k_1,\cdots,k_m;j}(t) = 0 \qquad (\nu = 2, 3, \cdots, N-1),$$

$$(6.13a)\qquad {}_\nu\lambda'_{k_1,\cdots,k_m;j}(t) = -\,{}_\nu\Gamma_j'^{k_1,\cdots,k_m}(t) \qquad (\nu = N, N+1, \cdots).$$

Thus $F_j(t)$ is representable by

$$(6.14)\qquad -F_j(t) = \sum_{\nu \geqq N} \sum_{k_1+\cdots+k_m=\nu} c_1^{k_1} \cdots c_m^{k_m} {}_\nu\Gamma_j'^{k_1,\cdots,k_m}(t).$$

In view of (5.21) one may replace w_{H-1}' in (6.10b) by

$$\bar{\omega} = \omega + p + 1. \tag{6.15}$$

Since

$$[t]^* \leqq |t|^{\xi} b_{\xi},$$

where ξ is positive, however small, the inequality being inferred from Definition 4, §2, we accordingly obtain

$$|{}_{\nu}\Gamma_j'^{k_1,\cdots,k_m}(t)| < g_{k_1,\cdots,k_m}(t)\,|t|^{-2\bar{\omega}+\xi} b_{\nu}' \qquad (t \text{ in } W), \tag{6.16}$$

where

$$g_{k_1,\cdots,k_m}(t) = |\exp[k_1 Q_1(t) + \cdots + k_m Q_m(t)]\, t^{k_1\bar{r}_1+\cdots+k_m\bar{r}_m}| \tag{6.16a}$$
$$(\bar{r}_1 = r_1 + \bar{\omega}, \cdots, \bar{r}_m = r_m + \omega).$$

Hence by (6.2) and (6.14) we have, for t in W,

$$|F_j(t)| < k_0 (c')^N |t|^{-2\bar{\omega}+\xi} \sum_{k_1+\cdots+k_m=N} g_{k_1,\cdots,k_m}(t) \tag{6.17}$$

(cf. (6.16a) and (6.15)), where k_0 is a constant independent of c'.

In the series (6.6a) the coefficients are given by (6.4b). In view of (1.1b) it accordingly follows that

$$\alpha_{\nu_1,\cdots,\nu_n;j}(t) = \sum_{\lambda_1,\cdots,\lambda_n} {}_{j}a_{\nu_1+\lambda_1,\cdots,\nu_n+\lambda_n}(t) C_{\lambda_1}^{\nu_1+\lambda_1} \cdots C_{\lambda_n}^{\nu_n+\lambda_n} Y_1^{\lambda_1}(t) \cdots Y_n^{\lambda_n}(t), \tag{6.18}$$

where

$$\lambda_1 \geqq 0, \cdots, \lambda_n \geqq 0;\ \lambda_1 + \cdots + \lambda_n \geqq 2 - (\nu_1 + \cdots + \nu_n). \tag{6.18a}$$

Since, in W, $Y_i(t) \sim 0$ $(i = 1, \cdots, n)$ and since by (6.18a) for $\nu_1 + \cdots + \nu_n = 1$ we have $\lambda_1 + \cdots + \lambda_n \geqq 1$, it is inferred from (6.18) that

$$\alpha_{\nu_1,\cdots,\nu_n;j}(t) \sim 0 \quad (t \text{ in } W;\ \nu_1 + \cdots + \nu_n = 1). \tag{6.19}$$

Furthermore

$$\alpha_{\nu_1,\cdots,\nu_n;j}(t) = {}_{j}a_{\nu_1,\cdots,\nu_n}(t) + {}_{j}\beta_{\nu_1,\cdots,\nu_n}(t) \qquad (\nu_1 + \cdots + \nu_n \geqq 2), \tag{6.19a}$$

where

$${}_{j}\beta_{\nu_1,\cdots,\nu_n}(t) = \sum_{\lambda_1+\cdots+\lambda_n \geqq 1} \cdots \sim 0 \qquad (t \text{ in } W). \tag{6.19b}$$

By (6.19) equation (6.6) may be written in the form

$$t^{-p}\rho_j^{(1)}(t) - l_j^*(t, \rho_1, \cdots, \rho_n) = q_j^*(t, \rho_1, \cdots, \rho_n) - F_j(t) \tag{6.20}$$
$$(j = 1, \cdots, n),$$

where

$$q_j^*(t, \rho_1, \cdots, \rho_n) = \sum_{\nu_1+\cdots+\nu_n \geqq 2} \alpha_{\nu_1,\cdots,\nu_n;j}(t)\rho_1^{\nu_1}\cdots\rho_n^{\nu_n} \tag{6.20a}$$

(cf. (6.19a), (6.19b)) and

$$l_j^*(t, \rho_1, \cdots, \rho_n) \sim l_j(t, \rho_1, \cdots, \rho_n) \qquad \text{(in } t \text{ for } t \text{ in } W\text{).}\dagger \tag{6.20b}$$

LEMMA 5. *The transformation* (6.1), (6.1a) *applied to the non-linear system* (A), §1 *will yield a system* (6.20). *In* (6.20) l_j^* *is linear in* $\rho_1, \cdots, \rho_n$, *the coefficients of* $\rho_1, \cdots, \rho_n$ *being analytic, for* $t \neq \infty$, *in* W; *moreover,* (6.20b) *holds. The* q_j^* *are given by* (6.20a) *and have coefficients analytic for* t *in* W $(t \neq \infty)$. *The series* (6.20a) *converge absolutely and uniformly (for* t *in* W $(|t| \geqq r')$ *and provided* (6.2) *is satisfied) whenever the inequalities* (6.5), (6.5a) *hold. The* $F_j(t)$ *are analytic in* W $(t \neq \infty)$ *and the* $|F_j(t)|$ *satisfy in* W *inequalities* (6.17). *These assertions are made under the supposition that* ρ', r', *and* c' *(of* (6.2)) *have been selected so that inequalities* (6.3a) *and* (6.3) *hold.*

7. **The First Existence Theorem.** We shall write for brevity

$$q_{k_1,\cdots,k_m;h}(t) = \exp\left[k_1Q_1(t) + \cdots + k_mQ_m(t)\right]t^{k_1\bar{r}_1+\cdots+k_m\bar{r}_m} \qquad (\bar{r}_i = r_i + h) \tag{7.1}$$

and, with $\nu > 0$,

$$q_{\nu;h}(t) = \sum_{k_1,\cdots,k_m} |\, q_{k_1,\cdots,k_m;h}(t)\,| \qquad (k_1 \geqq 0, \cdots, k_m \geqq 0;\ k_1 + \cdots + k_m = \nu). \tag{7.1a}$$

If we recall that $Q_i(\tau) = q_i\tau^{l_i/k_i} + \cdots$ $(i = 1, \cdots, m)$, *where* $q_i \neq 0$ *and* $l_i/k_i > 0$, *and take account of*

$$RQ_1(\tau) \leqq RQ_2(\tau) \leqq \cdots \leqq RQ_m(\tau) \qquad (\tau \text{ in } R),\ddagger \tag{7.1b}$$

it is inferred that necessarily

$$l' = \frac{l_1}{k_1} = \frac{l_2}{k_2} = \cdots = \frac{l_{\bar{m}}}{k_{\bar{m}}} > \frac{l_{\bar{m}+1}}{k_{\bar{m}+1}} \geqq \cdots \geqq \frac{l_m}{k_m}, \tag{7.2}$$

where $1 \leqq \bar{m} \leqq m$. In fact, if the contrary were the case, for some $\nu (1 \leqq \nu < m)$ one would have

$$\frac{l_\nu}{k_\nu} < \frac{l_{\nu+1}}{k_{\nu+1}}. \tag{7.2a}$$

We shall write $\bar{\tau} =$ angle of τ. Let $\bar{\tau} = \bar{\tau}_0$ be some ray, interior to R, with $\bar{\tau}_0$

† (6.20b) is to mean that $l_j^* - l_j = b_1(t)\rho_1 + \cdots + b_n(t)\rho_n$, where $b_i(t) \sim 0$ for $i = 1, \cdots, n$ and for t in W.

‡ R is the region of Lemma 4, §5.

distinct from the limiting directions of the regular curves bounding R. Such a ray exists since by hypothesis R is of positive opening. We have

$$\lambda'_\nu = \cos\left(\bar{q}_\nu + \frac{l_\nu}{k_\nu}\bar{\tau}_0\right) < 0, \quad \lambda'_{\nu+1} = \cos\left(\bar{q}_{\nu+1} + \frac{l_{\nu+1}}{k_{\nu+1}}\bar{\tau}_0\right) < 0 \tag{7.3}$$

$$(\bar{q}_i = \angle\, q_i;\ i = 1, \cdots, m)$$

since the above cosines can vanish only when $\bar{\tau}_0$ is replaced by the limiting directions of the regular curves

$$RQ_\nu(\tau) = 0, \qquad RQ_{\nu+1}(\tau) = 0,$$

respectively, and since in R

$$e^{Q_\nu(\tau)} \sim 0, \qquad e^{Q_{\nu+1}(\tau)} \sim 0.$$

Now (7.1b) necessitates that along $\bar{\tau} = \bar{\tau}_0$ we should have

$$|\, q_\nu |\, \lambda'_\nu |\, \tau\,|^{l_\nu/k_\nu} + \cdots \leqq |\, q_{\nu+1}|\, \lambda'_{\nu+1}|\, \tau\,|^{l_{\nu+1}/k_{\nu+1}} + \cdots;$$

that is,

$$|\, q_\nu |\, \lambda'_\nu |\, \tau\,|^{-\alpha} + \cdots \leqq |\, q_{\nu+1}|\, \lambda'_{\nu+1} + \cdots \qquad (\alpha > 0),$$

which can be seen in view of (7.2a). In the limit, as $|\tau| \to \infty$ along the ray $\bar{\tau} = \bar{\tau}_0$, this will yield the inequality

$$0 \leqq |\, q_{\nu+1}|\, \lambda'_{\nu+1},$$

which contradicts (7.3). Hence (7.2a) is impossible and the truth of (7.2) has been made evident.

DEFINITION 7. *Let R be a region of the type specified in Lemma* 4, §5. *Let W be a subregion of R of the character indicated in the same lemma. Thus, $W = R(\alpha_1, \alpha_2)$* (cf. Definition 3, §2), *where $0 < \alpha_2 - \alpha_1 < \pi/l'$* (cf. (7.2)). *Let W_ξ denote a subregion of W of the form*

$$W_\xi = R(\alpha_1 + \xi, \alpha_2 - \xi) \qquad (\xi > 0 \text{ and sufficiently small}) \tag{7.4}$$

where the two regular curves bounding W_ξ are the rays

$$\angle\, t = \alpha_1 + \xi, \qquad \angle\, t = \alpha_2 - \xi \qquad (|\, t| \geqq r'_\xi),$$

respectively.†

In consequence of the definition of W_ξ we have

$$-|\, q_i| \cos\left(\bar{q}_i + \frac{l_1}{k_1}\bar{\tau}\right) \geqq \epsilon = \epsilon(\xi) > 0 \quad (i = 1, \cdots, \bar{m}) \tag{7.4a}$$

for τ in W_ξ.

† The rest of the boundary of W_ξ consists of an arc of the circle $|\, t| = r'_\xi$.

DEFINITION 8. *If G is a regular region such that, whenever t is in G, all the points of the ray $\angle\tau = \angle t$ ($|\tau| \geqq |t|$) necessarily belong to G, and if $q(\tau)$ is positive and defined in G, it will be said that $q(\tau)$ is monotone in G, provided the upper bound of $q(\tau)$ ($\angle\tau = \angle t$; $|\tau| \geqq |t|$) is attained at t for every t in G.*

LEMMA 6. *Let W_ξ be a region as given in Definition 7. With*

$$Q_i(\tau) = q_i\tau^{l'} + \cdots \qquad (|q_i| \neq 0;\ i = 1, \cdots, \bar{m})$$

(cf. (7.2)) *and with $\epsilon = \epsilon(\xi)$ denoting the number involved in* (7.4a), *let $\nu(\xi)$ be the least integer equal to or greater than*

$$\frac{\epsilon + |q_\lambda|}{\epsilon} \tag{7.5}$$

$(\lambda = 1, \cdots, \bar{m} - 1)$.

Write

$$_\lambda\Lambda_{k_1,\cdots,k_{\bar{m}};h}(\tau) = |q_{k_1,\cdots,k_{\bar{m}};h}(\tau)|\,|e^{-Q_\lambda(\tau)\tau^{-\bar{r}_\lambda}}| \tag{7.6}$$

(cf. (7.1)), *where*

$$k_1 \geqq 0, \cdots, k_- \geqq 0;\quad k_1 + \cdots + k_- = \nu \geqq \nu(\epsilon);\quad \lambda = 1, 2, \cdots, n. \tag{7.6a}$$

There exists then a number $r' = r'(\xi, h)$, independent of $\lambda, k_1, \cdots, k_{\bar{m}}$, such that the functions (7.6) *are all monotone* (cf. Definition 8) *in W_ξ ($|t| \geqq r'$).*

To prove this lemma first *take $r' = r'_{\xi,k}$ sufficiently great so that the functions*

$$|e^{Q_j(\tau)\tau^{\bar{r}_j}}| \qquad (j = 1, \cdots, \bar{m}), \tag{7.7}$$

$$|\exp\,[Q_i(\tau) - Q_\lambda(\tau)]\tau^{\bar{r}_i - \bar{r}_\lambda}| \qquad (i \leqq \bar{m};\ i < r) \tag{7.7a}$$

are all monotone in W_ξ ($|t| \geqq r'$). The possibility of such a choice follows without difficulty if we note that in W_ξ

$$RQ_j(\tau) < 0, \qquad R(Q_i(\tau) - Q_\lambda(\tau)) < 0 \tag{7.7b}$$

for the values of the subscripts indicated in (7.7), (7.7a), while on the other hand the following is true. If $\bar{r}_1$ denotes the limiting direction of any one of the regular curves

$$RQ_j(\tau) = 0, \qquad R(Q_i(\tau) - Q_\lambda(\tau)) = 0 \qquad (j \leqq \bar{m};\ i \leqq \bar{m};\ i < \lambda \tag{7.7c}$$

then, for τ (with $\angle\tau = \bar{r}$) in W_ξ, we have $|\bar{r}_1 - \bar{r}| \geqq \zeta(\xi) > 0$.

In view of the statement in connection with (7.7) we infer that *the functions*

$$|q_{\nu_1,\cdots,\nu_{\bar{m}};h}(\tau)| \qquad (\nu_1 \geqq 0, \cdots, \nu_{\bar{m}} \geqq 0) \tag{7.8}$$

(cf. (7.1)) *are monotone in W_ξ ($|t| \geqq r'$),*† these functions being expressible

† The statement in connection with (7.8) can be made also when $\bar{m}$ is replaced by m.

as products of non-negative powers of functions monotone in W_ξ ($|t| \geqq r'$).

Let $\lambda > \bar{m}$. By (7.6a) some k_i ($i \leqq \bar{m}$) will be positive. Thus, if we write

$$(7.9) \qquad {}_\lambda\Lambda_{k_1,\cdots,k_{\bar{m}}^-;h}(\tau) = \left| q_{k_1,\cdots,k_i-1,\cdots,k_{\bar{m}}^-;h}(\tau) \right| \left| e^{Q_i(\tau)-Q_\lambda(\tau)\tau^{\bar{r}_i-r_\lambda}} \right|,$$

we observe that the statements made in connection with (7.8) and (7.7a) are applicable to the two factors of the second member of (7.9). *Hence the functions* (7.6) *with* $\lambda > \bar{m}$ *are monotone in* W_ξ ($|t| \geqq r'$).

Suppose $\lambda \leqq \bar{m}$ *and* $k_\lambda > 0$. Then

$$(7.9a) \qquad {}_\lambda\Lambda_{k_1,\cdots,k_{\bar{m}}^-;h}(\tau) = \left| q_{k_1,\cdots,k_\lambda-1,\cdots,k_{\bar{m}}^-;h}(\tau) \right|.$$

The statement in connection with (7.8) is again applicable, and *the functions* (7.9a) ($\lambda \leqq \bar{m}$, $k_\lambda > 0$) *are seen to be monotone in* W_ξ ($|t| \geqq r'$).

Suppose $\lambda \leqq \bar{m}$, $k_\lambda = 0$, *and some* $k_i > 0$ ($i < \lambda$). We then write (7.9) and demonstrate, as before, that *the functions* (7.6) *are monotone in* W_ξ (under the conditions just stated).

It remains to consider the *case when* $\lambda \leqq \bar{m}$ *and* $k_1 = k_2 = \cdots = k_\lambda = 0$. Necessarily one must have $\lambda < \bar{m}$ since otherwise $k_1 + \cdots + k_{\bar{m}}^- = 0$, contrary to (7.6a). One may write

$$(7.10) \qquad {}_\lambda\Lambda_{k_1,\cdots,k_{\bar{m}}^-;h}(\tau) = \left| e^{Q(\tau)t^{-\bar{r}_\lambda + h_{\lambda+1}\bar{r}_{\lambda+1} + \cdots + h_{\bar{m}}\bar{r}_{\bar{m}}}} \right| \cdot \left| q_{0,\cdots,0;k_{\lambda+1}-h_{\lambda+1},\cdots,k_{\bar{m}}^- - h_{\bar{m}}^-}(\tau) \right|,$$

where

$$(7.10a) \qquad h_{\lambda+1} \geqq 0, \cdots, h_{\bar{m}}^- \geqq 0;\ h_{\lambda+1} + \cdots + h_{\bar{m}} = \nu(\xi);\ h_i \leqq k_i \qquad (i = \lambda+1, \cdots, \bar{m})$$

and

$$(7.10b) \qquad Q(\tau) = -Q_\lambda(\tau) + h_{\lambda+1}Q_{\lambda+1}(\tau) + \cdots + h_{\bar{m}}Q_{\bar{m}}(\tau).$$

Sets of integers $(h_{\lambda+1}, \cdots, h_{\bar{m}})$, satisfying (7.10a) exist if (7.6a) is assumed (as it now is). *Moreover, there is only a finite number of such sets.* The second factor in the second member of (7.10) is monotone in W_ξ in consequence of the statement in connection with (7.8). It will be necessary to consider the other factor. The real part of the function (7.10b) is of the form

$$(7.11) \qquad RQ(\tau) = q(\bar{\tau}) \left| \tau \right|^{l'} + \cdots,$$

where

$$(7.11a) \qquad \begin{aligned} -q(\bar{\tau}) &= \left| q_\lambda \right| \cos(\bar{q}_\lambda + l'\bar{\tau}) \\ &\quad + [-h_{\lambda+1} \left| q_{\lambda+1} \right| \cos(\bar{q}_{\lambda+1} l'\bar{\tau}) - \cdots - h_{\bar{m}}^- \left| q_{\bar{m}}^- \right| \cos(\bar{q}_{\bar{m}}^- + l'\bar{\tau})]. \end{aligned}$$

By (7.4a) and (7.10a)

$$[\cdots] \geqq (h_{\lambda+1} + \cdots + h_{\bar{m}}^-)\epsilon = \nu(\xi)\epsilon.$$

Thus from (7.11a), in view of the definition of $\nu(\xi)$ given in Lemma 6, we get

$$-q(\bar{\tau}) \geqq \nu(\xi)\epsilon - |q_\lambda| \geqq \frac{\epsilon + |q_\lambda|}{\epsilon}\cdot\epsilon - |q_\lambda| = \epsilon \qquad (\tau \text{ in } W_\xi). \tag{7.11b}$$

If (7.11) and (7.11b) are used it can be shown that

$$RQ(\tau) = q(\bar{\tau})\,|\tau|^{\nu}(1 + \gamma(|\tau|, \bar{\tau}))\,|\tau|^{-\beta} \qquad (\beta > 0),$$

where $|\gamma(|\tau|, \bar{\tau})| < \gamma(\tau \text{ in } W_\xi)$. Hence there exists a number

$$r = r(\xi, h; h_{\lambda+1}, \cdots, h_{\bar{m}}) \tag{7.12}$$

such that the factor of the second member of (7.10) is monotone in W_ξ ($|t| \geqq r$). By virtue of the italicized statement subsequent to (7.10b) it is concluded that the lemma holds, with $r' = r'(\xi, h)$ defined as the greatest of all numbers (7.12) ($\lambda = 1, \cdots, \bar{m}-1$; $h_{\lambda+1}, \cdots, h_{\bar{m}}$ such that $h_{\lambda+1} \geqq 0, \cdots, h_m \geqq 0$ and $h_{\lambda+1} + \cdots + h_{\bar{m}} = \nu(\xi)$).

Consider now the functions (6.20a); they are represented by series satisfying the convergence conditions stated in Lemma 5, §6. Write

$$|\rho_i| \leqq \rho^*, \qquad |w_i| \leqq w^* \qquad (i = 1, \cdots, n), \tag{7.13}$$

where

$$\rho^* + w^* \leqq \rho''. \tag{7.13a}$$

We then have $|\rho_i| \leqq \rho''$, $|\rho_i + w_i| \leqq \rho''$ ($i = 1, \cdots, n$). With the conditions of Lemma 5 satisfied one may write

$$\begin{aligned} q_j^*(t, \rho_1 + w_1, \cdots, \rho_n + w_n) - q_j^*(t, \rho_1, \cdots, \rho_n) &= \sum_{i=1}^{n} \frac{\partial q_j^*}{\partial \rho_i} w_i \\ &+ \sum_{\nu_1, \cdots, \nu_n} q_{j;\nu_1, \cdots, \nu_n} w_1^{\nu_1} \cdots w_n^{\nu_n} \\ &(\nu_1 \geqq 0, \cdots, \nu_n \geqq 0;\ \nu_1 + \cdots + \nu_n = 2), \end{aligned} \tag{7.14}$$

where the functions $q_{j;\nu_1, \cdots, \nu_n}$ satisfy

$$|q_{j;\nu_1, \cdots, \nu_n}| \leqq q', \tag{7.15}$$

provided (7.13) and (7.13a) hold. In view of the character of the coefficients $\alpha_{\nu_1, \cdots, \nu_n\, j}(t)$ in the series (6.20a), it is seen that q' can be selected as a constant independent of t. Since in (6.20a) $\nu_1 + \cdots + \nu_n \geqq 2$ it follows that

$$\frac{\partial q_j^*}{\partial \rho_i} = \sum_{k_1 + \cdots + k_n \geqq 1} \alpha_{k_1, \cdots, k_n}^{i,j}(t) \rho_1^{k_1} \cdots \rho_n^{k_n}$$

and, by (7.13),

$$\left|\frac{\partial q_j^*}{\partial \rho_j}\right| < q''\rho^* \qquad (q'' \text{ independent of } t). \tag{7.15a}$$

By (7.15), (7.15a), and (7.13)

$$|\, q_j^*(t, \rho_1 + w_1, \cdots) - q_j^*(t, \rho_1, \cdots)\,| < nq''\rho^* w^* + q(w^*)^2 \tag{7.16}$$

(q independent of t). Thus the following is true.

Let t be in W ($|t| \geqq r'$) (cf. Lemma 5, §6). *Assume that* (7.13), (7.13a) *are satisfied. Then*

$$|\, q_j^*(t, \rho_1 + w_1, \cdots, \rho_n + w_n) - q_j^*(t, \rho_1, \cdots, \rho_n)\,| < (q'\rho^* + qw^*)w^*, \tag{7.17}$$

where q', q are independent of t. These inequalities will continue to hold when the ρ_i and the w_i are functions of t, provided that the inequalities (7.13), (7.13a) continue to hold.

With the aid of the systems

$$t^{-p}\overset{(1)}{\rho}_{j:k}(t) - l_j^*(t, \rho_{1:k}, \cdots, \rho_{n:k}) = q_j^*(t, \rho_{1:k-1}, \cdots, \rho_{n:k-1}) - F_j(t) \qquad (j = 1, \cdots, n;\ k = 0, 1, \cdots), \tag{7.18}$$

we shall seek to determine in succession the functions $\rho_{j:k}$ ($j=1, \cdots, n$; $k=0, 1, \cdots$). In (7.18) let

$$\rho_{j:-1} = 0 \qquad (j = 1, \cdots, n). \tag{7.19}$$

Write

$$\rho_{j:k} - \rho_{j:k-1} = w_{j:k} \qquad (j = 1, \cdots, n;\ k = 0, 1, \cdots). \tag{7.20}$$

Thus

$$\rho_{j:k} = w_{j:0} + w_{j:1} + \cdots + w_{j:k}. \tag{7.20a}$$

The set (7.18) is equivalent to the sequence of systems

$$t^{-p}\overset{(1)}{w}_{j:0}(t) - l_j^*(t, w_{1:0}, \cdots, w_{n:0}) = -F_j(t) \qquad (j = 1, \cdots, n), \tag{7.21}$$

$$t^{-p}\overset{(1)}{w}_{j:k}(t) - l_j^*(t, w_{1:k}, \cdots, w_{n:k}) = g_{j:k} \qquad (j = 1, \cdots, n;\ k = 1, 2, \cdots), \tag{7.21a}$$

where

$$g_{j:k} = q_j^*(t, \rho_{1:k-2} + w_{1:k-1}, \cdots, \rho_{n:k-2} + w_{n:k-1}) - q_j^*(t, \rho_{1:k-2}, \cdots, \rho_{n:k-2}). \tag{7.21b}$$

Under suitable convergence conditions the series

$$\rho_j(t) = w_{j;0} + w_{j;1} + \cdots \qquad (j = 1, \cdots, n) \tag{7.22}$$

will constitute a set of "actual" solutions of the transformed system referred to in Lemma 5, §6.

In view of (6.20b) it follows that, from an asymptotic point of view, the solutions of the linear system, obtained from (6.20) by replacing the second members by zeros, are identical with the solutions of the system (LA_1), §2. Thus the systems (7.21), (7.21a) may be written in the form

$$w_{j;k}(t) = \sum_{\lambda=1}^{n} c_\lambda^{j;k}(t) e^{Q_\lambda(t)} t^{r_\lambda} \gamma^{\lambda,j}(t) \qquad (j = 1, \cdots, n;\ k = 0, 1, \cdots), \tag{7.23}$$

where

$$c_\lambda^{j;k}(t) = \sum_{\lambda_1=1}^{n} \int^t g_{\lambda_1;k} e^{-Q_\lambda(\tau)} t^{-r_\lambda+\omega+p} \bar{\gamma}^{\lambda_1,\lambda}(\tau)\, d\tau. \tag{7.23a}$$

Here

$$g_{j;0} = -F_j(t) \tag{7.23b}$$

and

$$\gamma^{\lambda,j}(t) = [t]^*, \qquad \bar{\gamma}^{\lambda_1,\lambda}(t) = [t]^* \qquad (t \text{ in } W). \tag{7.23c}$$

In view of (7.23c)

$$|\gamma^{\lambda,j}(t)|, \qquad |\bar{\gamma}^{\lambda_1,\lambda}(t)| < |t|^{\epsilon'\gamma} \qquad (\epsilon' > 0;\ t \text{ in } W). \tag{7.23d}$$

Thus from (7.23) and (7.23a) it follows that, if $|g_{j;k}| \leqq g_k$ $(j = 1, \cdots, n)$,

$$|w_{j;k}(t)| < n\gamma^2 \sum_{\lambda=1}^{n} \left| e^{Q_\lambda(t)} t^{r_\lambda+\epsilon'} \right| \left| \int^t g_k \left| e^{-Q_\lambda(\tau)} \tau^{-r_\lambda+\bar{\omega}+2\epsilon'} \right| \right| \tau^{-1-\epsilon'} d\tau \Big| \tag{7.24}$$

for t in W and provided the integrals involved in the second members of (7.24) exist.

Henceforth, unless the contrary is stated, *t and τ will be in W_ξ* (cf. Definition 7) *and we shall suppose that all the previous results (including equations* (7.21), (7.21a)) *have been established with m replaced by $\bar{m}$, where $\bar{m}$ is the integer occurring in* (7.2). Moreover, *we shall take* $N \geqq \nu(\xi)$ (cf. Lemma 6).

By (7.1a), (7.1), and (6.17),

$$|F_j(\tau)| < g_0^*(\tau) = k_0(c')^N |\tau|^{-2\bar{\omega}+\xi} q_{N;\bar{\omega}}(\tau) \qquad (\bar{\omega} = \omega + p + 1). \tag{7.25}$$

In view of (7.25) and (7.23b) it is seen that the function, obtained by disregarding the factor $|\tau^{-1-\epsilon'} d\tau|$ in the integrand displayed in (7.24) $(k = 0)$, is less than

$$[k_0(c')^N |\tau|^{\xi+2\epsilon'}][q'_{N;\bar{\omega}}(\tau) \,|\, e^{-Q_\lambda(\tau)} \tau^{-(r_\lambda+\bar{\omega})} \,|\,] = i_\lambda^*(\tau) \qquad (\tau \text{ in } W_\xi). \tag{7.26}$$

In consequence of (7.1a), (7.1)

$$i_\lambda^*(\tau) = k_0(c')^N q_{N;\,h_0}(\tau)\,\big|\,e^{-Q_\lambda(\tau)}\tau^{-(r_\lambda+h_0)}\,\big|, \tag{7.26a}$$

where

$$h_0 = \bar{\omega} + \alpha_0, \qquad \alpha_0 = (\xi + 2\epsilon')/(N-1). \tag{7.26b}$$

By (7.6)

$$i_\lambda^*(\tau) = k_0(c')^N \sum_{k_1,\cdots,k_m} {}_\lambda\Lambda_{k_1,\cdots,k_m}\,{}_{;h_0}(\tau) \qquad (k_1 + \cdots + k_m = N \geqq \nu(\xi)). \tag{7.26c}$$

From (7.26c), with the aid of Lemma 6, it is inferred that $i_\lambda^*(\tau)$ *is monotone* (cf. Definition 8) *in* W_ξ ($|t| \geqq r'(\xi, h_0)$).

In the sequel, whenever t *is in* W_ξ *and* τ *is the variable of integration, the path will be selected along the ray*

$$\angle\,\tau = \angle\,t \qquad (|\tau| \geqq |t|).$$

The integral displayed in (7.24) (for $k=0$) is less than

$$i_\lambda^*(t)\int_t^\infty |\tau|^{-1-\epsilon'}\,|d\tau| = \frac{1}{\epsilon'}\,i_\lambda^*(t)\,|t|^{-\epsilon'} \tag{7.27}$$

when t is in W_ξ ($|t| \geqq r'(\xi, h_0)$). With the aid of (7.24) (for $k=0$), (7.27), and (7.26a) we obtain

$$|\,w_{j;\,0}(t)\,| < w_0^*(t) = w_0\,|t|^{-h_0} q_{N;\,h_0}(t), \qquad w_0 = \frac{n^2\gamma^2}{\epsilon'}\,k_0(c')^N \tag{7.28}$$

$$(j = 1, \cdots, n;\ t \text{ in } W_\xi\ (|t| \geqq r'(\xi, h_0))).$$

Write

$$h_1 = h_0 + \alpha_1, \qquad \alpha_1 = \frac{1}{N}(-\alpha_0 + 2\epsilon') \tag{7.29}$$

(cf. (7.26b));

$$w_1 = \frac{n^2\gamma^2}{\epsilon'}\,b \qquad (b = q'(1+\rho'') + q) \tag{7.29a}$$

(cf. (7.17));

$$\Gamma(t) = w_0 w_1 q_{N;\,h_1}(t) \tag{7.30}$$

(cf. (7.28), (7.1a), and (7.1)).

One will have

$$w_0^*(t) \leqq \frac{\rho''}{1+\rho''}, \qquad \Gamma(t) \leqq \frac{\rho''}{1+\rho''} \qquad (t \text{ in } W_\xi(|t| \geqq r^*)), \tag{7.31}$$

where r^* is some number $\geqq r'(\xi, h_1)$, $\geqq r'(\xi, h_0)$; (7.31) can also be obtained for t in W_ξ ($|t| \geqq r'(\xi, h_1)$) provided c' is taken sufficiently small.

Since (7.21b) (for $k=1$), (7.19), and (7.28) are valid, and since, by (7.31), $w_0^*(t) < \rho''$, it follows from (7.17) that†

$$|g_{j:1}(t)| < b(w_0^*(t))^2 = g_1^*(t) \qquad (t \text{ in } W_\xi(|t| \geqq r^*)). \tag{7.32}$$

It will now be proved that

$$|w_{j:1}(t)| < \Gamma(t) w_0^*(t) = w_1^*(t) \qquad (t \text{ in } W_\xi(|t| \geqq r^*)). \tag{7.33}$$

In fact, if we disregard the factor $|\tau^{-1-\epsilon'} d\tau|$, the integrand displayed in (7.24) (for $k=1$) is less than

$$b(w_0^*(\tau))^2 \left| e^{-Q_\lambda(\tau)} \tau^{-r_\lambda + \bar{\omega} + 2\epsilon'} \right| = b w_0^2 |\tau|^{-h_0 + \bar{\omega} + 2\epsilon'} q_{N:h_0}(\tau) \cdot \left[q_{N:h_0}(\tau) e^{-Q_\lambda(\tau)} \tau^{-r_\lambda - h_0} \right]. \tag{7.34}$$

By (7.26b) and (7.29), $-h_0 + \bar{\omega} + 2\epsilon' = N\alpha_1$. Hence, for $h_1 = h_0 + \alpha$, one has

$$|\tau|^{-h_0 + \bar{\omega} + 2\epsilon'} q_{N:h_0}(\tau) = q_{N:h_1}(\tau).$$

Thus the function of (7.34) is equal to

$$b w_0^2 q_{N:h_1}(\tau) \sum_{k_1 + \cdots + k_m^- = N} {}_\lambda\Lambda_{k_1, \cdots, k_m^-; h_0}(\tau). \tag{7.34a}$$

In view of the inequalities satisfied by r^* and in consequence of Lemma 6 it is observed that the functions (7.34a) $\lambda = 1, \cdots, n$ are monotone in W_ξ ($|t| \geqq r^*$). Hence the integral displayed in (7.24) (for $k=1$) is less than the product of the function of (7.34a) (with τ replaced by t) by

$$\int^t |\tau^{-1-\epsilon'} d\tau|; \tag{7.34b}$$

that is, by $(1/\epsilon')\,|t|^{-\epsilon'}$. Making use of this fact, we infer from (7.24) (for $k=1$) that

$$|w_{j:1}(t)| < \frac{n\gamma^2}{\epsilon'} \sum_{\lambda=1}^{n} b w_0^2 q_{N:h_1}(t) \sum_{k_1 + \cdots + k_m^- = N} {}_\lambda\Lambda_{k_1, \cdots, k_m^-; h_0}(t) \cdot |e^{Q_\lambda(t)} t^{r_\lambda}| = w_1^*(t).$$

By (7.6) and (7.29a)

$$w_1^*(t) = (w_0 w_1) q_{N:h_1}(t) (w_0 |t|^{-h_0}) \sum_{k_1 + \cdots + k_m^- = N} q_{k_1, \cdots, k_m^-; h_0}(t).$$

With the aid of (7.30), (7.1a), and (7.28), the truth of (7.33) is deduced from this relation.

† In the second member of (7.32) b could be replaced by $q(<b)$; this, however, is avoided in order to simplify subsequent developments.

By (7.33) and (7.31)

$$w_1^*(t) \leqq \left(\frac{\rho''}{1+\rho''}\right)^2, \quad w_1^*(t) \leqq w_0^*(t), \quad w_0^*(t) + w_1^*(t) < \rho'' \tag{7.35}$$

for t in W_ξ ($|t| \geqq r^*$).

In consequence of (7.21b) (for $k=2$), (7.35), and (7.17)

$$\begin{aligned} |g_{j:2}(t)| &= |q_j^*(t, w_{1:0}(t) + w_{1:1}(t), \cdots) - q_j^*(t, w_{1:0}(t), \cdots)| \\ &< (q' w_0^*(t) + q w_1^*(t)) w_1^*(t) \leqq b w_0^*(t) w_1^*(t) = g_2^*(t) \end{aligned} \tag{7.36}$$

for t in W_ξ ($|t| \geqq r^*$).

Assume now that for t in W_ξ ($|t| \geqq r^*$) and for $i = 1, 2, \cdots, k$ ($\geqq 2$) we have

$$|g_{j:i}(t)| < g_i^*(t) = b w_0^*(t) w_{i-1}^*(t), \tag{7.37}$$

$$|w_{j:i-1}(t)| < w_{i-1}^*(t) = \Gamma^{i-1}(t) w_0^*(t) \qquad (j = 1, \cdots, n). \tag{7.37a}$$

In (7.32), (7.36), and (7.33) this has been established for $k=2$.

By (7.37) and (7.37a) (for $i=k$)

$$g_k^*(t) = b\Gamma^{k-1}(t)(w_0^*(t))^2. \tag{7.37b}$$

Thus, by (7.37) (for $i=k$)

$$\begin{aligned} |g_{j:k}(\tau)|\,|e^{-Q_\lambda(\tau)\tau^{-r_\lambda+\bar{\omega}+2\epsilon'}}| &< b\Gamma^{k-1}(\tau)(w_0^*(\tau))^2\,|e^{-Q_\lambda(\tau)\tau^{-r_\lambda+\bar{\omega}+2\epsilon'}}| \\ &= b(w_0 w_1)^{k-1}\, {}^{k-1}q_{N:h_1}(\tau) w_0^2\, {}^{2}q_{N:h_0}(\tau)\,|e^{-Q_\lambda(\tau)\tau^{-(r_\lambda+h_0)}}|\,|\tau^{\bar{\omega}+2\epsilon'-h_0}| \\ &= b w_0^2 (w_0 w_1)^{k-1}\, {}^{k-1}q_{N:h_1}(\tau) \sum_{k_1+\cdots+k_m^-=N} {}_\lambda\Lambda_{k_1,\cdots,k_m^-:h_0}(\tau)\,(|\tau|^{\bar{\omega}'+2\epsilon''-h_0}\, q_{N:h_0}(\tau)). \end{aligned}$$

Making use of the formula subsequent to (7.34), one obtains

$$\begin{aligned} &|g_{j:k}(\tau)|\,|e^{-Q_\lambda(\tau)\tau^{-r_\lambda+\bar{\omega}+2\epsilon'}}| \\ &\qquad < b w_0^{k+1} w_1^{k-1}\, {}^{k}q_{N:h_1}(\tau) \sum_{k_1+\cdots+k_m^-=N} {}_\lambda\Lambda_{k_1,\cdots,k_m^-:h_0}(\tau). \end{aligned} \tag{7.38}$$

The functions ${}^{k}q_{N:h_1}(\tau)$, ${}_\lambda\Lambda_{k_1,\cdots,k_m^-:h_0}(\tau)$ are monotone† in W_ξ ($|t| \geqq r^*$). Hence the integral displayed in (7.24) is less than the product of the second member of (7.38) (with τ replaced by t) by the function of (7.34b), that is, by $(1/\epsilon')|t|^{-\epsilon'}$. In view of this fact and (7.6) it is deduced from (7.24) that

$$|w_{j:k}(t)| < \left(\frac{n^2\gamma^2}{\epsilon'} b\right) w_0^{k+1} w_1^{k-1}\, {}^{k}q_{N:h_1}(t)\,|t|^{-h_0} \sum_{k_1+\cdots+k_m^-=N} |q_{k_1,\cdots,k^-:h_0}(t)|. \tag{7.39}$$

By (7.29a) and (7.1a)

$$|w_{j:k}(t)| < w_k^*(t) = (w_0 w_1 q_{N:h_1}(t))^k (w_0 |t|^{-h_0} q_{N:h_0}(t)) \tag{7.40}$$

† Because $r^* \geqq r'(\xi, h_0)$, $r^* \geqq r'(\xi, h_1)$.

for t in W_ξ ($|t| \geqq r^*$) and for $j = 1, \cdots, n$. Making use of (7.30) and (7.28) one obtains

$$w_k^*(t) = \Gamma^k(t) w_0^*(t). \tag{7.40a}$$

Furthermore, by (7.31), (7.37a) (for $i = 1, \cdots, k$), and (7.40a)

$$w_i^*(t) \leqq \left(\frac{\rho''}{1+\rho''}\right)^i w_0^*(t) \qquad (i = 0, 1, \cdots, k). \tag{7.41}$$

Hence

$$\rho_{k-1}^*(t) = w_0^*(t) + \cdots + w_{k-1}^*(t) < (1+\rho'') w_0^*(t). \tag{7.42}$$

Also†

$$w_k^*(t) \leqq \left(\frac{\rho''}{1+\rho''}\right)^k w_0^*(t) < w_0^*(t) \qquad (t \text{ in } W_\xi(|t| \geqq r^*)) \tag{7.42a}$$

and, by (7.31),

$$\rho_{k-1}^*(t) + w_k^*(t) \leqq w_0^*(t) \sum_{i=0}^{k} \left(\frac{\rho''}{1+\rho''}\right)^i < (1+\rho'') w_0^*(t) \leqq \rho''. \tag{7.43}$$

By (7.21b) (cf. (7.21a)), in consequence of the statement in connection with (7.17),

$$\begin{aligned} |\, g_{j;\,k+1}(t)\,| &= |\, q_j^*(t, \rho_{1:\,k-1}(t) + w_{1:\,k}(t), \cdots) - q_j^*(t, \rho_{1:\,k-1}(t), \cdots, \rho_{n:\,k-1}(t))\,| \\ &< (q'\rho_{k-1}^*(t) + q w_k^*(t)) w_k^*(t). \end{aligned}$$

Thus, by virtue of (7.43) and (7.42a), we have in W_ξ ($|t| \geqq r^*$)

$$|\, g_{j;\,k+1}(t)\,| < [q!(1+\rho'') w_0^*(t) + q w_0^*(t)] w_k^*(t) = g_{k+1}^*(t). \tag{7.44}$$

Whence, in view of (7.29a),

$$g_{k+1}^*(t) = b w_0^*(t) w_k^*(t). \tag{7.44a}$$

In consequence of (7.40), (7.40a), (7.44), and (7.44a) it is observed that the relations (7.37), (7.37a) will necessarily hold when k is replaced by $k+1$. *It has therefore been proved that these relations hold for* $k = 2, 3, \cdots$. Thus, in particular, in view of (7.31) one has

$$|\, w_{j;\,k}(t)\,| < w_k^*(t) \leqq \left(\frac{\rho''}{1+\rho''}\right)^k w_0^*(t) \tag{7.45}$$

$$(j = 1, \cdots, n;\ k = 0, 1, \cdots;\ t \text{ in } W_\xi(|t| \geqq r^*)).$$

† (7.42), (7.42a) may be replaced by more precise inequalities and the developments could be given with a smaller b.

Accordingly it is seen that *the series* (7.22) *are absolutely and uniformly convergent, and*

$$|\rho_j(t)| < (1+\rho'')w_0^*(t)(\leqq \rho'') \qquad (j=1,\cdots,n), \tag{7.46}$$

when t *is in* W_ξ ($|t| \geqq r^*$).

From (7.21a), in view of (7.37), (7.37b), (7.40), (7.40a), we obtain

$$|t^{-p}w_{j:k}^{(1)}(t)| = |g_{j:k}(t) + l_j^*(t, w_{1:k}, \cdots, w_{n:k})| \leqq |g_{j:k}(t)|$$

$$+ K\sum_{i=1}^{n}|w_{j:k}(t)| < [b\Gamma^{k-1}(t)w_0^*(t) + Kn\Gamma^k(t)]w_0^*(t).$$

Thus, by (7.31),

$$|t^{-p}w_{j:k}^{(1)}(t)| < b'\left(\frac{\rho''}{1+\rho''}\right)^k w_0^*(t) \qquad (b'=b+Kn).\dagger \tag{7.47}$$

Hence the derived series

$$\sum_{k=0}^{\infty} w_{j:k}^{(1)}(t) \qquad (j=1,\cdots,n) \tag{7.48}$$

are absolutely and uniformly convergent in W_ξ. These series represent the derivatives of the functions $\rho_j(t)$ (involved in (7.46)), respectively. By (7.47)

$$|\rho_j^{(1)}(t)| < b'(1+\rho'')w_0^*(t)|t|^p \qquad (t \text{ in } W_\xi(|t| \geqq r^*)). \tag{7.49}$$

We are now ready to formulate the following existence theorem.

THE FIRST EXISTENCE THEOREM. *Let* R *be a region referred to in* §2 *in connection with* (i), (ii), (7.9a), (7.9b). *With* $Q_i(\tau)$ $(i=1,\cdots,n)$ *denoting the polynomials involved in the formal matrix solution* (2.5) *of the linear system* (LA_1), §2, *as a matter of notation we have in* R

$$RQ_1(t) \leqq RQ_2(t) \leqq \cdots \leqq RQ_n(t); \quad e^{Q_i(t)} \sim 0 \qquad (i=1,\cdots,m).$$

Moreover, the $Q_i(\tau)$ *will be supposed to be so selected that, on writing* $Q_i(\tau)=q_i\tau^{l_i/k_i}+\cdots$ $(i=1,\cdots,m;\ q_i\neq 0)$, *we have*

$$l' = \frac{l_1}{k_1} = \frac{l_2}{k_2} = \cdots = \frac{l_{\bar{m}}}{k_{\bar{m}}} > \frac{l_{\bar{m}+1}}{k_{\bar{m}+1}} \geqq \cdots \geqq \frac{l_m}{k_m} \qquad (1 \leqq \bar{m} \leqq m).$$

Let W *be a subregion of* R *of the form* $R(\alpha_1, \alpha_2)$ (cf. Definition 3, §2) *where* $0<\alpha_2-\alpha_1<\pi/l'$. *With* $\xi>0$, *however small,* W_ξ *will denote a subregion of* W

† Here K is a number equal to or greater than the absolute values of the coefficients of the linear differential operators l_j^* $(j=1,\cdots,n)$; these coefficients, being asymptotically identical with the corresponding ones in the l_j are therefore bounded for t in W_ξ.

of the form $W_\xi = R(\alpha_1+\xi, \alpha_2-\xi)$ (cf. Definition 7). *Let* N *be an integer* $\geqq \nu(\xi)$, *where* $\nu(\xi)$ *is the number defined in Lemma* 6.

The non-linear system (A), §1 *will possess a solution* $y_j(t)$ $(j=1, \cdots, n)$, *whose elements* $y_j(t)$ *are analytic for* $t \neq \infty$ *in* W_ξ $(|t| \geqq r^*)$, *while at* $t=\infty$ *they will generally be singular. Moreover,*

$$y_j(t) \sim s_j(t) \qquad (j = 1, \cdots, n;\ t \text{ in } W_\xi(|t| \geqq r^*)). \tag{7.50}$$

Here the $s_j(t)$ *are the formal series given by* (7.23), (7.23a), (7.24) *of Lemma* 4, §5 *under the supposition that in that lemma* m *has been replaced*† *by* $\bar{m}$. *The asymptotic relations* (7.50) *have the following significance. The* $y_j(t)$ *are functions of the form*

$$y_j(t) = \sum_{\nu=1}^{N-1} \sum_{k_1+\cdots+k_{\bar{m}}=\nu} c_1^{k_1} c_2^{k_2} \cdots c_{\bar{m}}^{k_{\bar{m}}} {}_\nu\eta_{k_1,\cdots,k_{\bar{m}};j}(t) + \rho_j(t), \tag{7.50a}$$

where $c_1, \cdots, c_{\bar{m}}$ *are arbitrary constants* $(|c_i| \leqq c';\ i=1, \cdots, \bar{m})$ *and*

$${}_\nu\eta_{k_1,\cdots,k_{\bar{m}};j}(t) = \exp\,(k_1Q_1(t) + \cdots + k_{\bar{m}}Q_{\bar{m}}(t))\,t^{k_1r_1+\cdots+k_{\bar{m}}r_{\bar{m}}}t^{(\nu-1)w_j'}[t]_{Q(\nu)} \qquad (t \text{ in } W_\xi(|t| \geqq r^*)) \tag{7.50b}$$

(cf. Definition 4, §2, and (5.21)), *where* $w_j' \leqq \omega+p+1$. *Furthermore, the functions* $\rho_j(t) = \rho_j(c_1, \cdots, c_{\bar{m}}, t)$ *are analytic for* $t \neq \infty$ *in* $W_\xi(|t| \geqq r^*)$ *and satisfy in this region the inequalities*

$$\begin{aligned} &|\rho_j(t)| < h'(c')^N |t|^{(N-1)h_0} \\ &\quad \cdot \sum_{k_1+\cdots+k_{\bar{m}}=N} |\exp\,[k_1Q_1(t) + \cdots + k_{\bar{m}}Q_{\bar{m}}(t)]\,t^{k_0r_0+\cdots+k_mr_m}| = {}_N\rho^*(t), \\ &|\rho_j^{(1)}(t)| < h''|t|^p {}_N\rho^*(t) \qquad (j = 1, \cdots, n;\ h_0 = \omega+p+1+\alpha_0) \end{aligned} \tag{7.50c}$$

(cf. (7.26b)), *provided* $|c_i| \leqq c'$ $(i=1, \cdots, \bar{m})$. *The above holds under the supposition that* c' *has been taken sufficiently small and that* r^* *is sufficiently great.*

Note. The term "asymptotic relations" applied to (7.50) is justifiable because in the neighborhood of $t=\infty$ the "remainder" term in (7.50a), that is to say, $\rho_j(t)$, is essentially of the same order of magnitude as

$$\sum_{k_1+\cdots+k_{\bar{m}}=N} c_1^{k_1} \cdots c_{\bar{m}}^{k_{\bar{m}}} {}_N\eta_{k_1,\cdots,k_{\bar{m}};j}(t).$$

In fact, for t in $W_\xi(|t| \geqq r^*)$, the absolute value of the above function is less than

$$h''|t|^{-\xi-2c'} {}_N\rho^*(t).$$

We consider it extremely likely that facts of the type of those stated in the

† This obviously has no effect on l'.

above theorem will hold in the region W itself, even if in the formal series (involved in Lemma 4, §5) *m is not replaced by* $\bar{m}$, *this being true at least when W is subject to the condition that its opening* (cf. Definition 3, §2) *is less than* $\pi/(2l')$. An attempt to prove this conjecture leads to analytic difficulties even greater than those encountered so far.

8. **Formal solutions for systems** (B). The First Existence Theorem (§7) is of course applicable to systems (B), §1. However, in view of the specialized (as compared with systems (A), §1) character of these systems, results of a more precise character, valid in more extensive regions, can be obtained by treating (B) directly. We have

$$\text{(B)} \qquad y_j^{(1)}(t) = a_j(y_1, \cdots, y_n) \qquad (j = 1, \cdots, n),$$

$$\text{(8.1)} \qquad a_j(y_1, \cdots, y_n) = l_j(y_1, \cdots, y_n) + q_j(y_1, \cdots, y_n),$$

$$\text{(8.1a)} \qquad l_j(y_1, \cdots, y_n) = l_{1,j}y_1 + \cdots + l_{n,j}y_n,$$

$$\text{(8.1b)} \qquad q_j(y_1, \cdots, y_n) = \sum_j a_{i_1,\cdots,i_n} y_1^{i_1} \cdots y_n^{i_n}$$

$$(i_1, i_2, \cdots, i_n \geqq 0;\ i_1 + \cdots + i_n \geqq 2;\ j = 1, \cdots, n),$$

the series in the second member of (8.1b) being convergent for

$$\text{(8.2)} \qquad |y_1|, \cdots, |y_n| \leqq \rho.\dagger$$

Associated with (B) there is a linear system

$$\text{(LB)} \qquad y_j^{(1)}(t) = l_j(y_1, \cdots, y_n) \qquad (j = 1, \cdots, n)$$

which is equivalent to the matrix equation

$$\text{(LB}_1\text{)} \qquad Y^{(1)}(t) = Y(t)B, \qquad Y(t) = (y_{i,j}(t)),$$

where

$$B = (l_{i,j}) \qquad (i, j = 1, \cdots, n).$$

Here elements of a row in $Y(t)$ will constitute a solution of (LB).

The characteristic equation‡

$$\text{(8.3)} \qquad E(\alpha) = |(l_{i,j} - \delta_{i,j}\alpha)| = 0$$

has roots

$$\text{(8.3a)} \qquad \alpha_1, \alpha_2, \cdots, \alpha_\lambda \qquad (1 \leqq \lambda \leqq n;\ \alpha_k \neq \alpha_\nu \text{ for } k \neq \nu)$$

of multiplicities

$$m_1, m_2, \cdots, m_\lambda \qquad (m_1 + m_2 + \cdots + m_\lambda = n),$$

† The same remark is made as in the footnote in connection with (1.2).

‡ Here $(\delta_{i,j})$ is the identity matrix; i.e., $\delta_{i,j}=0$, for $i \neq j$, and $\delta_{i,i}=1$.

respectively. The equation (LB_1) will possess a matrix solution

$$Y(t) = (e^{q_i t} p_{i,j}(t)) \qquad (i, j = 1, \cdots, n), \tag{8.4}$$

where

$$\begin{aligned} &q_i = \alpha_1 \ (i = 1, \cdots, m_1); \ q_i = \alpha_2 \ (i = m_1 + 1, \cdots, m_1 + m_2); \ \cdots, \\ &q_i = \alpha_\lambda \qquad (i = m_1 + \cdots + m_{\lambda-1} + 1, \cdots, n), \end{aligned} \tag{8.4a}$$

and the $p_{i,j}(t)$ are polynomials in t of degree at most

$$\begin{aligned} &m_1 - 1 \quad (\text{for } i = 1, \cdots, m_1), \\ &m_2 - 1 \quad (\text{for } i = m_1 + 1, \cdots, m_1 + m_2), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &m_{\lambda-1} - 1 \ (\text{for } i = m_1 + \cdots + m_{\lambda-1} + 1, \cdots, n). \end{aligned} \tag{8.4b}$$

It will be assumed that not all of the roots of the characteristic equation (8.3) *are zero.* Thus in (8.4) at least one of the q_i will be distinct from zero.

In the sequel there will be occasion to consider the matrix

$$Y^{-1}(t) = \overline{Y}(t). \tag{8.5}$$

Since it satisfies the equation

$$\overline{Y}^{(1)}(t) = - B\overline{Y}(t),$$

whose characteristic equation is $E(-\alpha) = 0$ (cf. (8.3a)), if we take account of the interchanged role of the rows and columns we conclude that

$$Y^{-1}(t) = (\bar{y}_{i,j}(t)) = (\bar{p}_{i,j}(t)e^{-q_j t}) \tag{8.5a}$$

where the $\bar{p}_{i,j}(t)$ are polynomials in t of degrees at most as stated in (8.4b) (with i replaced by j). Moreover,

$$(\bar{p}_{i,j}(t))(p_{i,j}(t)) = (\delta_{i,j}) = I. \tag{8.5b}$$

DEFINITION 9. *$S(\alpha_1, \alpha_2; r)$ will denote the sector*

$$\alpha_1 \leqq \angle\, t \leqq \alpha_2 \qquad (|t| \geqq r).$$

Let the B_i denote the rays along which $R(q_i t) = 0$ (whenever the constant q_i, under consideration, $\neq 0$). Let

$$S_1, S_2, \cdots \tag{8.6}$$

denote the sectors separated by the B_i rays and not containing any of these rays in their interiors.

In the above set of sectors (8.6) there is certain to exist at least one sector S such that the following holds.

In the set of numbers q_i $(i=1, \cdots, n)$ *there are some, say*

$$q_{i_1}, q_{i_2}, \cdots, q_{i_m} \qquad (1 \leqq m \leqq n), \tag{8.7}$$

all distinct from zero, corresponding to which

$$R(q_{i_\tau} t) \leqq 0 \qquad (\tau = 1, \cdots, m;\ t \text{ in } S). \tag{8.7a}$$

As a matter of notation the functions $q_i t$ *involved in* (8.7a) *will be designated as*

$$q_1 t, q_2 t, \cdots, q_m t. \tag{8.7b}$$

Henceforth, unless the contrary is stated or implied, we consider a particular sector S *of the above description. It will be designated as*

$$S(\alpha_1, \alpha_2; r) \qquad (\alpha_1 < \alpha_2) \tag{8.8}$$

(cf. Definition 9).

Let $c_1, \cdots, c_m$ denote arbitrary constants, and consider a solution of (LB) of the form

$${}_1y_j(t) = \sum_{\lambda=1}^{m} c_\lambda e^{q_\lambda t} p_{\lambda,j}(t) \qquad (j = 1, \cdots, n). \tag{8.9}$$

We shall seek to satisfy the non-linear system (B) with the aid of the formal series

$$y_j(t) = {}_1y_j(t) + {}_2y_j(t) + \cdots + {}_\nu y_j(t) + \cdots \qquad (j = 1, \cdots, n), \tag{8.10}$$

where

$${}_\nu y_j(t) = \sum_{k_1, \cdots, k_m} c_1^{k_1} \cdots c_m^{k_m}\, {}_\nu\eta_{k_1, \cdots, k_m; j}(t) \qquad (k_1, \cdots, k_m \geqq 0;\ k_1 + \cdots + k_m = \nu). \tag{8.10a}$$

For $\nu=1$ the coefficients in (8.10a) are already known. In fact, by (8.9),

$$e^{q_\lambda t} p_{\lambda,j}(t) = {}_1\eta_{k_1, \cdots, k_m; j}(t) \qquad (\lambda = 1, \cdots, m;\ j = 1, \cdots, n), \tag{8.10b}$$

where $k_i=0$ for $i \neq \lambda$ and $k_\lambda = 1$.

On substituting (8.10), (8.10a) in (B) it is found that the coefficients involved in (8.10a) satisfy the linear non-homogeneous systems (S_H), §2 $(H=2, 3, \cdots)$. Furthermore, in place of (2.30) and (2.30a) we now deduce in consequence of (8.5a) and (8.4)

$${}_H\eta_{k_1, \cdots, k_m; j}(t) = \sum_{\lambda=1}^{n} c_\lambda(t) e^{q_\lambda t} p_{\lambda,j}(t), \tag{8.11}$$

where

$$c_\lambda(t) = \sum_{\lambda_1=1}^{n} \int^t {}_H\Gamma_{\lambda_1}^{h_1,\cdots,h_m}(\tau)\bar{p}_{\lambda',\lambda}(\tau)e^{-q_\lambda\tau}d\tau. \tag{8.11a}$$

Here the

$${}_H\Gamma_{\lambda_1}^{h_1,\cdots,h_m}(\tau)$$

are given by (2.26).

LEMMA 7. *Consider the problem* (B) *at the beginning of this section. Let* $Y(t)$, *as given by* (8.4), *be a matrix solution of* (LB_1). *A mechanism for construction of formal solutions* (8.10) (cf. (8.10a)) *of the system* (B) *is given by* (8.11), (8.11a) (cf. (8.5a) and (2.26)).

It will be convenient to introduce the following definition.

DEFINITION 10. *Generically* $[t, \nu]$ *is to denote a polynomial in* t *of at most the* ν*th degree.*

By (8.10b)

$${}_1\eta_{k_1^r,\cdots,k_m^r;\alpha}(\tau) = \exp\,[(k_1^r q_1 + \cdots + k_m^r q_m)\tau][\tau, m'] \tag{8.12}$$

(m' *the greater of the numbers* (8.4b)). In view of (8.12) the left member of (3.3a) becomes

$$\exp\,[(\delta_1 q_1 + \cdots + \delta_m q_m)\tau][\tau, i_\alpha m']. \tag{8.13}$$

Applying the summation displayed subsequent to (3.3a) to (8.13), we get

$$F_\alpha' = \text{generic form of (8.13)}. \tag{8.13a}$$

In view of (2.24), (2.19b) we have

$$\prod_{\alpha=1}^{n} F_\alpha' = \exp\,[(h_1 q_1 + \cdots + h_m q_m)\tau][\tau, sm'] \qquad (s = i_1 + \cdots + i_n). \tag{8.13b}$$

By (3.4)

$$\begin{aligned} {}_2\Gamma_j^{h_1,\cdots,h_m}(\tau) &= \sum_{i_1+\cdots+i_n=2} \sum_{\gamma_1+\cdots+\gamma_n=2} \sum^{(3)}{}_{,\delta_1,\cdots,\delta_m} {}_j a_{i_1,\cdots,i_n} \prod_{\alpha=1}^{n} F_\alpha' \\ &= \exp\,[(h_1 q_1 + \cdots + h_m q_m)\tau][\tau, 2m']. \end{aligned} \tag{8.14}$$

Since

$$\bar{p}_{i,j}(\tau) = [\tau, m'] \qquad (i, j = 1, \cdots, n)$$

in consequence of (8.14), it is observed that the integrand displayed in (8.11a) is of the form

$$\exp\,[(h_1 q_1 + \cdots + h_m q_m - q_\lambda)\tau][\tau, 3m']. \tag{8.14a}$$

Hence the integral displayed in (8.11a) is expressible as

(8.14b) $$\exp\,[(h_1q_1+\cdots+h_mq_m-q_\lambda)t][t,\,3m'+1],$$

where $3m'+1$ may be replaced by $3m'$ whenever $h_1q_1+\cdots+h_mq_m-q_\lambda\neq 0$. One further finds that $c_\lambda(t)$ has the generic form of (8.14b). Hence, by (8.11) (for $H=2$),

(8.15) $$_2\eta_{h_1,\cdots,h_m;j}(t)=\exp\,[(h_1q_1+\cdots+h_mq_m)t][t,\,4m'+1].$$

In view of (8.12) and (8.15)

(8.16) $$_\nu\eta_{k_1,\cdots,k_m;\alpha}(t)=\exp\,[(k_1q_1+\cdots+k_mq_m)\tau][\tau,\,\rho_\nu]$$
$$(\rho_\nu=\nu(3m'+1)-(2m'+1);\ \nu=1,2).$$

On writing $k_i=k_i^r$ $(i=1,\cdots,m)$, $\nu=\nu_r$, substitute (8.16) in (2.26) (for $H=3$). The product (5.8) will now be of the form

(8.17) $$\exp\,[(\delta_1q_1+\cdots+\delta_mq_m)\tau][\tau,\,\rho_\alpha'],$$

where

(8.17a) $$\rho_\alpha'=\sum_{r=1}^{i_\alpha}\rho_\nu=\gamma_\alpha(3m'+1)-i_\alpha(2m'+1)$$

(cf. (5.8a)). Applying to (8.17) the summation symbol subsequent to (5.8a) one obtains

(8.18) $$F_\alpha'=\text{generic form of (8.17)}.$$

In view of (5.8c)

(8.18a) $$\prod_{\alpha=1}^{n}F_\alpha'=\exp\,[(h_1q_1+\cdots+h_mq_m)\tau][\tau,\,\rho_s''],$$

where, by (8.17a),

(8.18b) $$\rho_s''=\sum\rho_\alpha'=3(3m'+1)-s(2m'+1)\qquad(s=i_1+\cdots+i_n).$$

Here s may assume only the values 2 and 3. Hence in (8.18a) we may replace $[\tau,\,\rho_s'']$ by $[\tau,\,5m'+1]$. Thus the functions involved in the left member of (5.10) will possess the generic form of (8.18a) (where the indicated replacement has been made). If now the first summation symbol of the last member in (2.26) is extended over the functions (5.10), it is seen that the generic form will remain unaltered. Thus

(8.19) $$_3\Gamma_j^{h_1,\cdots,h_m}=\exp\,[(h_1q_1+\cdots+h_mq_m)\tau][\tau,\,5m'+1].$$

Assume that, with $\rho_q=q(3m'+1)-(2m'+1)$,

$$(8.20)\qquad {}_{q+1}\Gamma_j^{h_1,\cdots,h_m} = \exp\,[(h_1q_1+\cdots+h_mq_m)\tau][\tau,\rho_q+m'] \qquad (h_1,\cdots,h_m\geqq 0;\ h_1+\cdots+h_m=q+1;\ j=1,\cdots,n),$$

and

$$(8.20a)\qquad {}_q\eta_{k_1,\cdots,k_m;\alpha}(\tau) = \exp\,[(k_1q_1+\cdots+k_mq_m)\tau][\tau,\rho_q] \qquad (k_1,\cdots,k_m\geqq 0;\ k_1+\cdots+k_m=q;\ \alpha=1,\cdots,n)$$

for $q=1, 2, \cdots, \text{H}-1$ ($\text{H}\geqq 3$).

Substitute (8.20) (for $q=\text{H}-1$) in (8.11a). The integrand involved in (8.11a) will be of the form

$$\exp\,[(h_1q_1+\cdots+h_mq_m-q_\lambda)\tau][\tau,\rho_{H-1}+2m'].$$

Thus the integral displayed in (8.11a) is

$$(8.21)\qquad \exp\,[(h_1q_1+\cdots+h_mq_m-q_\lambda)t][t,\rho_{H-1}+2m'+1].$$

Here $2m'+1$ may be replaced by $2m'$ whenever $h_1q_1+\cdots+h_mq_m-q_\lambda\neq 0$. Whence $c_\lambda(t)$, as given by (8.11a), will be of the form (8.21). Finally, from (8.11) one obtains (8.20a) with q replaced by H. By making use of the relations (8.20a) (for $q=1,\cdots,\text{H}$) we are going to prove that (8.20) holds for $q=\text{H}$. For this purpose use will be made of (2.26) (with H replaced by H+1). In view of the equalities of (2.16b) it follows that

$$(8.22)\qquad \prod_{r=1}^{i_\alpha}\cdots = \exp\,[(\delta_1q_1+\cdots+\delta_mq_m)\tau][\tau,\rho_\alpha'],$$

where, since $\nu_1+\cdots+\nu_{i_\alpha}=\gamma_\alpha$ (cf. (2.26)), ρ_n' is given by (8.17b). Next it is inferred that

$$(8.23)\qquad \sum_{\nu_1,\cdots,\nu_{i_\alpha}}\ \sum_{k_1^1,\cdots,k_m^{i_\alpha}}^{(2)}\ \prod_{r=1}^{i_\alpha}\cdots = \text{generic form of (8.22)}.$$

Denoting the first member above by F_α', in view of (2.19b), it is deduced that

$$(8.23a)\qquad \prod_{\alpha=1}^{n}F_\alpha' = \exp\,[(h_1q_1+\cdots+h_mq_m)\tau][\tau,\rho_s''] \quad (s=i_1+\cdots+i_n),$$

where, by (8.17a)

$$\rho_s'' = \sum\rho_\alpha' = (3m'+1)\sum\gamma_\alpha-(2m'+1)\sum i_\alpha.$$

Now, since H in (2.26) has been replaced by H+1, in consequence of (2.24) one has $\gamma_1+\cdots+\gamma_n=\text{H}+1$. Thus $\rho_s''=(3m'+1)(\text{H}+1)-(2m'+1)s$, where $s=2, 3, \cdots, \text{H}+1$. Whence $\rho_s''\leqq\rho_2''=\rho_H+m'$ and, accordingly,

$$ {}_{j}a_{i_1,\cdots,i_n}\prod_{\alpha=1}^{n}F_{\alpha}' = \exp\left[(h_1q_1+\cdots+h_mq_m)\tau\right][\tau,\rho_H+m']. $$

Extending over the left member of the above relation the summation symbol with the superscript (4), one finally obtains

$$ {}_{H+1}\Gamma_j^{h_1,\cdots,h_m} = \exp\left[(h_1q_1+\cdots+h_mq_m)\tau\right][\tau,\rho_H+m'] $$
$$ (h_1,\cdots,h_m\geqq 0;\ h_1+\cdots+h_m=H+1;\ j=1,\cdots,n). $$

Thus it is seen that the relations (8.20), (8.20a) hold for $q=1, 2, \cdots$. The following result has been established.

LEMMA 8. *Consider the non-linear system* (B), *the associated linear system* (LB) (*matrix system* (LB_1)) *and the matrix solution* (8.4) *of* (LB_1) (cf. (8.3), the statement regarding the zeros of $E(\alpha)$, (8.4a), (8.4b)). *With* (8.6) *denoting the sectors in each of which the* $R(q_it)$ $(i=1,\cdots,n)$ *do not change sign, let* S *be a particular sector of the set* (8.6) *such that there are some* q_i, *say* $q_{i_1}, q_{i_2},\cdots, q_{i_m}$ $(1\leqq m\leqq n)$, *all distinct from zero, for which* $R(q_{i_\tau}t)\leqq 0$ $(\tau=1,\cdots,m;\ t\ in\ S)$. *As a matter of notation designate the* $q_{i_\tau}t$ *involved in these inequalities as* $q_1t, q_2t,\cdots, q_mt$. *The system* (B) *possesses a formal solution*

$$ s_j(t) = {}_1y_j(t)+{}_2y_j(t)+\cdots \qquad (j=1,\cdots,n) \tag{8.24} $$

where

$$ {}_{\nu}y_j(t) = \sum_{k_1,\cdots,k_m} c_1^{k_1}\cdots c_m^{k_m}\exp\left[(k_1q_1+\cdots+k_mq_m)t\right][t,\rho_\nu] $$
$$ (k_1\geqq 0,\cdots,k_m\geqq 0;\ k_1+\cdots+k_m=\nu;\ \nu=1,2,\cdots). \tag{8.24a} $$

Here $c_1,\cdots,c_m$ *are arbitrary constants,* $\rho_\nu=\nu(3m'+1)-(2m'+1)$ (m' *the greatest of the numbers* (8.4b) *and* $[t,\rho_\nu]$ *represents generically a polynomial in* t *of at most the* ρ_ν*th degree.*

Note. For $\nu\geqq\nu'$ (ν' sufficiently great) the ρ_ν can be taken smaller compared to the ρ_ν given in the lemma. In fact, one may take $\rho_\nu=\nu(3m')+a'(\nu\geqq\nu')$. This is because $h_1q_1+\cdots+h_mq_m-q_\lambda$ $(h_1\geqq 0,\cdots,h_m\geqq 0;\ h_1+\cdots+h_m=\nu)$ will certainly be distinct from zero (for $\lambda=1,\cdots,n$) whenever ν is sufficiently great.

9. **The Second Existence Theorem.** In general, the series (8.24) will not converge in S or in any given regular subregion (cf. Definition 3, §2) of S. However, as seen from the Existence Theorem of §7 there exist "actual solutions," asymptotic to the formal solutions (7.24) (in the sense of that theorem) in certain regular subregions of S.

We shall follow up the method previously employed in establishing the

theorem of §7, in order to obtain certain simplifications not directly obtainable from that theorem; more precisely, in order to obtain better results for the problem now under consideration.

The c_i will be again subjected to the inequalities (6.2). Applying transformation (2.1) and (2.1a) to the system (B) of §8 the inequalities (6.3) and (6.3a) should be satisfied. Now, in general, for t in S (6.3) will not hold, no matter how small c' is taken (unless $c_1=c_2=\cdots=c_m=0$, which is a case of no interest). This statement is seen to be true because of the presence of the polynomials $[t, \rho_\nu]$. When the roots of $E(\alpha)=0$ (cf. (8.3)) are all distinct, (6.3) can be made to hold in S by taking c' sufficiently small. We have $S=S(\alpha_1, \alpha_2; r)$ (Definition 9, §8). *There exists a regular subregion of* S,

$$R' = R(\alpha_1, \alpha_2) \tag{9.1}$$

(Definition 3, §2), *such that*

$$\left| e^{q_i t t^{3m'+1}} \right| \leqq 1 \qquad (i = 1, \cdots, m; t \text{ in } R'). \dagger \tag{9.1a}$$

In R' (6.3) *can be satisfied with* c' (>0) *sufficiently small.* Assume, for the present, that c' has been so chosen. In place of (6.10b) one now has

$$ {}_H\Gamma_j^{\prime h_1, \cdots, h_m}(t) = \exp\left[(h_1 q_1 + \cdots + h_m q_m)t\right][t, \rho_{H-1} + m']. \tag{9.2}$$

Thus (6.14) will become

$$\begin{aligned} -F_j(t) &= \sum_{\nu \geqq N} \sum_{k_1+\cdots+k_m=\nu} c_1^{k_1} \cdots c_m^{k_m} \exp\left[(k_1 q_1 + \cdots \right. \\ &\qquad \left. + k_m q_m)t\right][t, \rho_{\nu-1} + m'] \\ &= t^{-4m'-2} \sum_{\nu \geqq N} \sum_{k_1+\cdots+k_m=\nu} c_1^{k_1} \cdots c_m^{k_m} \exp\left[(k_1 q_1 + \cdots \right. \\ &\qquad \left. + k_m q_m)t\right] t^{\nu(3m'+1)} \left[\frac{1}{t}, \cdots\right], \end{aligned} \tag{9.3}$$

where the $[(1/t), \cdots]$ are polynomials in $(1/t)$. Hence, in view of (9.1a),

$$\left|F_j(t)\right| < k_0 (c')^N \left| t \right|^{-4m'-2} \sum_{k_1+\cdots+k_m=N} g_{k_1, \cdots, k_m}(t) \qquad (t \text{ in } R') \tag{9.4}$$

with

$$g_{k_1, \cdots, k_m}(t) = \left| \exp\left[(k_1 q_1 + \cdots + k_m q_m)t\right] t^{N(3m'+1)} \right| \qquad (k_1 + \cdots + k_m = N). \tag{9.4a}$$

For t in R', in place of (6.19), (6.19a), and (6.19b) one may merely assert that the

$$\alpha_{\nu_1, \cdots, \nu_n; j}(t) \qquad (\nu_1 + \cdots + \nu_n = 1) \tag{9.5}$$

† The second member in (9.1a) can be replaced by any constant $(\neq \infty)$.

and the

$$(9.5a) \qquad {}_j\beta_{\nu_1,\cdots,\nu_n}(t) \qquad (\nu_1 + \cdots + \nu_n \geqq 2)$$

are bounded.

However the following is noted. *There exists a regular subregion R of R'* (cf. the statement made in connection with (9.1) and (9.1a)) *of the form*

$$(9.6) \qquad R = R(\alpha_1, \alpha_2)$$

(cf. Definition 3, §2) *such that*

$$(9.6a) \qquad e^{q_i t} \sim 0 \qquad (i = 1, \cdots, m; t \text{ in } R).$$

Since in R, $Y_j(t) \sim 0$ $(j = 1, \cdots, n)$ it is observed that the functions (9.5), (9.5a) are asymptotic to zero for t in R.

LEMMA 9. *The transformation* (6.1) *and* (6.1a) *applied to the system* (B) *of* §8 *will yield a system*

$$(9.7) \qquad \rho_j^{(1)}(t) - l_j^*(t, \rho_1, \cdots, \rho_n) = q_j^*(t, \rho_1, \cdots, \rho_n) - F_j(t) \quad (j = 1, \cdots, n),$$

where

$$(9.7a) \qquad q_j^*(t, \rho_1, \cdots, \rho_n) \sim q_j(\rho_1, \cdots, \rho_n), \quad l_j^*(t, \rho_1, \cdots, \rho_n) \sim l_j(\rho_1, \cdots, \rho_n) \quad (j = 1, \cdots, n; t \text{ in } R)$$

(cf. (9.6)).† *For t in R and $|c_i| \leqq c'$ $(i = 1, \cdots, m)$ the $F_j(t)$ satisfy* (9.4) *and* (9.4a). *Also, for t in R ($|t| \geqq r$) and $|c_i| \leqq c'$ the series* (6.20a), *representing q_j^*, converge absolutely and uniformly.*‡

In order to obtain a solution of the system (9.7) by methods of the type of those used in §7, it is necessary to apply the methods of Trjitzinsky (cf. (T_2)) to establish the character of the solutions (within appropriate regions) of the linear system

$$(9.8) \qquad \rho_j^{(1)}(t) - l_j^*(t, \rho_1, \cdots, \rho_n) = 0 \qquad (j = 1, \cdots, n).$$

Now (9.8) is equivalent to the matrix equation

$$(9.8a) \qquad (\rho_{i,j}^{(1)}(t)) = (\rho_{i,j})(l_{i,j}^*(t)) \qquad (i, j = 1, \cdots, n),$$

where, by (9.7a),

$$(9.8b) \qquad (l_{i,j}^*(t)) \sim (l_{i,j}) \qquad (t \text{ in } R).$$

† In (9.7a) $q_j^* \sim q_j$, for instance, is to mean that in q_j^* and q_j the corresponding coefficients of the powers of $\rho_1, \cdots, \rho_n$ differ by functions of t, asymptotic to zero in R.

‡ With proper choice of r and c'.

In view of (9.8b) the formal matrix solution of (9.8a) is the same as that of (LB_1) of §8; that is, it is given by (8.4). While the matrix (9.4) happens to constitute an "actual" matrix solution of (LB_1), in general of course it is not an "actual" solution of (9.8a). Regarding "actual" solutions, we recall the pertinent facts derivable from the Fundamental Existence Theorem of (T_2) and briefly outlined in §2 (cf. the text in connection with (2.6)–(2.8c)). To make applications to (9.8a) one needs to write

$$S(t) = (e^{q_i t} p_{i,j}(t)) = (e^{q_i t t^{r_i}} {}_1p_{i,j}(t)) \tag{9.9}$$

and let $Q_i(t) = q_i t$ $(i = 1, \cdots, n)$. Here the r_i are certain integers and the ${}_1p_{i,j}(t)$ are polynomials in $(1/t)$. All the $B_{i,j}$ curves, that is the regular curves along which

$$R[(q_i - q_j)t] = 0 \qquad \text{(whenever } q_i \not\equiv q_j\text{)} \tag{9.10}$$

and the B_i curves

$$R(q_i t) = 0 \qquad \text{(whenever } q_i \not\equiv 0\text{)}, \tag{9.10a}$$

are rays of the form $\angle t = $ a constant.

DEFINITION 11. *S' is to denote any sector which can be constructed as follows. Consider the sectors separated by the rays* (9.10) $(i, j = 1, \cdots, n)$, *no such rays extending interior to any of those sectors. There will be some sectors (of the above set) containing subsectors, interior to each of which a number of the functions $R(q_i t)$ is negative.*† *Take a particular such subsector S'',*

$$S'' = S''(\gamma_1, \gamma_2; r) \qquad (\gamma_1 < \gamma_2)$$

(cf. Definition 9, §8). *As a matter of notation we may now write*

$$R(q_1 t) \leqq \cdots \leqq R(q_m t) \leqq R(q_{m+1} t) \leqq \cdots \leqq R(q_n t), \tag{9.11}$$

$$R(q_i t) \leqq 0 \quad (q_i \not\equiv 0;\ i = 1, \cdots, m;\ t \text{ in } S'). \tag{9.11a}$$

A sector $S' = S(\gamma_1 + \epsilon_1,\ \gamma_2 - \epsilon_2;\ r)$ is defined as follows. If the ray $\angle t = \gamma_1$ (ray $\angle t = \gamma_2$) is not coincident with any of the rays

$$R(q_i t) = 0 \quad (i = 1, \cdots, m), \qquad R[(q_i - q_\lambda)t] = 0 \quad (i \leqq m), \tag{9.12}$$

take $\epsilon_1 = 0$ (take $\epsilon_2 = 0$); in the contrary case we take ϵ_1 (take ϵ_2) positive, however small.

It is observed that in S' (9.11) holds and

$$e^{q_i t} \sim 0 \qquad (i = 1, \cdots, m;\ t \text{ in } S'); \tag{9.13}$$

moreover,

† This is because, by hypothesis, not all q_i are zero.

$$(9.13a) \qquad |q_i| \cos(\bar{q}_i + \bar{t}) \leqq -\epsilon < 0 \qquad (i = 1, \cdots, m; t \text{ in } S').$$

In following the developments of §7, as applied to the present problem, with t in the sector S', it is noted that *the number $\bar{m}$, introduced in connection with* (7.2), *can now be taken equal to m.* The inequalities (9.13a) will correspond to (7.4a). *Lemma* 6, §4 (*with $\bar{m}=m$) will be valid for t in S'* ($|t| \geqq r'$, where $r'=r'(\epsilon_1, \epsilon_2, h)$). The latter fact can be inferred by noting that the functions (7.7) and (7.7a) (for $\bar{m}=m$) are monotone (cf. Definition 8, §7) in S' (sufficiently far from the origin). This fact is deduced by observing that the statement made in connection with (7.7b) holds for the problem under consideration when t is in S' if we take into account the remarks made in connection with the rays (9.12).

With $S'=S(\delta_1, \delta_2; r)$ ($\delta_1<\delta_2$) denoting a sector satisfying the conditions of Definition 11, in consequence of (T_2) one may assert the following.

The system (9.8a) possesses the matrix solutions

$$(9.14) \qquad (\rho'_{i,j}(t)), \qquad (\rho''_{i,j}(t)),$$

with elements analytic in S' (for $t \neq \infty$), such that with $\epsilon>0$, however small,

$$(9.15) \qquad (\rho'_{i,j}(t)) \sim (e^{q_i t} p_{i,j}(t)) \qquad (t \text{ in } S'_1 = S(\delta_1, \delta_2 - \epsilon; r)),$$

$$(9.15a) \qquad (\rho''_{i,j}(t)) \sim (e^{q_i t} p_{i,j}(t)) \qquad (t \text{ in } S'_2 = S(\delta_1 + \epsilon, \delta_2; r)).$$

Here the $p_{i,j}(t)$ are the polynomials of (8.4).

The existence theorem of §7 is applicable to the problem (B) for t in S'_1 and, also, for t in S'_2. However, with the aid of the following lemma it will be shown that an existence theorem for the problem (B) can be stated for t in the more extensive region S' as well.

LEMMA 10. *Let S' be a sector of the type specified in Definition* 11. *The system* (9.8a) *possesses a matrix solution $(\rho_{i,j}(t))$, with elements analytic in S' ($t \neq \infty$), such that on writing*

$$(9.16) \qquad (\rho_{i,j}(t)) = (e^{q_i t} t^{m'} b_{i,j}(t)),$$

$$(9.16a) \qquad (\rho_{i,j}(t))^{-1} = (\bar{\rho}_{i,j}(t)) = (t^{m'} \bar{b}_{i,j}(t) e^{-q_j t})$$

(m' = greatest of numbers (9.4b)) *we have*

$$(9.16b) \qquad |b_{i,j}(t)| < \gamma, \quad |\bar{b}_{i,j}(t)| < \gamma \qquad (i, j = 1, \cdots, n; t \text{ in } S').$$

To prove this we note that the matrix solutions involved in (9.15), (9.15a) are connected by the relation

$$(9.17) \qquad (\rho'_{i,j}(t)) = (c_{i,j})(\rho''_{i,j}(t)) = \left(\sum_{\lambda} c_{i,\lambda} \rho''_{\lambda,j}(t) \right),$$

where the $c_{i,j}$ are certain constants. *Let $o(t)$ generically denote a function which in the indicated region is asymptotic to zero. By* (9.15a) *and* (8.5a) *we have*

$$(\rho''_{i,j}(t))^{-1} = (\bar{p}''_{i,j}(t)) = ((\bar{p}_{i,j}(t) + o(t))e^{-q_j t}) \qquad (in\ S'_2). \tag{9.17a}$$

From (9.17) and (9.15), in view of (9.17a),

$$(c_{i,j}) = (\rho'_{i,j}(t))(\bar{p}''_{i,j}(t)) = \left(e^{(q_i-q_j)t}\left[\sum_{\lambda} p_{i,\lambda}(t)p_{\lambda,j} + o(t)\right]\right) \tag{9.18}$$

$$(\text{in } S'_3 = S(\delta_1 + \epsilon, \delta_2 - \epsilon; r)).$$

By (8.5b)

$$(c_{i,j}) = (e^{(q_i-q_j)t}[\delta_{i,j} + o(t)]) \qquad (\text{in } S'_3). \tag{9.18a}$$

Letting $t \to \infty$ in S'_3 we infer that $c_{i,i} = 1 (i = 1, \cdots, n)$. Moreover, by (9.11) $c_{i,j} \to 0$ $(i<j)$ when $|t| \to \infty$ in S'_3; that is,

$$c_{i,j} = 0 \qquad (i < j).$$

Thus, by (9.17) and (9.15a),

$$\begin{aligned} \rho'_{i,j}(t) &= \rho''_{i,j}(t) + \sum_{\lambda=1}^{i-1} c_{i,\lambda}\rho''_{\lambda,j}(t) \\ &= e^{q_i t}\left[p_{i,j}(t) + o(t) + \sum_{\lambda=1}^{i-1} c_{i,\lambda}e^{(q_\lambda - q_i)t}(p_{\lambda,j}(t) + o(t))\right] \quad (\text{in } S'_2). \end{aligned} \tag{9.19}$$

In consequence of (9.11)

$$|e^{(q_\lambda - q_i)t}| \leqq 1 \qquad (\lambda = 1, \cdots, i-1; t \text{ in } S'_2);$$

moreover, the $p_{i,j}(t)$ $(i, j = 1, \cdots, n)$ are of the form $[t, m']$. Hence the $\rho'_{i,j}(t)$ are of the form (9.16), (9.16b) for t in S'_2. By virtue of (9.15) it is seen that they are of the form (9.16), (9.16b) in S' as well. With ϵ sufficiently small $S' = S'_1 + S'_2$. Accordingly, for t in S', the matrix $(\rho'_{i,j}(t))$ will satisfy the conditions (9.16), (9.16b) of the lemma.

By (9.15), (9.15a), and (8.5a)

$$(\rho'_{i,j}(t))^{-1} = (\bar{p}_{i,j}(t)) = ((\bar{p}_{i,j}(t) + o(t))e^{-q_j t}) \qquad (\text{in } S'_1), \tag{9.20}$$

$$(\rho''_{i,j}(t))^{-1} = (\bar{p}''_{i,j}(t)) = ((\bar{p}_{i,j}(t) + o(t))e^{-q_j t}) \qquad (\text{in } S'_2). \tag{9.20a}$$

If we write

$$(c_{i,j})^{-1} = (\bar{c}_{i,j}),$$

it follows that

$$\bar{c}_{i,j} + \sum_{\lambda=j+1}^{n} \bar{c}_{i,\lambda}c_{\lambda,j} = \delta_{i,j}. \tag{9.21}$$

From (9.21) (for $j=n$) we deduce that $\bar{c}_{n,n}=1$ and $\bar{c}_{i,n}=0$ $(i<n)$. For $j=n-1$ (9.21) will yield $\bar{c}_{n-1,n-1}=1$, $\bar{c}_{i,n-1}=0$ $(i<n-1)$; and so on. Thus

$$\bar{c}_{i,i} = 1 \qquad (i = 1, \cdots, n), \qquad \bar{c}_{i,j} = 0 \qquad (i < j). \tag{9.21a}$$

By (9.17)

$$(\bar{p}'_{i,j}(t)) = (\bar{p}''_{i,j}(t))(\bar{c}_{i,j}) = \left(\sum_{\lambda} \bar{p}''_{i,\lambda}(t)\bar{c}_{\lambda,j} \right). \tag{9.22}$$

In view of (9.21a) and (9.20a)

$$\begin{aligned} \bar{p}'_{i,j}(t) &= \bar{p}''_{i,j} + \sum_{\lambda=j+1}^{n} (\bar{p}_{i,\lambda}(t) + o(t))e^{-q_\lambda t}\bar{c}_{\lambda,j} \\ &= e^{-q_j t}\left[\bar{p}_{i,j}(t) + o(t) + \sum_{\lambda=j+1}^{n} (\bar{p}_{i,\lambda}(t) + o(t)) \exp\left[(q_j - q_\lambda)t\right]\bar{c}_{\lambda,j} \right], \end{aligned} \tag{9.22a}$$

for t in S'_2. Here, by (9.11),

$$\left| \exp\left[(q_j - q_\lambda)t\right] \right| \leqq 1 \qquad (\lambda = j+1, \cdots, n;\ t \text{ in } S'_2). \tag{9.22b}$$

Now, as we have noted before, the $\bar{p}_{i,j}(t)$ are of the form $[t, m']$. Hence, in consequence of (9.22a), (9.22b), it is observed that

$$\left| \bar{p}'_{i,j}(t) \right| < \gamma' \left| t \right|^{m'} e^{-q_j t} \qquad (i, j = 1, \cdots, n;\ t \text{ in } S'_2). \tag{9.23}$$

By (9.20) the $\left| \bar{p}_{i,j}(t) \right|$ satisfy inequalities of the same type in S'_1. Hence, with a suitable γ' (9.23) will hold in S'. This completes the proof of the lemma.†

A solution of the system (9.7) will be obtained in the form of a convergent series (7.22), the terms of this series being defined by the relations (7.19)–(7.21b) (with $p=0$). In place of (7.23) we now have

$$w_{j;k}(t) = \sum_{\lambda=1}^{n} c_\lambda^{j,k}(t)\rho_{\lambda,j}(t), \tag{9.24}$$

where

$$c_\lambda^{j,k}(t) = \sum_{\lambda_1=1}^{n} \int^{t} g_{\lambda_1;k}(\tau)\bar{p}_{\lambda_1,\lambda}(\tau)d\tau \tag{9.24a}$$

(cf. (2.28) – (2.30b)). Here the $\rho_{i,j}(t)$ and the $\bar{p}_{i,j}(\tau)$ are the functions referred to in Lemma 10. With the aid of (9.16), (9.16a), (9.16b) one obtains from (9.24), (9.24a), provided $\left| g_{j;k} \right| \leqq g_k$,

† Either of the matrices $(\rho'_{i,j})$, $(\rho''_{i,j})$ can be taken for the matrix $(\rho_{i,j})$ of the lemma.

$$(9.25) \qquad |w_{j;k}(t)| < n\gamma^2 \sum_{\lambda=1}^{n} |e^{q_\lambda t} t^{m'}| \int^t |g_k| \, |e^{-q_\lambda \tau} \tau^{m'+1+\epsilon'}| \, |\tau^{-1-\epsilon'} d\tau|$$

for t in S' (provided that the involved integrals converge).

It is essential to note that, for t in S', the inequalities (9.25) are of the same type as those of (7.24) and that Lemma 6, §7, as we remarked subsequent to (9.13a), is applicable. Repeating the processes of §7, integrating along rays $\angle\tau = \angle t$ ($|\tau| \geqq |t|$; t in S'), the following theorem is now deduced.

THE SECOND EXISTENCE THEOREM. *Consider the system* (B), §8. (LB_1), §8 *will represent a related linear system. Let the q_i denote the roots of the characteristic equation* (9.3) (*a root of multiplicity m_j repeated m_j times*). *It is assumed that not all the q_i are zero. Let S' denote a sector satisfying the conditions of Definition* 11. *Thus, as a matter of notation, in S' we have* (9.11), (9.13), (9.13a). *Let N be any integer $\geqq \nu(\epsilon)$; here, with ϵ denoting the number in the second members of* (9.13a), *$\nu(\epsilon)$ is the least integer equal to or greater than $(\epsilon + |q_\lambda|)/\epsilon$ ($\lambda = 1, \cdots, m-1$).*

The system (B), §8 *possesses a solution, $y_j(t)$ ($j = 1, \cdots, n$), whose elements are analytic in S' ($t \neq \infty$). At $t = \infty$ they will be singular. Moreover,*

$$(9.26) \qquad y_j(t) \sim s_j(t) \qquad (j = 1, \cdots, n; \; t \text{ in } S'(|t| \geqq r^*))$$

where the $s_j(t)$ are the formal series (8.24) (cf. (8.24a)) *of Lemma* 8, §8. *The asymptotic relations* (9.26) *are to be interpreted as follows. The $y_j(t)$ are functions of the form*

$$(9.26a) \quad y_j(t) = \sum_{\nu=1}^{N-1} \sum_{k_1+\cdots+k_m=\nu} c_1^{k_1} \cdots c_m^{k_m} \exp\left[(k_1 q_1 + \cdots + k_m q_m)t\right] {}_\nu p_{k_1,\cdots,k_m;j}(t) + \rho_j(t)$$

where $c_1, \cdots, c_m$ are arbitrary constants ($|c_i| \leqq c'$; $i = 1, \cdots, m$) and the ${}_\nu p_{k_1,\cdots,k_m;j}(t)$ are polynomials in t of at most the $[\nu(3m'+1) - (2m'+1)]$st degree.† *The $\rho_j(t) = \rho_j(c_1, \cdots, c_m; t)$ are analytic, for $t \neq \infty$, in $S'(|t| \geqq r^*)$ and satisfy in this sector the inequalities*

$$(9.26b) \quad |\rho_j(t)| < h'(c')^N |t|^{Nm''} \sum_{k_1+\cdots+k_m=N} |\exp\left[(k_1 q_1 + \cdots + k_m q_m)t\right]|,$$

provided $|c_i| \leqq c'$ ($i = 1, \cdots, m$); here m'' is a constant independent of N. In the above we have assumed that c' is sufficiently small and r^* is sufficiently great.

Note. In this theorem the term "asymptotic" is justified for reasons of the same type as stated in the note subsequent to the theorem of §7.

10. **Uniqueness theorem.** Turning our attention to the theorem of §7 we observe that there is a manifold of solutions, depending on $\bar{m}$ parameters c_i,

† m' is the greatest of the numbers (8.4b).

such that the elements of the solutions, as well as all the derivatives of these elements, approach zero as $|t| \to \infty$ within W_ξ. This set of solutions has been constructed corresponding to some $N(\geqq \nu(\xi))$. We shall write

$$(10.1) \quad y_j(t) \underset{N}{\sim} s_j(t) \quad (j = 1, \cdots, n; t \text{ in } W_\xi(|t| \geqq r_N); |c_i| \leqq c'),$$

where $\underset{N}{\sim}$ is to signify (7.50a) and (7.50c); this symbol will be read as "asymptotic to N terms." It appears at the first glance that the $y_j(t)$ may depend on N.

Let the $y_j'(t)$ be elements of a solution of (A), §1 such that

$$(10.2) \quad y_j'(t) \underset{N+1}{\sim} s_j(t) \quad (j = 1, \cdots, n; t \text{ in } W_\xi(|t| \geqq r_{N+1}); |c_i| \leqq c').$$

We may have $r_{N+1} \geqq r_N$. Form the difference

$$(10.3) \quad z_j(t) = y_j'(t) - y_j(t) = {}_N y_j(t) + \rho_j'(t) - \rho_j(t);$$

here ${}_N y_j(t)$ is given by (5.23a) and (5.24a), and $\rho_j'(t)$ are the $\rho_j(t)$ of the theorem of §7, formed with N replaced by $N+1$. If we take account of (7.50c) and of the note of §7, from (10.3) we infer that

$$(10.3a) \quad |z_j(t)| < h^*{}_N\rho^*(t) + {}_{N+1}\rho^*(t) \quad (t \text{ in } W_\xi; |t| \geqq r_{N+1}; |c_i| \leqq c').\dagger$$

By (10.3a) and (7.50c)

$$(10.4) \quad \begin{aligned} |z_j(t)| &< \zeta \cdot {}_N\rho^*(t) \\ &= \zeta\zeta'(c')^N |t|^{(N-1)\bar{\omega}+\xi+2\epsilon'} \sum_{k_1+\cdots+k_{\bar{m}}=N} |\exp [k_1 Q_1(t) + \cdots \\ &\qquad + k_{\bar{m}} Q_{\bar{m}}(t)] t^{k_1 r_1 + \cdots + k_{\bar{m}} r_{\bar{m}}}| = z_1^*(t) \quad (t \text{ in } W_\xi(|t| \geqq r_{N+1})); \end{aligned}$$

also, by the theorem of §7,

$$(10.4a) \quad |y_j(t)| < \eta_1 \rho^*(t) |t|^{-\xi-2\epsilon'} = y^*(t) \quad (t \text{ in } W_\xi(|t| \geqq r_{N+1})).$$

It will be assumed that r_{N+1} has been taken sufficiently great so that

$$(10.4b) \quad z_1^*(t) + y^*(t) \leqq \rho^* < \rho \quad (t \text{ in } W_\xi; |t| \geqq r_{N+1})$$

(cf. (1.2)).

By (10.4), (10.4a), (10.4b), in consequence of the statement in connection with (7.17),‡ we have

$$(10.4c) \quad \begin{aligned} &|q_j(t, y_1 + z_1, \cdots, y_n + z_n) - q_j(t, y_1, \cdots, y_n)| \\ &\qquad < (q' y^*(t) + q z_1^*(t)) z_1^*(t) \end{aligned}$$

† Take $r_{N+1} \geqq r_N$.

‡ One now has q_j in place of q_j^*; however q_j is of the same description as q_j^*.

for the indicated values of the variables. Hence

$$(10.5)\qquad |\, q_j(t, y_1 + z_1, \cdots) - q_j(t, y_1, \cdots)\,| < a'_1\rho^*(t)_N\rho^*(t)\,|\,t\,|^{-\xi-2\epsilon'} \qquad (t \text{ in } W_\xi;\ |\,t\,| \geqq r_{N+1}).$$

Subtracting the corresponding members of the relations

$$(10.6)\qquad t^{-p}y_j'^{(1)}(t) - l_j(t, y_1', \cdots, y_n') = q_j(t, y_1', \cdots, y_n'),$$

$$(10.6a)\qquad t^{-p}y_j(t) - l_j(t, y_1, \cdots, y_n) = q_j(t, y_1, \cdots, y_n) \qquad (j = 1, \cdots, n)$$

(cf. (A), §1, (1.1a), (1.1b)), we find that in consequence of (10.3)

$$(10.7)\qquad z_j^{(1)}(t) - t^p l_j(t, z_1(t), \cdots, z_n(t)) = t^p T_j(t) \quad (j = 1, \cdots, n),$$

where

$$(10.7a)\qquad T_j(t) = q_j(t, y_1 + z_1, \cdots) - q_j(t, y_1, \cdots).$$

Thus, by (10.5),

$$(10.7b)\qquad |\, T_j(t)\,| < a'_1\rho^*(t)_N\rho^*(t)\,|\,t\,|^{-\xi-2\epsilon'} \qquad (t \text{ in } W_\xi;\ |\,t\,| \geqq r_{N+1}).\dagger$$

It is noted that the left members in (10.7) are of the form

$$L_j(t, z_1(t), \cdots, z_n(t)),$$

where the differential operator L_j is precisely the same as the one so denoted in (S_H), §2 (also cf. (2.14a)). If we note that (S_H), §2 can be solved with the aid of (2.30) and (2.30a) and that these formulas may be written in the form (5.4), (5.4a), and (5.4b), the relations (10.7) may be written as follows:

$$(10.8)\qquad z_j(t) = \sum_{\lambda=1}^{n} c_\lambda(t)e^{Q_\lambda(t)}t^{r_\lambda}\gamma_{\lambda,j}(t),$$

where

$$(10.8a)\qquad c_\lambda(t) = \sum_{\lambda_1=1}^{n}\int^t T_{\lambda_1}(\tau)e^{-Q_\lambda(\tau)}\tau^{-r_\lambda+\omega+p}\bar{\gamma}_{\lambda_1,\lambda}(\tau)d\tau,$$

$$(10.8b)\qquad \gamma_{\lambda,j}(\tau) = [\tau]^*, \quad \bar{\gamma}_{\lambda_1,\lambda}(\tau) = [\tau]^* \qquad (\tau \text{ in } W_\xi;\ |\,t\,| \geqq r_{N+1}).$$

The formulas (10.8), (10.8a), with the integrals taken as indefinite, would give a general solution of (10.7) if $z_1(t), \cdots, z_n(t)$ were considered as the unknowns and the $T_j(t)$ were considered as known functions of t. In the case on hand, however, the $z_j(t)$ by definition are known functions. Hence (10.8),

† Throughout the discussion we keep $|c_i| \leqq c'$ $(i = 1, \cdots, \bar{m})$.

(10.8a) will be equivalent to (10.7) only if the integrations in (10.8a) are suitably specified. One may write

$$c_\lambda(t) = d_{j,\lambda} + \sum_{\lambda_1=1}^{n} \int_\infty^t T_{\lambda_1}(\tau) e^{-Q_\lambda(\tau)} \tau^{-r_\lambda+\omega+p} \bar{\gamma}_{\lambda_1,\lambda}(\tau) d\tau, \tag{10.9}$$

where the path of integration (for t in W_ξ) is within W_ξ and the $d_{j,\lambda}$ are certain constants. The integrals here involved exist in view of (10.7b). The integrand displayed in (10.9) being analytic in W_ξ ($t \neq \infty$), the path of integration will be chosen as the ray

$$\angle \tau = \angle t \qquad (|\tau| \geqq |t|; t \text{ in } W_\xi).$$

Substituting (10.9) in (10.8) one has

$$z_j(t) = \zeta_j(t) + \zeta_j'(t), \tag{10.10}$$

where

$$\zeta_j'(t) = \sum_{\lambda=1}^{n} d_{j,\lambda} e^{Q_\lambda(t)} t^{r_\lambda} \gamma_{\lambda,j}(t) \qquad (j = 1, \cdots, n); \tag{10.10a}$$

on the other hand the $\zeta_j(t)$ will satisfy inequalities,† provided $|T_{\lambda_1}(\tau)| < T(\tau)$,

$$|\zeta_j(t)| < n\gamma_0^2 \sum_{\lambda=1}^{n} \left| e^{Q_\lambda(t)} t^{r_\lambda+\epsilon'} \right| \left| \int_\infty^t |T(\tau) e^{-Q_\lambda(\tau)} \tau^{-r_\lambda+\bar{\omega}+2\epsilon'}| \, |\tau^{-1-\epsilon'} d\tau| \right|. \tag{10.10b}$$

In consequence of (10.7b) and in view of the notation introduced at the beginning of §7 (cf. (7.1)–(7.6))

$$\begin{aligned} |T(\tau) e^{-Q_\lambda(\tau)} \tau^{-r_\lambda+\bar{\omega}+2\epsilon'}| &< a'_1 \rho^*(\tau)_N \rho^*(\tau) |\tau|^{-\xi-2\epsilon'} |e^{-Q_\lambda(\tau)} \tau^{-r_\lambda+\bar{\omega}+2\epsilon'}| \\ &= a'(\zeta')^2 (c')^{N+1} q_{1;\bar{\omega}+2\epsilon'}(\tau) \sum_{k_1+\cdots+k_m=N} {}_\lambda\Lambda_{k_1,\cdots,k_m;h_N}(\tau), \end{aligned} \tag{10.11}$$

where $h_N = \bar{\omega} + (\xi + 2\epsilon')/(N-1)$. Hence, by Lemma 6, §7, the second member of (10.11) is monotone (cf. Definition 8, §7) in

$$W_\xi \qquad (|t| \geqq r'(\xi, h) \geqq r_{N+1}), \tag{10.11a}$$

where h is the greater one of the numbers $\bar{\omega}+2\epsilon'$ and h_N. Hence the integral displayed in (10.10b) is less than the second member of (10.11) (with $\tau = t$) multiplied by $|t|^{-\epsilon'}/\epsilon'$, provided t is in the region (10.11a). Thus, by (10.10b),

$$|\zeta_j(t)| < \frac{a' n^2 \gamma_0^2}{\epsilon'} |t|^{\bar{\omega}-\xi} {}_1\rho^*(t)_N \rho^*(t) = z_2^*(t) = g(t) z_1^*(t), \tag{10.12}$$

$$g(t) = a'' |t|^{\bar{\omega}-\xi} {}_1\rho^*(t), \qquad a'' = \frac{a' n^2 \gamma_0'}{\epsilon' \zeta}, \tag{10.12a}$$

† Note that, by (10.8b), $|\gamma_{\epsilon,j}(t)|$, $|\bar{\gamma}_{\epsilon,j}(t)| < |t|^{\epsilon'} \gamma_0$ ($\epsilon' > 0$) in the indicated region; $\bar{\omega} = \omega + p + 1$, as in §7.

for t in the region (10.11a). In (10.11a) take $r'(\xi, h)$ also sufficiently great so that in the indicated region

$$g(t) \leqq \delta < 1. \tag{10.12b}$$

Since, except for a constant factor, $g(t)$ is of the form

$$|t|^{\bar{\omega}+2\epsilon'} q_{1:0}(t)$$

(cf. (7.1a)), by virtue of the definition of h it follows that $g(t)$ is monotone in the region (10.11a).

If we make use of (10.10), (10.4), (10.12), and (10.12b) we conclude that

$$|\zeta_j'(t)| < (1+\delta) z_1^*(t) \qquad (t \text{ in } W_\xi;\ |t| \geqq r'(\xi, h)). \tag{10.13}$$

Let R be some fixed ray interior to W_ξ. On R

$$\begin{aligned} RQ_1(t) = \cdots = RQ_{n_1}(t) < RQ_{n_1+1}(t) = \cdots = RQ_{n_2}(t) < \cdots \\ < RQ_{n_{\sigma-2}+1}(t) = \cdots = RQ_{n_{\sigma-1}}(t) < RQ_{n_{\sigma-1}+1}(t) = \cdots = RQ_{n_\sigma}(t) \end{aligned} \tag{10.14}$$

$$(n_\sigma = n)$$

and

$$\exp\,[Q_i(t) - Q_j(t)] \sim 0 \qquad (t \text{ on } R) \tag{10.14a}$$

for $i<j$, whenever $Q_i(t) \not\equiv Q_j(t)$. In view of (10.10a) one may write

$$\zeta_j'(t) = e^{Q_{n_\sigma}(t)} \zeta'_{j,\sigma}(t) + e^{Q_{n_{\sigma-1}}(t)} \zeta'_{j,\sigma-1}(t) + \cdots + e^{Q_{n_1}(t)} \zeta'_{j,1}(t), \tag{10.15}$$

where

$$\zeta'_{j,\nu}(t) = \sum_{\lambda=n_{\nu-1}+1}^{n_\nu} d_{j,\lambda} t^{r_\lambda} \gamma_{\lambda,j}(t) \qquad (n_0 = 0;\ \nu = 1, \cdots, \sigma). \tag{10.15a}$$

With $o(t)$ denoting generically a function ~ 0 (for the indicated values of t), in consequence of (10.14a), (10.15), and (10.15a) we have

$$\zeta_j'(t) = e^{Q_{n_\sigma}(t)} [\zeta'_{j,\sigma}(t) + o(t)] \qquad (t \text{ on } R). \tag{10.16}$$

By virtue of (10.13)

$$|\zeta'_{j,\sigma}(t) + o(t)| < (1+\delta) z_1^*(t) e^{-Q_{n_\sigma}(t)} \qquad (t \text{ on } R). \tag{10.16a}$$

If note is taken of the properties of the functions (7.6), for $k_1 + \cdots + k_{\bar{m}} = \nu \geqq N \geqq \nu(\xi)$, it follows that the second member in (10.16a) is ~ 0 along R. Hence the function of (10.15a) for $\nu = \sigma$ must be ~ 0 for t on R. In view, however, of the origin of the functions involved in the second member of (10.15a) for $\nu = \sigma$ this would imply that necessarily

$$\zeta'_{j,\sigma}(t) \equiv 0 \qquad (j = 1, \cdots, n). \tag{10.16b}$$

We substitute (10.16b) in (10.15) and prove that $\zeta'_{j,\sigma-1}(t)\equiv 0$ $(j=1,\cdots,n)$ in a way analogous to that used in establishing (10.16b). Thus, step by step it is shown that the $\zeta'_{j,\nu}(t)$ are all identically zero. Hence, from (10.10) and (10.12) it follows that

$$|z_j(t)| < g(t)z_1^*(t) = z_2^*(t) \qquad (j=1,\cdots,n) \tag{10.17}$$

in the region (10.11a) (where $r'(\xi, h)$ is assumed to satisfy the condition stated in connection with (10.12b)).

We repeat the procedure beginning with (10.4), replacing (10.4) by (10.17), and *letting t remain in the region* (10.11a). It is observed that (10.4b) will be satisfied, with $z_1^*(t)$ replaced by $z_2^*(t)$, since $g(t)<1$. Moreover, (10.4c) with $z_1^*(t)$ replaced by $z_2^*(t)$ will hold. Now

$$(q'y^*(t) + qz_2^*(t))z_2^*(t) < [(q'y^*(t) + qz_1^*(t))z_1^*(t)]g(t),$$

so that (10.5) becomes

$$|q_j(t, y_1+z_1,\cdots) - q_j(t, y_1,\cdots)| < a'_1\rho^*(t)_N\rho^*(t)\,|t|^{-\xi-2\epsilon'}g(t). \tag{10.5'}$$

Whence in the second member of (10.7b) we may introduce $g(t)$ as a factor. The argument is the same until we obtain the relation (10.11). The second and the third members of (10.11) are to be multiplied by $g(t)$. Designate the resulting inequalities by (10.$\overline{11}$). Now, as remarked before, $g(\tau)$ is monotone in the region (10.11a) (with $r'(\xi, h)$ chosen so that (10.12b) holds). Along the path of integration indicated in (10.10b) $g(\tau)\leqq g(t)$. Hence in place of (10.12) one now obtains

$$|\zeta_j(t)| < g^2(t)z_1^*(t) = z_3^*(t) \qquad (t \text{ in the region (10.11a)}). \tag{10.18}$$

As established before, we have $\zeta'_j(t)\equiv 0$. Thus in (10.18) $\zeta_j(t)$ may be replaced by $z_j(t)$.

It is not difficult to prove by induction that

$$|z_j(t)| < g^k(t)z_1^*(t) \qquad (j=1,\cdots,n;\ k=1,2,\cdots) \tag{10.19}$$

for t in the region (10.11a) (*such that* (10.12b) *holds*). The latter region is independent of k. Let $t=t_0$ be any point in the above region; since by (10.12b) and (10.19)

$$|z_j(t_0)| < \delta^k z_1^*(t_0) \qquad (j=1,\cdots,n;\ k=1,2,\cdots),$$

where the second member can be made as small as desired by a suitable choice of k, it follows that $z_j(t_0)=0$ $(j=1,\cdots,n)$. Thus

$$z_j(t) \equiv 0 \qquad (j=1,\cdots,n). \tag{10.20}$$

Whence, by (10.3), it is concluded that the solutions referred to in (10.1) and

(10.2) are identical. It is thus inferred that the elements of the solution described in the First Existence Theorem are independent of N so long as $N \geqq \nu(\xi)$. The same result will hold for the theorem of §9, even though the latter theorem is not a special case of the first. One needs only to apply the result established for the theorem of §7 to a suitable subsector (a ray, even, is sufficient) of the sector for which the theorem of §9 has been stated; (10.20) will then hold in the original sector in consequence of a well known property of analytic functions.

UNIQUENESS THEOREM. *The solutions of the existence theorems of §§7 and 9 are independent of N. Thus, if*

$$y_1(t), y_2(t), \cdots, y_n(t)$$

are the elements of a solution of the non-linear differential system under consideration, and if

$$s_1(t), s_2(t), \cdots, s_n(t)$$

are the corresponding elements of a formal solution, we have

$$y_j(t) \underset{N}{\sim} s_j(t) \qquad (j = 1, \cdots, n; t \text{ in the indicated region}), \tag{10.21}$$

where $N = \nu(\xi), \nu(\xi)+1, \nu(\xi)+2, \cdots$. For any fixed N ($\geqq \nu(\xi)$) the "asymptotic" relations (10.21) *are to signify that the $y_j(t)$ are functions possessing the properties described in the existence theorems referred to above.*

Note. The above theorem, then, implies that the solutions mentioned in the theorems of §§7 and 9 are asymptotic to the corresponding formal series not merely "to N terms," as stated in those theorems, but also in the ordinary sense; that is, to infinitely many terms.

11. **Systems** (C). Our purpose will now be to investigate in the complex neighborhood of the singular point $\lambda = \infty$ the analytic character of some of the solutions of systems (C), mentioned in §1. A system (C) will be assumed to be of the following form:

$$\lambda^{-p} y_j^{(1)}(x) = a_j(\lambda, x, y_1, \cdots, y_n) \quad (j = 1, \cdots, n; p \text{ an integer} \geqq 0), \dagger \tag{11.1}$$

where λ is a parameter and

$$a_j(\lambda, x, y_1, \cdots, y_n) = l_j(\lambda, x, y_1, \cdots, y_n) + q_j(\lambda, x, y_1, \cdots, y_n), \tag{11.1a}$$

$$l_j(\lambda, x, y_1, \cdots, y_n) = l_{1,j}(\lambda, x) y_1 + \cdots + l_{n,j}(\lambda, x) y_n, \tag{11.1b}$$

$$q_j(\lambda, x, y_1, \cdots, y_n) = \sum {}_j a_{i_1, \cdots, i_n}(\lambda, x) y_1^{i_1} \cdots y_n^{i_n} \tag{11.1c}$$

$$(i_1, \cdots, i_n \geqq 0; i_1 + \cdots + i_n \geqq 2; j = 1, \cdots, n).$$

† Indicated derivatives are with respect to x.

Here the $l_{i,j}(\lambda, x)$, ${}_ja_{i_1,\cdots,i_n}(\lambda, x)$ are functions representable by series of the form

$$c_0(x) + c_1(x)\lambda^{-1} + c_2(x)\lambda^{-2} + \cdots, \tag{11.2}$$

where the $c_\nu(x)$ $(\nu = 0, 1, \cdots)$ are continuous, together with their derivatives of all orders, for x on the closed interval (α, β). Moreover, the series (11.2) are assumed to converge absolutely and uniformly† for x in (α_1, β_1) and for

$$|\lambda| \geqq r. \tag{11.3}$$

The series involved in the second member of (11.1c) are assumed to be absolutely and uniformly convergent‡ for

$$\alpha_1 \leqq x \leqq \beta_1;\ |\lambda| \geqq r;\ |y_1|, |y_2|, \cdots, |y_n| \leqq \rho. \tag{11.4}$$

Moreover, *it will be assumed that the system obtained by replacing the q_j by zeros is actually of order n.*

Associated with the problem (11.1) there is a linear system

$$\lambda^{-p} y_j^{(1)}(x) = l_j(\lambda, x, y_1, \cdots, y_n) \qquad (j = 1, \cdots, n). \tag{11.5}$$

A general analytic theory for systems of this type has been given by Trjitzinsky.§ It will be assumed that the reader is familiar with the results and methods contained in (T_3). In connection with the earlier contributions to the theory of linear systems containing a parameter, we shall mention papers of outstanding importance by G. D. Birkhoff,‖ J. D. Tamarkin¶ and P. Noaillon.†† Some additional references to the contributions of these and some other writers may be found in (T_3).

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† The functions under consideration will be analytic in λ for $|\lambda| \geqq r$ ($\lambda = \infty$ included) when x has any fixed value in (α_1, β_1), *if it is assumed (as it is) that r is sufficiently great.* For every fixed λ ($|\lambda| \geqq r$) these functions are continuous in x for x in (α_1, β_1), in consequence of the assumed uniformity of convergence of the series (11.2). This continuity is uniform with respect to λ ($|\lambda| \geqq r$).

‡ *r is taken sufficiently great and ρ sufficiently small* so that, for x in (α_1, β_1), the functions $q_j(\lambda, x, y_1, \cdots, y_n)$ are analytic in $(\lambda, y_1, \cdots, y_n)$, at $(\lambda = \infty, y_1 = y_2 = \cdots = y_n = 0)$, for $(|\lambda| \geqq r;\ |y_1|, |y_2|, \cdots, |y_n| \leqq \rho)$. For any set of values, such that (11.4) holds, the q_j will be continuous in x for x in (α_1, β_1).

§ W. J. Trjitzinsky, *Theory of linear differential equations containing a parameter*, Acta Mathematica, vol. 67 (1936), pp. 1–50. In the sequel this paper will be referred to as (T_3).

‖ G. D. Birkhoff, *On the asymptotic character of solutions of certain linear differential equations*, these Transactions, vol. 9 (1908), pp. 219–231.

¶ J. D. Tamarkin, *Some general problems of the theory of linear differential equations and expansions* . . . , Mathematische Zeitschrift, vol. 27 (1927), pp. 1–54.

†† P. Noaillon, *Développements asymptotiques dans les équations différentielles linéaires à paramètre variable*, Mémoires de la Société des Sciences de Liege, (3), vol. 9 (1912), 197 pp.

12. **Preliminaries for systems (C).** The linear system (11.5) is equivalent to the matrix equation

$$Y^{(1)}(x) = Y(x)A(\lambda, x), \qquad Y(x) = (y_{i,j}(x)), \tag{12.1}$$

$$A(\lambda, x) = (\lambda^{p_{i,j}}(\lambda, x)) \qquad (i, j = 1, \cdots, n) \tag{12.1a}$$

(cf. (11.1b)). A solution of (12.1) will consist of the n elements in a row of $Y(t)$. Now, this system is of the type to which the results of (T_3) apply. It is to be noted that the linear system on hand possesses a singularity at $\lambda = \infty$.

It will be convenient to introduce the following definition.

DEFINITION 1. *Generically* $\{\lambda, x\}$ *(integer* $q \geqq 0$*) is to denote a series possibly divergent for all* $\lambda \neq \infty$*, of the form*

$$\sigma_0(x) + \sigma_1(x)\lambda^{-1/k} + \cdots + \sigma_\nu(x)\lambda^{-\nu/k} + \cdots$$

(integer $k > 0$*). Here* $\sigma_\nu(x)$ $(\nu = 0, 1, \cdots)$ *is, together with the derivatives of all orders (which are assumed to exist), continuous on some real interval.*

The system (12.1) possesses a formal matrix solution†

$$S(\lambda, x) = (s_{i,j}(\lambda, x)) = (e^{Q_i(\lambda, x)}\sigma_{i,j}(\lambda, x)). \tag{12.2}$$

In (12.2)

$$Q_i(\lambda, x) = \sum_{\nu=0}^{l_i - 1} q_\nu{}^i(x)\lambda^{(l_i - \nu)/k_i} \qquad \text{(integers } k_i \geqq 1), \tag{12.2a}$$

$$\sigma_{i,j}(\lambda, x) = \{\lambda, x\} \qquad (i, j = 1, \cdots, n) \tag{12.2b}$$

(cf. Definition 1), the functions of x in the second members of (12.2a) and (12.2) being, together with the derivatives of all orders, continuous on a subinterval (α_2, β_2) of (α_1, β_1).‡ *Henceforth, unless stated otherwise, attention will be confined to a particular such interval* (α_2, β_2) *and its subintervals. The interval* (α_2, β_2) *is taken so that for no* x *in* (α_2, β_2) *do any two roots of the characteristic equation of* (12.1) *coincide, unless these two roots coincide throughout* (α_2, β_2).§ The intervals under consideration are closed.

By the Fundamental Existence Theorem of (T_3) in particular the following can be asserted.

Let (α_2, β_2) *be an interval as stated above. Let the* $B_{i,j}^{1,x}$ *designate curves in the* λ*-plane, extending to infinity, along which*

† That is, the determinant $|S(x, \lambda)|$ does not formally reduce to zero.

‡ It is to be noted that (α_1, β_1) may contain values of $x = x_0$, such that in two intervals contained in (α_1, β_1) on different sides of x_0 the system (12.1) will possess distinct formal matrix solutions (12.2).

§ That is, (α_2, β_2) is such that roots which are distinct for some x in (α_2, β_2) remain distinct throughout (α_2, β_2); while any two roots coincident for some x in (α_2, β_2) will coincide throughout this interval.

$$RQ_i^{(1)}(\lambda, x) = RQ_j^{(1)}(\lambda, x). \dagger \tag{12.3}$$

There exist subintervals of (α_2, β_2) *such that , when x is restricted to one of them, there exists a regular* (cf. Definition 3, §2) *region R in the* λ*-plane so that there are no curves* $B_{i,j}^{1,x}$ *interior to R. Let* (α', β') *denote any particular subinterval mentioned above. As a matter of notation we may write*

$$RQ_1^{(1)}(\lambda, x) \leqq RQ_2^{(1)}(\lambda, x) \leqq \cdots \leqq RQ_n^{(1)}(\lambda, x) \qquad (x \text{ in } (\alpha', \beta'); \lambda \text{ in } R). \tag{12.4}$$

Let α *be an integer, however large. The system* (12.1) *possesses a matrix solution* ${}_\alpha Y(\lambda, x)$*, with elements analytic in* λ $(\lambda \neq \infty)$ *and continuous in x for x in* (α', β') *and* λ *in R; moreover, for the indicated values of the variables*‡

$${}_\alpha Y(\lambda, x) \underset{\alpha}{\sim} S(\lambda, x) \tag{12.5}$$

(cf. (12.2)).

Consider a matrix solution satisfying (12.5). Form another matrix solution

$$Y(\lambda, x) = (y_{i,j}(\lambda, x)) = (\delta_{i,j} e^{-Q_j(\lambda, \alpha')})_\alpha Y(\lambda, x), \tag{12.6}$$

where $(\delta_{i,j})$ is the identity matrix. We shall have

$$Y(x, \lambda) \underset{\alpha}{\sim} S^*(\lambda, x) = (e^{G_i(\lambda, x)} \sigma_{i,j}(\lambda, x)) \qquad (x \text{ in } (\alpha', \beta'); \lambda \text{ in } R) \tag{12.6a}$$

(cf. (12.2b)), and

$$G_i(\lambda, x) = Q_i(\lambda, x) - Q_i(\lambda, \alpha'). \tag{12.6b}$$

Since,§ by (12.6b),

$$G_i(\lambda, x) = \int_{u=\alpha'}^{x} G_i^{(1)}(\lambda, u) du$$

and

$$RG_i(\lambda, x) = \int_{u=\alpha'}^{x} RQ_i^{(1)}(\lambda, u) du,$$

---

† Whenever $Q_i(\lambda, x) \equiv Q_j(\lambda, x)$, the corresponding $B_{i,j}^{1,x}$ curves will be said not to exist. Throughout it will be understood that these curves extend to infinity and that sufficiently far from the origin they are simple; moreover, at infinity they will possess limiting directions. A particular curve (12.3) will depend on x; as x varies in the interval this curve may vary, remaining always interior to a sector; the angle of this sector can be made as small as desired by taking the interval for x suitably small (cf. (T_3)).

‡ $\sim$ is to denote an asymptotic relation in the ordinary sense (unless the contrary is stated); that is, to infinitely many terms. $\underset{\alpha}{\sim}$ denotes an asymptotic relation, to α terms. These relations are with respect to λ and are "uniform" with respect to x (x in (α', β')) (cf. (T_3)).

§ Throughout, differentiation is never with respect to λ.

if it is noted that

$$R(G_\nu(\lambda, x) - G_k(\lambda, x)) = \int_{u=\alpha'}^{x} R(Q_\nu^{(1)}(\lambda, u) - Q_k^{(1)}(\lambda, u))du,$$

in view of (12.4) it is concluded that

$$RG_\nu(\lambda, x) \leqq RG_k(\lambda, x),$$

whenever $\nu < k$, x is in (α', β') and λ is in R. Thus, for the indicated values of the variables

$$RG_1(\lambda, x) \leqq RG_2(\lambda, x) \leqq \cdots \leqq RG_n(\lambda, x); \tag{12.7}$$

moreover, if for x in (α', β') and λ in R and some i, we have $RQ_i^{(1)}(\lambda, x) \leqq 0$, it will follow that for the same values of the variables

$$RG_i(\lambda, x) \leqq 0. \tag{12.7a}$$

The case of interest is when not all the $Q_i^{(1)}(\lambda, x)$ are identically zero. This will be assumed.

DEFINITION 2. *$\Gamma(\alpha', \beta'; R)$ will denote the aggregate of the values of x and λ such that*

$$\alpha' \leqq x \leqq \beta' \text{ and } \lambda \text{ is in } R \tag{12.8}$$

($\alpha' < \beta'$; R a regular region (cf. Definition 3, §2)) *$\Gamma(\alpha', \beta'; R)$ will be said to be proper if*

(i) *The interval (α', β') possesses the properties attributed to (α_2, β_2) in the italicized statement subsequent to* (12.2b).

(ii) *With suitable notation, we may assert the validity of* (12.4) *in $\Gamma(\alpha', \beta'; R)$.*

It is observed that if we take the interval for x sufficiently small, proper regions $\Gamma(\alpha', \beta'; R)$ can always be constructed. By the Fundamental Existence Theorem of (T_3) it follows that, given any particular proper region $\Gamma(\alpha', \beta'; R)$ and any α, however large, there exists a matrix solution of (12.1), say ${}_\alpha Y(\lambda, x)$, which satisfies in $\Gamma(\alpha', \beta'; R)$ the conditions of the italicized statement made in connection with (12.5).

Since not all the $RQ_i^{(1)}(\lambda, x)$ are identically zero, in view of the statement in connection with (12.7a), it is inferred that there exist proper regions with the additional property

$$RQ_1^{(1)}(\lambda, x) \leqq \cdots \leqq RQ_m^{(1)}(\lambda, x) \leqq 0; \tag{12.9}$$

that is,

$$RG_1(\lambda, x) \leqq \cdots \leqq RG_m(\lambda, x) \leqq 0, \tag{12.9a}$$

satisfied in the region under consideration $(1 \leqq m \leqq n)$.

Hence it is not difficult to see that there exist regions satisfying the following definition.

DEFINITION 3. *A region* $\Gamma(c, d; R')$ (notation of Definition 2) *will be said to be admissible if it has the following properties*:

(a) *There exists a region* $\Gamma(\alpha', \beta'; R)$, *proper according to Definition* 2, *such that the interval* (c, d) *is contained in* (α', β') *and* R' *is contained in* R. *With suitable notation we may then write*

$$RQ_1^{(1)}(\lambda, x) \leqq RQ_2^{(1)}(\lambda, x) \leqq \cdots \leqq RQ_n^{(1)}(\lambda, x) \tag{12.10}$$

in $\Gamma(\alpha', \beta'; R)$ (*we may also write* (12.7); cf. (12.6b)).

(b) *For some* m $(1 \leqq m \leqq n)$

$$RQ_i^{(1)}(\lambda, x) \leqq 0 \qquad (i = 1, \cdots, m; \text{ in } \Gamma(c, d; R')). \tag{12.10a}$$

It is observed that (12.10a) *implies*

$$RG_i(\lambda, x) \leqq 0 \qquad (i = 1, \cdots, m; \text{ in } \Gamma(c, d; R')). \tag{12.10b}$$

(c) *Whenever the number* m *employed in* (b) *exceeds unity we also have*

$$RQ_1^{(1)}(\lambda, x) \geqq 2RQ_m^{(1)}(\lambda, x) \qquad (\text{in } \Gamma(c, d; R')). \tag{12.10c}$$

Admissible regions for which $m=1$ (that is, (a) and (b) are satisfied and (c) is deleted) can be always constructed. Admissible regions with $m>1$ will exist in a wide variety of problems.

Consider now a particular admissible region $\Gamma(c, d; R')$, *subset of a proper region* $\Gamma(\alpha', \beta'; R)$. *Let*

$$Y(\lambda, x) = ({}_0y_{i,j}(\lambda, x)) \tag{12.11}$$

be a matrix solution of (12.1) *satisfying* (12.6a) (cf. (12.6b)) *and* (12.10) *in* $\Gamma(\alpha', \beta'; R)$; *moreover, in* $\Gamma(c, d; R')$ *we shall have* (12.10a), (12.10b), (12.10c).†

On turning to the system (11.5) it is noted that a solution can be taken in the form

$${}_1y_j(\lambda, x) = \sum_{\sigma=1}^{m} c_\sigma \, {}_0y_{\sigma,j}(\lambda, x) \qquad (j = 1, \cdots, n), \tag{12.12}$$

where $c_1, c_2, \cdots, c_m$ are arbitrary constants (which may depend on λ). We shall seek to satisfy the non-linear system (11.1) with the aid of the formal series

$$y_j(\lambda, x) = {}_1y_j(\lambda, x) + {}_2y_j(\lambda, x) + \cdots + {}_\nu y_j(\lambda, x) + \cdots \qquad (j = 1, \cdots, n), \tag{12.13}$$

† (12.10c) is deleted if $m=1$.

where ${}_1y_j(\lambda, x)$ is given by (12.12). The ${}_\nu y_j(\lambda, x)$ will be determined as functions of the form

$$(12.13a)\qquad {}_\nu y_j(\lambda, x) = \sum_{k_1,\cdots,k_m} c_1^{k_1}\cdots c_m^{k_m}\,{}_\nu\eta_{k_1,\cdots,k_m;j}(\lambda, x)$$
$$(k_1, \cdots, k_m \geqq 0;\ k_1 + \cdots + k_m = \nu).$$

For $\nu = 1$ we may write

$$(12.13b)\qquad {}_0y_{\sigma,j}(\lambda, x) = {}_1\eta_{k_1,\cdots,k_m;j}(\lambda, x),$$

where

$$k_i = 0 \quad (\text{for } i \neq \sigma), \qquad k_\sigma = 1.$$

Substitution of (12.13) in (11.1) will result in a set of differential relations; the coefficients involved in the second members of (12.13a) must satisfy these relations. Formulas of this kind have been calculated in §2. It will be convenient to collect together, with some slight modifications, those of the formulas which will be needed in the sequel. We have

$$(12.14)\qquad L_j(\lambda, x, {}_H\eta_{h_1,\cdots,h_m;1}, \cdots, {}_H\eta_{h_1,\cdots,h_m;n}) = \lambda^p\,{}_H\Gamma_j^{h_1,\cdots,h_m}$$
$$(j = 1, \cdots, n;\ H = h_1 + \cdots + h_n;\ h_0 \geqq 0, \cdots, h_m \geqq 0;\ H = 2, 3, \cdots),$$

where

$$(12.14a)\qquad L_j \equiv {}_H\overset{(1)}{\eta}_{h_1,\cdots,h_m;j} - \lambda^p l_j(\lambda, x, {}_H\eta_{h_1,\cdots,h_m;1}, \cdots, {}_H\eta_{h_1,\cdots,h_m;n})$$

(cf. (11.1b)) and

$$(12.15)\qquad {}_H\Gamma_j^{h_1,\cdots,h_m} = \sum{}^{(4)}\,{}_ja_{i_1,\cdots,i_n}(\lambda, x)\prod_{\alpha=1}^{n}\sum_{\nu_1+\cdots+\nu_{i_\alpha}=\gamma_\alpha}\ \sum_{k_1^1,\cdots,k_m^{i_\alpha}}^{(2)}\ \prod_{r=1}^{i_\alpha}{}_\nu\eta_{k_1^r,\cdots,k_m^r;i_\alpha}$$
$$(h_1 + \cdots + h_m = H;\ h_1 \geqq 0, \cdots, h_m \geqq 0).$$

In (12.15)

$$(12.16)\qquad \sum{}^{(4)} = \sum_{s=2}^{H}\ \sum_{i_1+\cdots+i_n=s}\ \sum_{\gamma_1+\cdots+\gamma_n=H}\ \sum{}^{(3)}_{{}_1\delta_1,\cdots,{}_n\delta_m} \quad (\gamma_1 \geqq i_1, \cdots, \gamma_n \geqq i_n)$$

with

$$(12.16a)\qquad \sum{}^{(3)} = \sum\big[{}_1\delta_q + \cdots + {}_n\delta_q = h_q, \qquad (q = 1, \cdots, m);$$
$${}_1\delta_1 + \cdots + {}_1\delta_m = \gamma_1; \cdots; {}_n\delta_1 + \cdots + {}_n\delta_m = \gamma_n\big];$$

moreover,

$$\sum^{(2)} = \sum [k_1^1 + \cdots + k_1^{i_\alpha} = \delta_1;\ k_2^1 + \cdots + k_2^{i_\alpha} = \delta_2; \cdots;$$

(12.17) $$k_m^1 + \cdots + k_m^{i_\alpha} = \delta_m;\ k_1^1 + \cdots + k_m^1 = \nu_1; \cdots;$$

$$k_1^{i_\alpha} + \cdots + k_m^{i_\alpha} = \nu_{i_\alpha}].$$

Also, formally,

(12.18) $$y_j^{(1)}(x) - \lambda^p a_j(\lambda, x, y_1, \cdots, y_n) = \sum_{H=2}^{\infty} \sum_{h_1+\cdots+h_m=H} c_1^{h_1} \cdots c_m^{h_m} {}_H\psi_{h_1,\cdots,h_m;j},$$

where

(12.18a) $${}_H\psi_{h_1,\cdots,h_m;j} = L_j(\lambda, x, {}_H\eta_{h_1,\cdots,h_m;1}, \cdots, {}_H\eta_{h_1,\cdots,h_m;n}) - \lambda^p {}_H\Gamma_j^{h_1,\cdots,h_m}.$$

Consider a non-homogeneous system corresponding to (11.5),

(12.19) $$y_j^{(1)}(x) - \lambda^p l_j(\lambda, x, y_1, \cdots, y_n) = g_j \qquad (j = 1, \cdots, n).$$

We then have

(12.20) $$y_j(\lambda, x) = \sum_{\beta=1}^{n} c_\beta(\lambda, x)\, {}_0y_{\beta,j}(\lambda, x) \qquad (j = 1, \cdots, n),$$

where

(12.20a) $$c_\beta(\lambda, x) = \sum_{\tau=1}^{n} \int^x g_\tau(u)\, {}_0\bar{y}_{\tau,\beta}(\lambda, u)du,$$

provided that the integrals here involved converge. Here the ${}_0y_{\sigma,j}(\lambda, x)$ are from the matrix solution (12.11), and the ${}_0\bar{y}_{\tau,\sigma}(\lambda, u)$ are the elements of the inverse of the matrix $Y(\lambda, u)$; that is,

(12.20b) $$({}_0\bar{y}_{i,j}(\lambda, u)) = ({}_0y_{i,j}(\lambda, u))^{-1}.$$

An application of (12.19)–(12.20b) to (12.14) yields

(12.21) $${}_H\eta_{h_1,\cdots,h_m;j}(\lambda, x) = \sum_{\beta=1}^{n} {}_Hc_\beta(\lambda, x)\, {}_0y_{\beta,j}(\lambda, x),$$

(12.21a) $${}_Hc_\beta(\lambda, x) = \sum_{\tau=1}^{n} \int^x \lambda^p {}_H\Gamma_\tau^{h_1,\cdots,h_m}(u)\, {}_0\bar{y}_{\tau,\beta}(\lambda, u)du.$$

13. **Formal solutions for systems (C).** We shall proceed now with $\Gamma(\alpha', \beta'; R)$, $\Gamma(c, d; R')$ and $Y(\lambda, x)$ having the meaning assigned to these symbols in the italicized statement made in connection with (12.11). Thus

(13.1) $${}_0y_{\beta,j}(\lambda, x) = e^{G_\beta(\lambda,x)}\gamma_{\beta,j}(\lambda, x),$$

where

(13.1a) $$\gamma_{\beta,j}(\lambda, x) \underset{\alpha}{\sim} \{\lambda, x\}$$

(cf. Definition 1, §12) in $\Gamma(\alpha', \beta'; R)$. Definition 1 here is applied, of course, for the interval (α', β'). Taking account of (12.20b) and noting (formally) that $|S^*(\lambda, u)| \not\equiv 0$ (cf. (12.6a)), we have

(13.2) $$_0\bar{y}_{\tau,\beta}(\lambda, u) = e^{-G_\beta(\lambda,u)}\lambda^\omega \bar{\gamma}_{\tau,\beta}(\lambda, u),$$

(13.2a) $$\bar{\gamma}_{\tau,\beta}(\lambda, u) \underset{\bar{\alpha}}{\sim} \{\lambda, u\} \qquad (\omega \text{ an integer} \geqq 0),$$

where $\bar{\alpha} \leqq \alpha$ and $\bar{\alpha} \to \infty$ with α. Also

(13.3) $$|\gamma_{\beta,j}(\lambda, x)|, \quad |\bar{\gamma}_{\tau,\beta}(\lambda, x)| < \gamma \qquad (\text{in } \Gamma(\alpha', \beta'; R)).$$

In place of (12.21) and (12.21a) we now may write

(13.4) $$_H\eta_{h_1,\cdots,h_m;j}(\lambda, x) = \sum_{\beta=1}^{n} e^{G_\beta(\lambda,x)}\gamma_{\beta,j}(\lambda, x) \sum_{\tau=1}^{n} \int^x {}_HI_{\tau,\beta}(\lambda, u)du$$

(cf. (13.1a)),

(13.4a) $$_HI_{\tau,\beta}(\lambda, u) = {}_H\Gamma_\tau^{h_1,\cdots,h_m}(u)e^{-G_\beta(\lambda,u)}\lambda^{p+\omega}\bar{\gamma}_{\tau,\beta}(\lambda, u)$$
$$(j = 1, \cdots, n;\ H = h_1 + \cdots + h_m;\ h_1 \geqq 0, \cdots, h_m \geqq 0;\ H = 2, 3, \cdots)$$

(cf. (13.2a)).

For $H = 2$ (12.15) becomes

(13.5) $$_2\Gamma_j^{h_1,\cdots,h_m}(u) = \sum{}^{(4)} {}_ja_{i_1,\cdots,i_n}(\lambda, u) \prod_{\alpha=1}^{n} \sum_{\nu_1+\cdots+\nu i_\alpha = i_\alpha} \sum{}^{(2)} \prod_{r=1}^{i_\alpha} {}_{\nu_r}\eta_{k_1^r,\cdots,k_m^r;\alpha}(\lambda, u).$$

On the other hand, in view of (12.13b),

(13.6) $$_1\eta_{k_1^r,\cdots,k_m^r;\alpha}(\lambda, u) = \exp\left[k_1^r\bar{\gamma}_1(\lambda, u) + \cdots + k_m^rG_m(\lambda, u)\right][\lambda, u]_\alpha$$
$$(k_1^r + \cdots + k_m^r = 1;\ \alpha = 1, \cdots, n;\ \lambda, u \text{ in } \Gamma(\alpha', \beta'; R)).$$

Here and in the sequel $[\lambda, u]_\alpha$ is defined as follows.

DEFINITION 4. *Generically* $[\lambda, x]_\alpha$ (λ, x *in a region* $\Gamma(a, b; R)$; cf. Definition 2, §12) *is to denote a function defined and such that*

$$[\lambda, x]_\alpha \underset{\alpha}{\sim} \{\lambda, x\}$$

(cf. Definition 1, §2) *for* λ, x *in the above region.*†

† That is, $[\lambda, x]_\alpha = \sigma_0(x) + \sigma_1(x)\lambda^{-1/k} + \cdots + \sigma_{\alpha-1}(x)\lambda^{-(\alpha-1)/k} + \sigma_\alpha(x, \lambda)\lambda^{-\alpha/k}$, where $|\sigma_\alpha(x, \lambda)| < b$ in $\Gamma(a, b; R)$.

Generically $[\lambda, x]_\alpha\ [\lambda, x]_\alpha = [\lambda, x]_\alpha$ and $[\lambda, x]_\alpha + [\lambda, x]_\alpha = [\lambda, x]_\alpha$.

In consequence of (12.17) from (13.6) we obtain

$$(13.7)\qquad \prod_{r=1}^{i_\alpha} {}_r\eta_{k_1^r,\cdots,k_m^r;\alpha}(\lambda, u) = \exp\,[\delta_1 G_1(\lambda, u) + \cdots + \delta_m G_m(\lambda, u)]\,[\lambda, u]_\alpha.$$

Designate the expression displayed in (13.5) subsequent to the first product symbol as F_α'. In view of (13.7)

$$(13.7a)\qquad F_\alpha' = \text{generic form of (13.7)} \qquad (\text{with } \delta_i = {}_\alpha\delta_i).$$

Thus

$$(13.7b)\qquad \prod_{\alpha=1}^{n} F_\alpha' = \exp\,[({}_1\delta_1 + \cdots + {}_n\delta_1)G_1(\lambda, u) + \cdots + ({}_1\delta_m + \cdots + {}_n\delta_m)G_m(\lambda, u)]\,[\lambda, u]_\alpha$$

and, by (12.16) and (12.16a),

$$(13.7c)\qquad \prod_{\alpha=1}^{n} F_\alpha' = \exp\,[h_1 G_1(\lambda, u) + \cdots + h_m G_m(\lambda, u)]\,[\lambda, u]_\alpha.$$

Now, we may write

$${}_j a_{i_1,\cdots,i_n}(\lambda, u) = [\lambda, u]_\infty$$

(cf. (11.2)). Also we observe that $[\lambda, u]_\alpha\ [\lambda, u]_\infty = [\lambda, u]_\alpha$. Hence

$$(13.8)\qquad {}_j a_{i_1,\cdots,i_n}(\lambda, u) \prod_{\alpha=1}^{n} F_\alpha' = \text{generic form of (13.7c)}.$$

It is noted, next, that in (13.5)

$$\sum{}^{(4)} = \sum_{i_1+\cdots+i_n=2} \sum_{\gamma_1+\cdots+\gamma_n=2} \sum{}^{(3)}_{{}_1\delta_{\gamma_1},\cdots,{}_n\delta_{\gamma_n}} \qquad (\gamma_1 = i_1, \cdots, \gamma_n = i_n).$$

Whence, by (13.5) and (13.8),

$$(13.9)\qquad {}_2\Gamma_j^{h_1,\cdots,h_m}(u) = \exp\,[h_1 G_1(\lambda, u) + \cdots + h_m G_m(\lambda, u)]\,[\lambda, u]_\alpha$$

for λ, u in $\Gamma(\alpha', \beta'; R)$. Thus, in view of (13.4a) (for $n=2$) and (13.2a),

$$(13.10)\qquad {}_2I_{\tau,\beta}(\lambda, u) = \exp\,[h_1 G_1(\lambda, u) + \cdots + h_m G_m(\lambda, u) - G_\beta(\lambda, u)]\,\lambda^{p+\omega}[\lambda, u]_{\bar\alpha} \qquad (h_1, \cdots, h_m \geqq 0;\ h_1 + \cdots + h_m = 2)$$

when λ, u is in $\Gamma(\alpha', \beta'; R)$.

The following lemma will now be proved.

LEMMA 1. *Let $\Gamma(c, d; R')$ denote an admissible region* (cf. Definition 3, §12). *Then, with suitable notation, we have in it* (12.10), (12.10a), and (12.10c).

Let

$$Q(\lambda, x) = h_1Q_1(\lambda, x) + h_2Q_2(\lambda, x) + \cdots + h_mQ_m(\lambda, x) - Q_\beta(\lambda, x) \tag{13.11}$$
$$(h_1, \cdots, h_m \geqq 0;\ h_1 + \cdots + h_m = \nu \geqq 2;\ \textit{the } h_i \textit{ integers}).$$

Assume, moreover, that in $Q^{(1)}(\lambda, x)$ *(unless* $Q^{(1)}(\lambda, x) \equiv 0$*) the coefficient of the highest power of* λ *does not vanish in the interval (closed)* (a, b)*. We then can evaluate the integral below so that, with* $w \geqq l+1$*, we have*

$$\int^x e^{Q(\lambda, u)}[\lambda, u]_w du = e^{Q(\lambda, x)}[\lambda, x]_{w-1} \tag{13.12}$$

(cf. Definition 4), *provided that the displayed integrand is of the stated form in* $\Gamma(c, d; R')$*. Formula* (13.12) *will be valid for* λ, x *in* $\Gamma(c, d; R')$*. Furthermore, if the highest power of* λ *in* $Q^{(1)}(\lambda, x)$ *is the* (l/k)*th* $(l/k > 0$*; i.e.,* $Q^{(1)}(\lambda, x) \not\equiv 0)$ *the coefficients of* $\lambda^{-\nu/k}$ $(\nu = 0, \cdots, l-1)$ *in the second member of* (13.12) *will all be zero.*

Note. If we assume that this lemma is true, it will follow that the facts stated therein will continue to hold when the $Q_i(\lambda, x)$ are replaced by the $G_i(\lambda, x)(=Q_i(\lambda, x) - Q_i(\lambda, \alpha'))$ $(i = 1, \cdots, n)$, respectively.

The truth of the above lemma would follow from the developments of §4 of (T_3) if it were shown that

$$RQ^{(1)}(\lambda, x) \leqq 0 \qquad (\lambda, x \text{ in } \Gamma(c, d; R')). \tag{13.13}$$

Suppose $\beta > m$. Some h_i will be positive. We then have

$$Q^{(1)}(\lambda, x) = [h_1Q_1^{(1)}(\lambda, x) + \cdots + (h_i - 1)Q_i^{(1)}(\lambda, x) + \cdots + h_mQ_m^{(1)}(\lambda, x)] + [Q_i^{(1)}(\lambda, x) - Q_\beta^{(1)}(\lambda, x)]; \tag{13.13a}$$

and (13.13) will follow in consequence of (12.10a) and (12.10).†

Consider now the case when $\beta \leqq m$. If $h_\beta > 0$ and if we write

$$Q^{(1)}(\lambda, x) = h_1Q_1^{(1)}(\lambda, x) + \cdots + (h_\beta - 1)Q_\beta^{(1)}(\lambda, x) + \cdots + h_mQ_m^{(1)}(\lambda, x), \tag{13.13b}$$

the truth of (13.13) will be inferred in consequence of (12.10a). If $h_\beta = 0$ but some h_i, with $i < \beta$, is positive, write (13.13a); the validity of (13.13) is then deduced in view of (12.10) and (12.10a). The remaining case is the case when

$$Q(\lambda, x) = -Q_\beta(\lambda, x) + h_{\beta+1}Q_{\beta+1}(\lambda, x) + \cdots + h_mQ_m(\lambda, x) \tag{13.14}$$
$$(h_{\beta+1}, h_{\beta+2}, \cdots, h_m \geqq 0;\ h_{\beta+1} + \cdots + h_m = \nu \geqq 2;\ \beta < m).$$

† $R(Q_i^{(1)} - Q_\beta^{(1)}) \leqq 0$ since $i < \beta$.

In treating the case when (13.14) holds we make use of (12.10c). In view of (12.10) and (12.10c), whenever

$$1 \leqq \beta < i \leqq j \leqq m, \tag{13.15}$$

one has

$$RQ_\beta^{(1)}(\lambda, x) \geqq [RQ_1^{(1)}(\lambda, x) \geqq RQ_m^{(1)}(\lambda, x) + RQ_m^{(1)}(\lambda, x) \geqq]RQ_i^{(1)}(\lambda, x) + RQ_j^{(1)}(\lambda, x) \quad (\text{in } \Gamma(c, d; R')). \tag{13.15a}$$

Since in (13.14) $h_{\beta+1} + \cdots + h_m \geqq 2 \quad (h_{\beta+1}, \cdots, h_m \geqq 0)$,

$$Q^{(1)}(\lambda, x) = [-Q_\beta^{(1)}(\lambda, x) + Q_i^{(1)}(\lambda, x) + Q_j^{(1)}(\lambda, x)] + k_{\beta+1}Q_\beta^{(1)}(\lambda, x) + k_{\beta+2}Q_{\beta+2}^{(1)}(\lambda, x) + \cdots + k_mQ_m^{(1)}(\lambda, x) \quad (k_{\beta+1}, \cdots, k_m \geqq 0) \tag{13.16}$$

for some pair of integers i, j satisfying (13.15). By (13.15a) the real part of the expression within the brackets displayed above is equal to or is less than zero for λ, x in $\Gamma(c, d; R')$. Hence, in view of (13.16) and (12.10a), (13.13) will necessarily hold. Thus (13.13), as well as the lemma, has been established.

Henceforth the developments will be given for *a region* $\Gamma(c', d'; R')$, *where* R' *is the same as in* $\Gamma(c, d; R')$, *and where* (c', d') *is an interval contained in* (c, d) *such that in each of the expressions*

$$Q^{(1)}(\lambda, x) = h_1Q_1^{(1)}(\lambda, x) + \cdots + h_mQ_m^{(1)}(\lambda, x) - Q_\beta^{(1)}(\lambda, x)$$
$$(h_1 \geqq 0, \cdots, h_m \geqq 0;\ h_1 + \cdots + h_m = \nu;\ \textit{the } h_i \textit{ integers};\ \beta = 1, \cdots, n),$$

formed for $\nu = 2, 3, \cdots, N$, *the coefficient of the highest power of* λ *is either* $=0$ *or* $\neq 0$ *throughout* (c', d').† *The choice of* (c', d') *may depend on* N.

In view of Lemma 1 and of the relation (13.10) it is concluded that

$$\int_x {}_2I_{\tau,\beta}(\lambda, u)du = \exp[h_1G_1(\lambda, x) + \cdots + h_mG_m(\lambda, x) - G_\beta(\lambda, x)]\lambda^{p+\omega}[\lambda, x]_{\bar{a}-1} \tag{13.17}$$

in $\Gamma(c', d'; R')$. Substitution of (13.17) in (13.4) (for $n = 2$) will yield, in consequence of (13.1a),

$${}_2\eta_{h_1,\cdots,h_m;j}(\lambda, x) = \exp[h_1G_1(\lambda, x) + \cdots + h_mG_m(\lambda, x)]\lambda^{p+\omega}[\lambda, x]_{\bar{a}-1} \quad (h_1 + \cdots + h_m = 2;\ \lambda, x \text{ in } \Gamma(c', d'; R')). \tag{13.18}$$

By virtue of (13.6) and (13.18)

† When the coefficient of the highest power of λ is $=0$ throughout (c', d'), $Q^{(1)}(\lambda, x) \equiv 0$ (for x in c', d').

$$
\begin{aligned}
{}_{\nu_r}\eta_{k_1^r,\cdots,k_m^r;\alpha}(\lambda, x) &= \exp\,[k_1^r G_1(\lambda, x) + \cdots \\
&\quad + k_m^r G_m(\lambda, x)]\lambda^{(\nu_r-1)(p+\omega)}[\lambda, x]_{\bar{\alpha}-1} \qquad (13.19)\\
&(k_1^r, \cdots, k_m^r \geqq 0;\ k_1^r + \cdots + k_m^r = \nu_r;\ \alpha = 1, \cdots, n;\ \nu_r = 1, 2)
\end{aligned}
$$

for λ, x in $\Gamma(c', d'; R')$. Substitute (13.19) in (12.15) (for $H=3$). Taking account of (12.17) we have

$$
\prod_{r=1}^{i_\alpha} \cdots = \exp\,[\delta_1 G_1(\lambda, x) + \cdots + \delta_m G_m(\lambda, x)]\lambda^{\nu(\gamma_\alpha, i_\alpha)(p+\omega)}[\lambda, x]_{\bar{\alpha}-1}, \qquad (13.20)
$$

where

$$
\nu(\gamma_\alpha, i_\alpha) = \sum_{r=1}^{i_\alpha} (\nu_r - 1) = \gamma_\alpha - i_\alpha. \qquad (13.20a)
$$

We next have

$$
F_\alpha' = \sum_{\nu_1+\cdots+\nu_{i_\alpha}=\gamma} \sum^{(2)} \prod_{r=1}^{i_\alpha} \cdots = \text{generic form of (13.20)} \qquad (\text{in } \Gamma(c', d'; R')).
$$

If we write $\delta_i = {}_\alpha\delta_i$, in consequence of the equalities of (12.16a), it is concluded that

$$
\prod_{\alpha=1}^{n} F_\alpha' = \exp\,[h_1 G_1(\lambda, x) + \cdots + h_m G_m(\lambda, x)]\lambda^{\nu'(H,s)(p+\omega)}[\lambda, x]_{\bar{\alpha}-1}, \qquad (13.21)
$$

where

$$
\nu'(3, s) = \sum_{\alpha=1}^{n} (\gamma_\alpha - i_\alpha) = 3 - s, \qquad (13.21a)
$$

by (12.16). The product of the functions of (13.21) by ${}_j a_{i_1,\cdots,i_n}(\lambda, x)$ is a function possessing the generic form of the second member of (13.21). Hence

$$
F_s'' = \sum_{i_1+\cdots+i_n=s} \sum_{\gamma_1+\cdots+\gamma_n=3} \sum_{{}_1\delta_1,\cdots}^{(3)} \cdots = \text{generic form of (13.21)}.
$$

Since,† for $s=2, 3$, we have

$$
\lambda^{(3-s)(p+\omega)}[\lambda, x]_{\bar{\alpha}-1} = \lambda^{p+\omega}[\lambda, x]_{\bar{\alpha}-1},
$$

from (12.15) (for $n=3$) and (12.16) it follows that

$$
\begin{aligned}
{}_3\Gamma_j^{h_1,\cdots,h_m} &= \sum_{s=2}^{3} F_s'' = \exp\,[h_1 G_1(\lambda, x) + \cdots + h_m G_m(\lambda, x)]\lambda^{p+\omega}[\lambda, x]_{\bar{\alpha}-1} \qquad (13.22)\\
&(h_1, \cdots, h_m \geqq 0;\ h_1 + \cdots + h_m = 3;\ \text{in } \Gamma(c', d'; R')).
\end{aligned}
$$

† $\lambda^{-(p+\omega)}[\lambda, x]_{\bar{\alpha}-1} = [\lambda, x]_{\bar{\alpha}-1}$.

Assume that† *within* $\Gamma(c', d'; R')$, *for some* μ $(3 \leqq \mu \leqq N)$, *we have*

$$(13.23)\quad {}_{q+1}\Gamma_j^{h_1,\cdots,h_m} = \exp\left[h_1G_1(\lambda, x) + \cdots + h_mG_m(\lambda, x)\right]\lambda^{(q-1)(p+\omega)}[\lambda, x]_{\bar{\alpha}-q+1}$$
$$(h_1, \cdots, h_m \geqq 0;\ h_1 + \cdots + h_m = q+1;\ j = 1, \cdots, n),$$

$$(13.24)\quad {}_q\eta_{k_1,\cdots,k_m;\alpha}(\lambda, x) = \exp\left[k_1G_1(\lambda, x) + \cdots + k_mG_m(\lambda, x)\right]\lambda^{(q-1)(p+\omega)}[\lambda, x]_{\bar{\alpha}-q+1}$$
$$(k_1, \cdots, k_m \geqq 0;\ k_1 + \cdots + k_m = q;\ \alpha = 1, \cdots, n)$$

for $q = 1, 2, \cdots, \mu-1$. For $\mu = 3$ this has been previously established in (13.6), (13.9), (13.18), and (13.22).

Substitution of (13.23) (for $q = \mu-1$) in (13.4a) will yield

$$(13.25)\quad {}_{\mu}I_{\tau,\beta}(\lambda, u) = \exp\left[h_1G_1(\lambda, u) + \cdots + h_mG_m(\lambda, u) - G_\beta(\lambda, u)\right]\lambda^{(\mu-1)(p+\omega)}[\lambda, u]_{\bar{\alpha}-\mu+2}$$
$$(h_1, \cdots, h_m \geqq 0;\ h_1 + \cdots + h_m = \mu;\ \lambda, u \text{ in } \Gamma(c', d'; R')).$$

Since $h_1 + \cdots + h_m = \mu \leqq N$, in view of the italicized statement preceding (13.17) it is observed that Lemma 1 (together with the appended note) is applicable for the evaluation of

$$(13.25a)\quad \int^x {}_{\mu}I_{\tau,\beta}(\lambda, u)du.$$

Accordingly (13.25a) will have the form

$$\exp\left[h_1G_1(\lambda, u) + \cdots + h_mG_m(\lambda, u) - G_\beta(\lambda, u)\right]\lambda^{(\mu-1)(p+\omega)}[\lambda, x]_{\bar{\alpha}-\mu+1}.$$

Substitution of this in (13.4) will give

$$(13.26)\quad {}_{\mu}\eta_{h_1,\cdots,h_m;j}(\lambda, x) = \exp\left[h_1G_1(\lambda, x) + \cdots + h_mG_m(\lambda, x)\right]\lambda^{(\mu-1)(p+\omega)}[\lambda, x]_{\bar{\alpha}-\mu+1}$$
$$(\lambda, x \text{ in } \Gamma(c', d'; R')).$$

Thus (13.24) holds for $q = 1, 2, \cdots, \mu$.

In (12.15) replace μ by $\mu+1$ and substitute (13.24) for $q = 1, 2, \cdots, \mu$. Since (12.17) holds, and since $\nu_1 + \cdots + \nu_{i_\alpha} = \gamma_\alpha$,

$$(13.27)\quad \prod_{r=1}^{i_\alpha} \cdots = \exp\left[\delta_1G_1(\lambda, x) + \cdots + \delta_mG_m(\lambda, x)\right]\lambda^{(\gamma_\alpha - i_\alpha)(p+\omega)}[\lambda, x]_{\bar{\alpha}-\mu+1}.$$

We next obtain

$$F_\alpha' = \sum_{\nu_1+\cdots+\nu_{i_\alpha}=\gamma} \sum^{(2)} \prod_{r=1}^{i_\alpha} \cdots = \text{generic form of (13.27)} \qquad (\delta_i = {}_\alpha\delta_i),$$

† We take α (hence $\bar{\alpha}$ also) suitably great.

and, by (12.16a) and (12.16) (with H replaced by $H+1$),

$$\prod_{\alpha=1}^{n} F_\alpha' = \exp\left[h_1 G_1(\lambda, x) + \cdots + h_m G_m(\lambda, x)\right]\lambda^{(H+1-s)(p+\omega)}[\lambda, x]_{\bar{\alpha}-H+1}$$

(13.27a)

$$= \exp\left[h_1 G_1(\lambda, x) + \cdots + h_m G_m(\lambda, x)\right]\lambda^{(H-1)(p+\omega)}[\lambda, x]_{\bar{\alpha}-H+1}$$

$$(s = 2, \cdots, H+1; \text{ in } \Gamma(c', d'; R')).$$

Finally, by (12.16) (with H replaced by $H+1$),

$${}_{H+1}\Gamma_j^{h_1,\cdots,h_m} = \sum^{(4)} {}_j a_{i_1,\cdots,i_n}(\lambda, x) \prod_{\alpha=1}^{n} F_\alpha'$$

$$= \text{generic form of third member of (13.27a).}$$

Thus relations (13.23), (13.24) *are valid in* $\Gamma(c', d'; R')$ *for* $q = 1, 2, \cdots, N$.

LEMMA 2. *Let N be an integer, however large. Let $\Gamma(c, d; R')$ be an admissible region* (cf. Definition 3, §12) *and $\Gamma(\alpha', \beta'; R)$ be a proper region* (cf. Definition 2, §12) *containing $\Gamma(c, d; R')$. With suitable notation we thus have* (12.10), (12.10a), (12.10b), and (12.10c).

Let (c', d') be an interval contained in (c, d) and satisfying the condition of the italicized statement preceding (13.17).

The system (11.1) *possesses a formal solution represented by the series $s_j(\lambda, x)$ $(j = 1, \cdots, n)$ given by* (12.13) *and* (12.13a). *The coefficients*

$$(13.28) \qquad {}_\nu\eta_{k_1,\cdots,k_m;j}(\lambda, x) \qquad (k_1, \cdots, k_m \geqq 0;\ k_1 + \cdots + k_m = \nu),$$

involved in (12.13a) *can be determined by means of* (13.4), (13.4a) *and the formulas* (12.15), (12.16), *and* (12.17). *For $\nu = 1, 2, \cdots, N$ the functions* (13.28) *may be taken of the form*

$${}_\nu\eta_{k_1,\cdots,k_m;j}(\lambda, x)$$

(13.28a)

$$= \exp\left[k_1 G_1(\lambda, x) + \cdots + k_m G_m(\lambda, x)\right]\lambda^{(\nu-1)(p+\omega)}[\lambda, x]_{\bar{\alpha}-\nu+1}$$

$$(k_1, \cdots, k_m \geqq 0;\ k_1 + \cdots + k_m = \nu;\ j = 1, \cdots, n),$$

formulas (13.28a) *being valid in $\Gamma(c', d'; R')$. The symbol $[\lambda, x]_{\bar{\alpha}-\nu+1}$ is defined (in $\Gamma(c', d'; R')$) according to Definition 4. We may take $\bar{\alpha}$ as great as desired; however, the functions* (13.28) *will in general depend on $\bar{\alpha}$.*

14. **Corresponding transformations.** The system (11.1) will be transformed with the aid of

$$(14.1) \qquad y_j(x) = Y_j(\lambda, x) + \rho_j(\lambda, x) \qquad (j = 1, \cdots, n),$$

where

$$(14.1a) \qquad Y_j(\lambda, x) = {}_1y_j(\lambda, x) + {}_2y_j(\lambda, x) + \cdots + {}_{N-1}y_j(\lambda, x).$$

Here the ${}_\nu y_j(\lambda, x)$ $(\nu = 1, \cdots, N-1)$ are the functions involved in the formal series solutions $s_j(\lambda, x)$, referred to in Lemma 2, §13. The arbitrary constants $c_1, \cdots, c_m$, involved in the ${}_\nu y_j(\lambda, x)$ may be taken as functions of λ. *We shall take the* $c_i = c_i(\lambda)$ *analytic in* R' $(\lambda \neq \infty)$ *and such that*

$$|c_i| \leqq \gamma(\lambda);\ \gamma(\lambda) \sim 0; \qquad \gamma(\lambda) \leqq \gamma_1 \qquad (i = 1, \cdots, m;\ \lambda \text{ in } R'). \tag{14.2}$$

In view of (12.10b) it follows from (13.28a) that

$$|{}_\nu\eta_{k_1,\cdots,k_m;j}(\lambda, x)| < |\lambda|^{(\nu-1)(p+\omega)} b_\nu' \tag{14.3}$$

$$(k_1, \cdots, k_m \geqq 0;\ k_1 + \cdots + k_m = \nu;\ j = 1, \cdots, n;\ \nu = 1, 2, \cdots, N-1)$$

for λ, x in $\Gamma(c', d'; R')$. Here b_ν' is independent of λ and x. By (14.2), (14.3), and (12.13a)

$${}_\nu y_j(\lambda, x) \sim 0; \quad |{}_\nu y_j(\lambda, x)| < \gamma^\nu(\lambda) |\lambda|^{(\nu-1)(p+\omega)} b_\nu'' \qquad (\text{in } \Gamma(c', d'; R')).$$

Hence, since $\gamma(\lambda) \sim 0$,

$$Y_j(\lambda, x) \sim 0; \quad |Y_j(\lambda, x)| < \gamma(\lambda) b' \qquad (\text{in } \Gamma(c', d'; R')). \tag{14.4}$$

Here b' is independent of x, λ. Either by taking $\gamma_1 (>0)$ sufficiently small or c' sufficiently great, we secure

$$|Y_j(\lambda, x)| \leqq \rho' \qquad (j = 1, \cdots, n;\ \text{in } \Gamma(c', d'; R');\ |\lambda| \geqq r'), \tag{14.4a}$$

where ρ' is independent of λ, x and

$$0 < \rho' < \rho, \tag{14.4b}$$

where ρ is defined as in (11.4).

In consequence of (11.1a) and (11.1b) substitution of (14.1) in (11.1) will yield

$$\begin{aligned} &\lambda^{-p} \rho_j^{(1)}(\lambda, x) - l_j(\lambda, x, \rho_1, \cdots, \rho_n) \\ &\quad = -\lambda^{-p} Y_j^{(1)}(\lambda, x) + l_j(\lambda, x, Y_1, \cdots, Y_n) \\ &\qquad + q_j(\lambda, x, Y_1 + \rho_1, \cdots, Y_n + \rho_n). \end{aligned} \tag{14.5}$$

Here

$$\begin{aligned} q_j(\lambda, x, Y_1 + \rho_1, \cdots) &= q_j(\lambda, x, Y_1, \cdots, Y_n) \\ &\quad + \sum_{\nu_1 + \cdots + \nu_n \geqq 1} \alpha_{\nu_1, \cdots, \nu_n; j}(\lambda, x) \rho_1^{\nu_1} \cdots \rho_n^{\nu_n}, \end{aligned} \tag{14.5a}$$

$$\nu_1! \cdots \nu_n! \alpha_{\nu_1, \cdots, \nu_n; j}(\lambda, x) = \frac{\partial^{\nu_1 + \cdots + \nu_n}}{\partial^{\nu_1} y_1 \cdots \partial^{\nu_n} y_n} q_j(\lambda, x, y_1, \cdots, y_n)\Big] \tag{14.5b}$$

$$(y_i = Y_i(\lambda, x);\ i = 1, \cdots, n),$$

the series above being absolutely and uniformly convergent in $\Gamma(c', d'; R')$ ($|\lambda| \geqq r'$) provided

$$|\rho_1|, \cdots, |\rho_n| \leqq \rho'' \qquad (0 < \rho'' < \rho - \rho') \tag{14.6}$$

(cf. (14.4b)).†

By (14.5b) and (11.1c)

$$\alpha_{\nu_1,\cdots,\nu_n}(\lambda, x) = \sum_{\lambda_1,\cdots,\lambda_n} {}_j a_{\nu_1+\lambda_1,\cdots,\nu_n+\lambda_n}(\lambda, x) C_{\lambda_1}^{\nu_1+\lambda_1} \cdots C_{\lambda_n}^{\nu_n+\lambda_n} Y_1^{\lambda_1} \cdots Y_n^{\lambda_n} \tag{14.7}$$

$$(\lambda_1, \cdots, \lambda_n \geqq 0; \lambda_1 + \cdots + \lambda_n \geqq 2 - (\nu_1 + \cdots + \nu_n)).$$

Thus, in particular,

$$\alpha_{\nu_1,\cdots,\nu_n;j}(\lambda, x) = \sum_{\lambda_1+\cdots+\lambda_n \geqq 1} \cdots \qquad (\nu_1 + \cdots + \nu_n = 1),$$

so that, by (14.7) and (14.4),

$$\alpha_{\nu_1,\cdots,\nu_n;j}(\lambda, x) \sim 0 \qquad (\text{in } \Gamma(c', d'; R'); \nu_1 + \cdots + \nu_n = 1). \tag{14.7a}$$

On the other hand,

$$\alpha_{\nu_1,\cdots,\nu_n;j}(\lambda, x) = {}_j a_{\nu_1,\cdots,\nu_n}(\lambda, x) + \sum_{\lambda_1+\cdots+\lambda_n \geqq 1} \cdots \qquad (\nu_1 + \cdots + \nu_n \geqq 2);$$

thus, in view of (14.7),

$$\alpha_{\nu_1,\cdots,\nu_n;j}(\lambda, x) = {}_j a_{\nu_1,\cdots,\nu_n}(\lambda, x) + {}_j\beta_{\nu_1,\cdots,\nu_n}(\lambda, x), \tag{14.7b}$$

where

$${}_j\beta_{\nu_1,\cdots,\nu_n}(\lambda, x) \sim 0 \qquad (\nu_1 + \cdots + \nu_n \geqq 2; \text{ in } \Gamma(c', d'; R')). \tag{14.7c}$$

By (14.5a), (14.7a), (14.7b), (14.7c), and (14.5)

$$\lambda^{-p} \rho_j^{(1)}(\lambda, x) - l_j^*(\lambda, x, \rho_1, \cdots, \rho_n) = q_j^*(\lambda, x, \rho_1, \cdots, \rho_n) - F_j(\lambda, x) \tag{14.8}$$

$$(j = 1, \cdots, n).$$

Here

$$F_j(\lambda, x) = \lambda^{-p} Y_j^{(1)}(\lambda, x) - l_j(\lambda, x, Y_1, \cdots, Y_n) - q_j(\lambda, x, Y_1, \cdots, Y_n) \tag{14.8a}$$

and

$$l_j^* \sim l_j; \quad q_j^* \sim q_j \qquad (\text{in } \Gamma(c', d'; R')). \tag{14.9}$$

The relations (14.9) *are to be understood as follows. The* $l_j^* - l_j$ *are linear in* $\rho_1, \cdots, \rho_n$ *and the* $q_j^* - q_j$ *are series in non-negative integral powers of* $\rho_1, \cdots, \rho_n$; *the coefficients of the various powers of* $\rho_1, \cdots, \rho_n$ *are functions of* x *and* λ *which are all* ~ 0 *in* $\Gamma(c', d'; R')$.

† ρ'' is independent of x and λ.

It remains to study the function (14.8a). By (12.18) and (12.18a) in consequence of (14.8a) we have

$$F_j(\lambda, x) = \lambda^{-p}[Y_j^{(1)}(\lambda, x) - \lambda^p a_j(\lambda, x, Y_1, \cdots, Y_n)] \tag{14.10}$$
$$= \sum_{H=2}^{\infty} \sum_{h_1+\cdots+h_m=H} c_1^{h_1} \cdots c_m^{h_m} \lambda^{-p}\, {}_H\psi^*_{h_1,\cdots,h_m;j},$$

where

$${}_H\psi^*_{h_1,\cdots,h_m;j} = {}_H\psi_{h_1,\cdots,h_m;j}, \tag{14.11}$$

provided that in the second member of (14.11) we replace the

$${}_\nu\eta_{k_1,\cdots,k_m;\alpha}(\lambda, x) \qquad (\nu \geqq N) \tag{14.12}$$

by zeros.† Thus by (12.18a)

$${}_H\psi^*_{h_1,\cdots,h_m;j} = {}_H\psi_{h_1,\cdots,h_m;j} \qquad (H \leqq N-1), \tag{14.12a}$$

$${}_N\psi^*_{h_1,\cdots,h_m;j} = -\lambda^p\, {}_N\Gamma_j^{h_1,\cdots,h_m}, \tag{14.12b}$$

$${}_H\psi^*_{h_1,\cdots,h_m;j} = -\lambda^p\, {}_H\Gamma_j^{h_1,\cdots,h_m,*} \qquad (H > N). \tag{14.12c}$$

In (14.12c)

$${}_H\Gamma_j^{h_1,\cdots,h_m,*} = {}_H\Gamma_j^{h_1,\cdots,h_m} \qquad (H > N), \tag{14.13}$$

provided that in the second members the functions (14.12) are replaced by zeros.

It is to be recalled that the

$${}_\nu\eta_{k_1,\cdots,k_m;\alpha}(\lambda, x) \qquad (\nu \leqq N-1)$$

have been previously determined so that

$${}_H\psi_{h_1,\cdots,h_m;j} = 0 \qquad (H \leqq N-1). \tag{14.14}$$

If we take account of the statement preceding Lemma 2, §13, the form of the function of (14.12b) is inferred with the aid of (13.23) (for $q = N-1$). Thus,

$${}_N\psi^*_{h_1,\cdots,h_m;j}$$
$$= \exp\,[h_1G_1(\lambda, x) + \cdots + h_mG_m(\lambda, x)]\lambda^{(N-2)(p+\omega)+p}[\lambda, x]_{\bar{\alpha}-N+2} \tag{14.14a}$$
$$(h_1 + \cdots + h_m = N;\ \text{in } \Gamma(c', d'; R')).$$

The form of the functions (14.13) can be established as follows.

† When the functions (14.12) have been replaced by zeros, the formal series $s_j(\lambda, x) = {}_1y_j(\lambda, x) + {}_2y_j(\lambda, x) + \cdots$ reduce to the functions $Y_j(\lambda, x)$, respectively. On the other hand, (12.18) and (12.18a) have been obtained as a result of substituting in $y_j^{(1)}(\lambda, x) - \lambda^p a_j(\lambda, x, y_1, \cdots, y_n)$ the series $s_4(\lambda, x)$.

Since, for $\nu \leqq N-1$, $\bar{\alpha}-\nu+1 \geqq \bar{\alpha}-N+2$, in consequence of (13.24) (for $q=1, \cdots, N-1$) we have

$$(14.15)\qquad {}_{\nu}\eta_{k_1,\cdots,k_m;\alpha}(\lambda, x) = \exp\left[k_1G_1 + \cdots + k_mG_m\right]\lambda^{(\nu-1)(p+\omega)}[\lambda, x]_{\bar{\alpha}-N+2}$$
$$(k_1 + \cdots + k_m = \nu;\ \nu = 1, \cdots, N-1;\ \text{in } \Gamma(c', d'; R')).$$

With the functions (14.12) replaced by zeros, we may write

$$(14.15a)\qquad {}_{\nu}\eta_{k_1,\cdots,k_m;\alpha}(\lambda, x) = 0 = \text{generic form of the second member of (14.15)}$$
$$(k_1 + \cdots + k_m = \nu;\ \nu = N, N+1, \cdots;\ \text{in } \Gamma(c', d'; R')).$$

If (14.15) and (14.15a) are substituted in (12.15), in view of the definition of the functions (14.13) it is inferred that

$$(14.16)\qquad {}_{H}\Gamma_j^{h_1,\cdots,h_m,*} = \exp\left[h_1G_1(\lambda, x) + \cdots + h_mG_m(\lambda, x)\right]\lambda^{(H-2)(p+\omega)}[\lambda, x]_{\bar{\alpha}-N+2}$$
$$(h_1 + \cdots + h_m = H;\ H = N+1, N+2, \cdots;\ \text{in } \Gamma(c', d'; R')).$$

By (14.14)

$$F_j(\lambda, x) = \sum_{H \geqq N} \sum_{h_1+\cdots+h_m=H} \lambda^{-p} c_1^{h_1} \cdots c_m^{h_m}\, {}_{H}\psi^*_{h_1,\cdots,h_m;j}.$$

Thus, in consequence of (14.2), (14.14a), (14.12c), and (14.16)

$$(14.17)\qquad |F_j(\lambda, x)| < k_0(\gamma(\lambda))^N |\lambda|^{(N-2)(p+\omega)} \sum_{h_1+\cdots+h_m=N} g_{h_1,\cdots,h_m}(\lambda, x)$$
$$(j = 1, \cdots, n;\ \text{in } \Gamma(c', d'; R')),$$

where

$$(14.18)\qquad g_{h_1,\cdots,h_m}(\lambda, x) = |\exp\left[h_1G_1(\lambda, x) + \cdots + h_mG_m(\lambda, x)\right]|.$$

LEMMA 3. *The transformation* (14.1), (14.1a) (cf. the italics subsequent to (14.1a)), *applied to the system* (11.1), *will yield the system*

$$(14.19)\qquad \lambda^{-p}\rho_j^{(1)}(\lambda, x) - l_j^*(\lambda, x, \rho_1, \cdots, \rho_n) = q_j^*(\lambda, x, \rho_1, \cdots, \rho_n) - F_j(\lambda, x)$$
$$(j = 1, \cdots, n).$$

The l_j^* and q_j^*, here involved, satisfy (14.9) *in the sense of the italicized statement subsequent to* (14.9). *The series*

$$(14.19a)\qquad q_j^*(\lambda, x, \rho_1, \cdots, \rho_n) = \sum_{\nu_1+\cdots+\nu_n \geqq 2} \alpha_{\nu_1,\cdots,\nu_n;j}(\lambda, x)\rho_1^{\nu_1} \cdots \rho_n^{\nu_n}$$

will converge absolutely and uniformly in $\Gamma(c', d'; R')$ $(|\lambda| \geqq r')$, *provided* (14.6) *holds. Moreover, the* $F_j(\lambda, x)$ *satisfy* (14.17), (14.18). *The above is asserted under the supposition that the arbitrary constants (functions of* λ) c_i *satisfy* (14.2); *furthermore,* γ_1 *and* r' *are to be suitably chosen so that* (14.4a), (14.4b) *hold.*

15. **The Third Existence Theorem.** We now have the following statement which is analgous to a statement in §7.

Let λ, x *be in* $\Gamma(c', d'; R')$ $(|\lambda| \geqq r')$. *Suppose*

$$|\rho_i| \leqq \rho^*, \qquad |w_i| \leqq w^* \qquad (i = 1, \cdots, n) \tag{15.1}$$

with

$$\rho^* + w^* \leqq \rho''. \tag{15.1a}$$

For the function (14.19a) *we then shall have*

$$\begin{aligned} &|q_j^*(\lambda, x, \rho_1 + w_1, \cdots, \rho_n + w_n) - q_j^*(\lambda, x, \rho_1, \cdots, \rho_n)| \\ &\quad < (q'\rho^* + qw^*)w^* \quad (\lambda, x \text{ in } \Gamma(c', d'; R'); q', q \text{ independent of } \lambda, x). \end{aligned} \tag{15.2}$$

We now write

$$\rho_{j:k} - \rho_{j:k-1} = w_{j:k} \qquad (k = 0, 1, \cdots; \rho_{j:-1} = 0), \tag{15.3}$$

$$\rho_{j:k} = w_{j:0} + w_{j:1} + \cdots + w_{j:k} \qquad (k = 0, 1, \cdots) \tag{15.3a}$$

and consider the succession of differential systems

$$\begin{aligned} \lambda^{-p} w_{j:k}^{(1)} - l_j^*(\lambda, x, w_{1:k}, \cdots, w_{n:k}) &= g_{j:k} \\ \equiv q_j^*(\lambda, x, \rho_{1:k-2} + w_{1:k-1}, \cdots, \rho_{n:k-2} + w_{n:k-1}) \\ - q_j^*(\lambda, x, \rho_{1:k-2}, \cdots, \rho_{n:k-2}) \\ (j = 1, \cdots, n; k = 0, 1, \cdots; g_{j:0} = -{}_jF(\lambda, x)). \end{aligned} \tag{15.4}$$

A solution will be obtained in the form of convergent series

$$\rho_j(\lambda, x) = w_{j:0}(\lambda, x) + w_{j:1}(\lambda, x) + \cdots \qquad (j = 1, \cdots, n). \tag{15.5}$$

The left member in (15.4), multiplied by λ^p, is "asymptotically" identical with the left member in (12.19) (cf. Lemma 3, §14). Hence in consequence of the statement in connection with (12.19)–(12.20b) one may write (15.4) in the form

$$w_{j:k}(\lambda, x) = \sum_{\beta=1}^{n} {}_kc_\beta(\lambda, x)\,{}_0y_{\beta,j}^*(\lambda, x) \qquad (j = 1, \cdots, n), \tag{15.6}$$

where

$${}_kc_\beta(\lambda, x) = \sum_{\tau=1}^{n} \int^x \lambda^p g_{\tau:k}\,{}_0\bar{y}_{\tau,\beta}^*(\lambda, u)du. \tag{15.6a}$$

Here the ${}_0y_{i,j}^*(\lambda, x)$ can be taken asymptotically the same (in $\Gamma(c', d'; R')$) as the ${}_0y_{i,j}(\lambda, x)$ $(i, j = 1, \cdots, n)$, involved in (12.20); moreover,

$$({}_0\bar{y}_{i,j}^*(\lambda, u)) = ({}_0y_{i,j}^*(\lambda, u))^{-1}.$$

If account is taken of (13.1), (13.2), and (13.3), it is concluded that

$$(15.7)\qquad {}_0y^*_{\beta,j}(\lambda, x) = e^{G_\beta(\lambda,x)}\gamma^*_{\beta,j}(\lambda, x), \qquad \gamma^*_{\beta,j}(\lambda, x) = [\lambda, x]_\sigma,$$

$$(15.7a)\qquad {}_0\bar{y}^*_{\tau,\beta}(\lambda, u) = e^{-G_\beta(\lambda,u)}\lambda^\omega\bar{\gamma}^*_{\tau,\beta}(\lambda, u), \qquad \bar{\gamma}^*_{\tau,\beta}(\lambda, u) = [\lambda, u]_{\bar\sigma},$$

$$(15.7b)\qquad |\gamma^*_{\beta,j}(\lambda, x)|, \quad |\bar{\gamma}^*_{\tau,\beta}(\lambda, x)| < \gamma_* \qquad (\text{in } \Gamma(c', d'; R')).$$

Moreover, σ can be taken as great as desired, and $\bar\sigma\to\infty$ with σ. However, the $y^*_{i,j}(\lambda, x)$ will, in general, depend on σ.

In view of the above (15.6) may be written in the form

$$(15.8)\qquad w_{j;k}(\lambda, x) = \sum_{\beta=1}^{n} {}_k\mathcal{C}_\beta(\lambda, x)e^{G_\beta(\lambda,x)}\gamma^*_{\beta,j}(\lambda, x),$$

$$(15.8a)\qquad {}_k\mathcal{C}_\beta(\lambda, x) = \sum_{\tau=1}^{n}\int^x \lambda^{p+\omega}g_{\tau;k}e^{-G_\beta(\lambda,u)}\bar{\gamma}^*_{\tau,\beta}(\lambda, u)du.$$

Finally, with the aid of (15.7b) we obtain

$$(15.9)\qquad |w_{j;k}(\lambda, x)| < n\gamma^2\sum_{\beta=1}^{n}|e^{G_\beta(\lambda,x)}|\int^x g_k|\lambda|^{p+\omega}|e^{-G_\beta(\lambda,u)}||du|$$

(in $\Gamma(c', d'; R)$), provided $|g_{j;k}| \leqq g_k$ $(j=1, \cdots, n)$ and the integrals here involved can be evaluated.

Write

$$(15.10)\qquad q_\nu(\lambda, x) = \sum_{h_1+\cdots+h_m=\nu} |\exp[h_1G_1(\lambda, x) + \cdots + h_mG_m(\lambda, x)]|,$$

$$(15.10a)\qquad \mu(\lambda) = \gamma(\lambda)|\lambda|^{p+\omega}.$$

Then, since $g_{j;0} = -{}_jF(\lambda, u)$, in view of (14.17) we may take

$$(15.11)\qquad g_0 = k_0\mu^N(\lambda)|\lambda|^{-2(p+\omega)}q_N{}^{(\lambda,u)}.$$

Whence, by (15.9),

$$(15.12)\qquad |w_{j;0}(\lambda, x)| < k_0n\gamma^2\mu^N(\lambda)|\lambda|^{-(p+\omega)}\sum_{\beta=1}^{n}|e^{G_\beta(\lambda,x)}| \cdot \sum_{h_1+\cdots+h_m=N}\int^x |e^{G(\lambda,u)}||du|,$$

where

$$(15.13)\qquad G(\lambda, u) = h_1G_1(\lambda, u) + \cdots + h_mG_m(\lambda, u) - G_\beta(\lambda, u).$$

Now

$$(15.13a)\qquad RG^{(1)}(\lambda, u) = h_1RQ_1^{(1)}(\lambda, u) + \cdots + h_mRQ_m^{(1)}(\lambda, u) - RQ_\beta^{(1)}(\lambda, u).$$

Applications of (12.10), (12.10a), and (12.10c) leads (by the same reasoning as employed in proving (13.13)) to the conclusion that the function of (15.13a) is $\leqq 0$ for λ, u in $\Gamma(c', d'; R')$.† Since

$$R[G(\lambda, x) - G(\lambda, u)] = \int_u^x RG^{(1)}(\lambda, v)dv,$$

and since, for

$$c' \leqq x \leqq v \leqq u \leqq d', \tag{15.14}$$

$dv \leqq 0$, the integrand just displayed is $\geqq 0$ for λ, x in $\Gamma(c', d'; R')$, provided (15.14) holds. Thus, under (15.14),

$$RG(\lambda, x) \geqq RG(\lambda, u) \tag{15.14a}$$

(cf. (15.13)). Accordingly,

$$\int_{d'}^{x} |e^{G(\lambda,u)}du| \leqq |e^{G(\lambda,x)}| \int_{d'}^{x} |du| = |e^{G(\lambda,x)}|\,(d' - c') \tag{15.15}$$

$$(\lambda, x \text{ in } \Gamma(c', d'; R')).$$

In the sequel integrations will be from d' to x (x in (c', d')).

By (15.15), (15.12), (15.13), and the definition of $q_N(\lambda, x)$

$$\begin{aligned} |w_{j,0}(\lambda, x)| &< k_0 k_1 \mu^N(\lambda)\,|\lambda|^{-(p+\omega)} q_N(\lambda, x) = k_1 |\lambda|^{p+\omega} g_0(\lambda, x) \\ &= w_0^*(\lambda, x) \qquad (j = 1, \cdots, n; \lambda, x \text{ in } \Gamma(c', d'; R')) \end{aligned} \tag{15.16}$$

(cf. (15.11)), where

$$k_1 = (d' - c')n^2\gamma^2. \tag{15.16a}$$

With q' and q denoting the constants of (15.2) *let b be the upper bound of the numbers*

$$q, \quad q'\left[1 + \frac{\rho''}{1+\rho''} + \cdots + \left(\frac{\rho''}{1+\rho''}\right)^{\nu-1}\right] + q\left(\frac{\rho''}{1+\rho''}\right)^{\nu}$$

($\nu = 1, 2, \cdots$); b will be finite. *The inequalities*

$$w_0^*(\lambda, x) \leqq \frac{\rho''}{1+\rho''}, \qquad \Gamma(\lambda, x) = k'b\,|\lambda|^{p+\omega} w_0^*(\lambda, x) \leqq \frac{\rho''}{1+\rho''} \tag{15.17}$$

$$(\lambda, x \text{ in } \Gamma(c', d'; R')(|\lambda| \geqq r'))$$

(cf. (15.16), (15.16a)) *will be obtained either by*

(i) *taking r' sufficiently great*;

or by

† Presently, however, it is immaterial whether the coefficient of the highest power of λ, in (15.13a), vanishes for any x in (c', d').

(ii) *taking the interval* (c', d') *sufficiently small.*

If we make use of (15.4) for $k=1$ (with $\rho_{j:-1}=0$) and apply (15.1), (15.2), with

$$\rho^* = \rho^*_{-1}(\lambda, x) = 0, \qquad w^* = w_0^*(\lambda, x),$$

it follows that

$$|\, g_{j:1}(\lambda, u)\,| < b(w_0^*(\lambda, u))^2 = g_1 \qquad (\lambda, u \text{ in } \Gamma(c', d'; R')). \tag{15.18}$$

In view of (15.16) and (15.14a) the upper bound of

$$w_0^*(\lambda, u)\,|\, e^{-G_\beta(\lambda, u)}\,| \qquad (\lambda, u \text{ in } \Gamma(c', d'; R')), \tag{15.19}$$

for

$$c' \leqq x \leqq u \leqq d', \tag{15.19a}$$

is attained at x; the function $w_0^*(\lambda, u)$, and hence

$$(w_0^*(\lambda, u))^2\,|\, e^{-G_\beta(\lambda, u)}\,|, \tag{15.20}$$

will possess the same property. By (15.9) (for $k=1$), (15.18), and the stated property of (15.20)

$$|\, w_{j:1}(\lambda, x)\,| < (d' - c')n^2\gamma^2\,|\,\lambda\,|^{p+\omega} b(w_0^*(\lambda, x))^2.$$

Thus, by (15.17)

$$|\, w_{j:1}(\lambda, x)\,| < \Gamma(\lambda, x) w_0(\lambda, x) = w_1^*(\lambda, x) \qquad (\text{in } \Gamma(c', d'; R')). \tag{15.21}$$

Making use of (15.4) for $k=2$ (with $\rho_{j:0}=w_{j:0}$) and applying (15.1), (15.2), with

$$\rho^* = \rho_0^*(\lambda, x) = w_0^*(\lambda, x), \qquad w^* = w_1^*(\lambda, x)\dagger$$

we infer that

$$\begin{aligned} |\, g_{j:2}(\lambda, x)\,| &< (q' w_0^*(\lambda, x) + q w_1^*(\lambda, x)) w_1^*(\lambda, x) \\ &= [q' + q\Gamma(\lambda, x)] w_0^*(\lambda, x) w_1^*(\lambda, x) \\ &< \left[q' + q \cdot \left(\frac{\rho''}{1+\rho''}\right)\right] w_0^*(\lambda, x) w_1^*(\lambda, x) < b w_0^*(\lambda, x) w_1^*(\lambda, x) \end{aligned} \tag{15.22}$$

(in $\Gamma(c', d'; R')$). ‡

Assume that for λ, x *in* $\Gamma(c', d'; R')$ $(|\lambda| \geqq r')$ *and for* $i=1, \cdots, k$ $(\geqq 2)$ *the following inequalities hold*

$$|\, g_{j:i}(\lambda, x)\,| < g_i = b w_0^*(\lambda, x) w_{i-1}^*(\lambda, x), \tag{15.23}$$

† $\rho^*+w^*=w_0^*+w_1^*<(\rho''/(1+\rho''))+(\rho''/(1+\rho''))^2<\rho''$ by (15.17) and (15.21).

‡ By definition of b.

(15.23a) $\quad |w_{j;\,i-1}(\lambda, x)| < w^*_{i-1}(\lambda, x) = \Gamma^{i-1}(\lambda, x)w_0^*(\lambda, x) \quad (j = 1, \cdots, n).$

For $k=2$ these relations have been established in (15.18), (15.22), (15.16), and (15.21).

In consequence of (15.23) for $i=k$, and (15.23a) for $i=k$

$$g_k = g_k(\lambda, u) = b\Gamma^{k-1}(\lambda, u)(w_0^*(\lambda, u))^2. \tag{15.24}$$

In view of the property previously stated in connection with the function (15.19) and $w_0^*(\lambda, u)$, and in view of the fact that $\Gamma(\lambda, u)$ enjoys the same property,† if we write

$$g_k(\lambda, u)\left|e^{-G_\beta(\lambda,u)}\right| = b\Gamma^{k-1}(\lambda, u)w_0^*(\lambda, u)\left[w_0^*(\lambda, u)e^{-G_\alpha(\lambda,u)}\right], \tag{15.24a}$$

we infer that the function on the left-hand side of (15.24a) will attain its upper bound at $u=x$, for λ, u in $\Gamma(c', d'; R')$ (provided (15.19a) is satisfied). Hence from (15.9) it follows that

$$|w_{j;\,k}(\lambda, x)| < n^2\gamma^2|\lambda|^{p+\omega}(d'-c')g_k(\lambda, x) = k_1|\lambda|^{p+\omega}b\Gamma^{k-1}(\lambda, x)(w_0^*(\lambda, x))^2.$$

Thus, by (15.17)

$$|w_{j;\,k}(\lambda, x)| < \Gamma^k(\lambda, x)w_0^*(\lambda, x) = w_k^*(\lambda, x) \tag{15.25}$$
$$(\text{in } \Gamma(c', d'; R'); j = 1, \cdots, n).$$

In other words, (15.23a) will hold for $i=1, \cdots, k+1$.

We are going to apply (15.4) with k replaced by $k+1$. We may write

$$|\rho_{j;\,k-1}(\lambda, x)| = |w_{j;\,0}(\lambda, x) + \cdots + w_{j;\,k-1}(\lambda, x)| < w_0^*(\lambda, x) + \cdots$$
$$+ w^*_{k-1}(\lambda, x) = \rho^*, \quad |w_{j;\,k}(\lambda, x)| < w_k^*(\lambda, x) = w^* \qquad (j = 1, \cdots, n).$$

Now, by (15.23a) ($i=1, \cdots, k+1$) and by virtue of (15.17)

$$\rho^* + w^* < \left(\frac{\rho''}{1+\rho''}\right) + \left(\frac{\rho''}{1+\rho''}\right)^2 + \cdots + \left(\frac{\rho''}{1+\rho''}\right)^{k+1} < \rho''.$$

Hence the statement in connection with (15.1), (15.2) is applicable. We have

$$\begin{aligned} |g_{j;\,k+1}(\lambda, x)| &< (q'\rho^* + qw^*)w^* \\ &= [q'(1 + \Gamma(\lambda, x) + \cdots + \Gamma^{k-1}(\lambda, x)) + q\Gamma^k(\lambda, x)]w_0(\lambda, x)w_k^*(\lambda, x) \\ &< [q'(1 + \delta + \cdots + \delta^{k-1}) + q\delta^k]w_0^*(\lambda, x)w_k^*(\lambda, x) \\ &\leqq bw_0^*(\lambda, x)w_k^*(\lambda, x) = g_{k+1} \qquad (\delta = \rho''/(1+\rho''); \text{ in } \Gamma(c', d'; R')). \end{aligned} \tag{15.26}$$

Whence it is concluded that the inequalities (15.23), (15.23a) *hold for* $i=1, 2, \cdots$. Consequently the series (15.5) converge absolutely and uniformly in $\Gamma(c', d'; R')$ ($|\lambda| \geqq r'$); in fact,

† $\Gamma(\lambda, u)$ is $w_0^*(\lambda, u)$ except for a factor independent of u.

$$
\begin{aligned}
|\rho_j(\lambda, x)| &\leqq |w_{j:0}(\lambda, x)| + |w_{j:1}(\lambda, x)| + \cdots \\
&< w_0^*(\lambda, x) + w_1^*(\lambda, x) + \cdots \\
&= w_0^*(\lambda, x)[1 + \Gamma(\lambda, x) + \Gamma^2(\lambda, x) + \cdots] \\
&\leqq w_0^*(\lambda, x)[1 + \delta + \delta^2 + \cdots] = w_0^*(\lambda, x)(1 + \rho'') \\
&\qquad\qquad \text{(in } \Gamma(c', d'; R'); j = 1, \cdots, n).
\end{aligned} \tag{15.27}
$$

The absolute values of the coefficients of $w_{1:i}$ $(i=1, \cdots, n)$ in

$$l_j^*(\lambda, x, w_{1:k}, \cdots, w_{n:k})$$

are $\leqq K$, where K is independent of λ and x (λ, x in $\Gamma(c', d'; R')$). By (15.4) for $k=0$

$$|w_{j:0}^{(1)}(\lambda, x)| \leqq |\lambda|^p nK w_0^*(\lambda, x) + |\lambda|^p g_0(\lambda, x).$$

Substituting the expression for $g_0(\lambda, x)$ from (15.16), we obtain

$$|w_{j:0}^{(1)}(\lambda, x)| < k''|\lambda|^p w_0^*(\lambda, x) \qquad (j = 1, \cdots, n; \text{ in } \Gamma(c', d'; R')).\dagger$$

By virtue of (15.4) for $k>1$, (15.25), and (15.24) and the fact that $\Gamma(\lambda, x)$, $w_0^*(\lambda, x) \leqq \delta$, it is seen that

$$
\begin{aligned}
|w_{j:k}^{(1)}(\lambda, x)| &\leqq |\lambda|^p nK w_k^*(\lambda, x) + |\lambda|^p b\Gamma^{k-1}(\lambda, x)(w_0^*(\lambda, x))^2 \\
&< (nK + b)|\lambda|^p \delta^k w_0^*(\lambda, x).
\end{aligned}
$$

Thus, with ${}_1k$ denoting the greater of the numbers k'', $nK+b$, we have

$$|w_{j:k}^{(1)}(\lambda, x)| < {}_1k|\lambda|^p \delta^k w_0^*(\lambda, x) \qquad (k = 0, 1, \cdots; j = 1, \cdots, n) \tag{15.28}$$

in $\Gamma(c', d'; R')$.

With the aid of (15.28) it is concluded that the series

$$\rho_j^{(1)}(\lambda, x) = w_{j:0}^{(1)}(\lambda, x) + w_{j:1}^{(1)}(\lambda, x) + \cdots$$

converge absolutely and uniformly. We have, for λ, x in $\Gamma(c', d'; R')$,

$$
\begin{aligned}
|\rho_j^{(1)}(\lambda, x)| &\leqq |w_{j:0}^{(1)}(\lambda, x)| + \cdots < (1 + \rho'')_1 k |\lambda|^p w_0^*(\lambda, x) \\
&\qquad\qquad (j = 1, \cdots, n).
\end{aligned} \tag{15.29}
$$

For λ, x in $\Gamma(c', d'; R')$ ($|\lambda| \geqq r'$; r' sufficiently great) the series (15.5) will truly represent a solution of the transformed system. The $\rho_j(\lambda, x)$ are analytic in λ for λ in R' ($\lambda \neq \infty$), provided x is in (c', d'). For any fixed λ in R' the $\rho_j(\lambda, x)$ will be continuous in x for x in (c', d').‡ Moreover, these functions are differentiable in $\Gamma(c', d'; R')$ with respect to x.

† k'' is independent of λ, x.

‡ The $\rho_j(\lambda, x)$ are uniformly convergent series of continuous functions.

THE THIRD EXISTENCE THEOREM. *Let $\Gamma(\alpha', \beta'; R)$ be a region proper according to Definition 2, §12. The functions $Q_i(\lambda, x)$, here involved, are from the formal matrix solution* (12.2) *satisfying the linear problem* (12.1) *(associated with the system* (11.1)). *Let the $G_i(\lambda, x)$ $(i=1, \cdots, n)$ be defined by* (12.6b). *Let $\Gamma(c, d; R')$ be an admissible* (cf. Definition 3, §12) *subregion of $\Gamma(\alpha', \beta'; R)$. With suitable notation we then have* (12.10), (12.10a), (12.10b), *and* (12.10c).

Let N (>0) be a fixed integer, however large. Form a region $\Gamma(c', d'; R')$, where R' is the same region as before and (c', d') is a subinterval of (c, d) such that the italicized statement preceding (13.17) *should hold for (c', d'). Let the c_i $(i=1, \cdots, m)$ be arbitrary functions of λ (independent of x), subject, however, to the condition stated in italics in connection with* (14.2).

The system (11.1) *will then possess a solution $y_j(\lambda, x)$ $(j=1, \cdots, n)$, whose elements are analytic in λ for λ in R' $(\lambda \neq \infty)$ (x in (c', d')) and satisfy asymptotic relations*

$$y_j(\lambda, x) \sim s_j(\lambda, x) \qquad (j = 1, \cdots, n; \text{ in } \Gamma(c', d'; R')), \tag{15.30}$$

where the $s_j(\lambda, x)$ are the series referred to in Lemma 2, §13. The relations (15.30) *are asymptotic in the following sense. The $y_j(\lambda, x)$ are of the form*

$$y_j(\lambda, x) = \sum_{\nu=1}^{N-1} \sum_{k_1+\cdots+k_m=\nu} c_1^{k_1} \cdots c_m^{k_m} {}_\nu\eta_{k_1,\cdots,k_m;j}(\lambda, x) + \rho_j(\lambda, x) \qquad (j = 1, \cdots, n), \tag{15.30a}$$

where

$${}_\nu\eta_{k_1,\cdots,k_m;j}(\lambda, x) = \exp\left[k_1 G_1(\lambda, x) + \cdots + k_m G_m(\lambda, x)\right] \lambda^{(\nu-1)(p+\omega)} [\lambda, x]_{\bar{\alpha}-\nu+1} \tag{15.30b}$$

$$(k_1, \cdots, k_m \geqq 0;\ k_1 + \cdots + k_m = \nu;\ \nu = 1, 2, \cdots, N-1;\ j = 1, \cdots, n)$$

for λ, x in $\Gamma(c', d'; R')$. The symbol $[\lambda, x]_{\bar{\alpha}-\nu+1}$, above, is defined in $\Gamma(c', d'; R')$ according to Definition 4, §13. The $\rho_j(\lambda, x)$ are functions defined, together with the derivatives $\rho_j^{(1)}(\lambda, x)$, in $\Gamma(c', d'; R')$ and are such that

$$|\rho_j(\lambda, x)| < \bar{k}\gamma^N(\lambda) |\lambda|^{(N-1)(p+\omega)} \sum_{h_1+\cdots+h_m=N} |\exp\left[h_1 G_1(\lambda, x) + \cdots + h_m G_m(\lambda, x)\right]| = \rho_N^*(\lambda, x), \tag{15.30c}$$

where $\gamma(\lambda)$ is defined as in (14.2) *and*

$$|\rho_j^{(1)}(\lambda, x)| < \bar{k}_1 \rho_N^*(\lambda, x) |\lambda|^p \qquad (j = 1, \cdots, n; \text{ in } \Gamma(c', d'; R')), \tag{15.30d}$$

$\bar{k}$, $\bar{k}_1$ being independent of λ, x. In the above, $\bar{\alpha}$, depending on N, is to be taken

suitably great. In R', $|\lambda| \geqq r'$ and r' is to be taken sufficiently great so that (14.4a) *and* (15.17)† *will hold.*

Note. We observe that $|\rho_j(\lambda, x)|$ is essentially of the order of

$$\sum_{k_1+\cdots+k_m=N} c_1^{k_1} \cdots c_m^{k_m} {}_N\eta_{k_1,\cdots,k_m;j}(\lambda, x).$$

In fact, the absolute value of the latter expression is less than

$$h\rho_N^*(\lambda, x) \qquad (\text{in } \Gamma(c', d'; R')),$$

where h is independent of λ, x.

The following general remarks will be made regarding the main results of this work.

The remaining problem is of interest, namely, *to determine under what specialized conditions the formal series solutions involved in the three existence theorems converge.* It is to be noted that in all of Horn's work, which relates to certain restricted first-order problems,‡ the formal solutions converge and thus represent actual solutions.§

Another problem of importance is to find *under what restrictions are the coefficients in the formal series solutions (in the case of the First Existence Theorem) representable with the aid of the fundamental methods of Nörlund* (Laplace integrals leadings to convergent factorial series developments—essentially by a method of exponential summability).‖

Finally, it is observed that the existence theorems of this work can be extended without any substantial difficulties by replacing the second members in the systems (A), (B), and (C) of §1 by suitable functions satisfying in a certain neighborhood of the singular point appropriate relations asymptotic with respect to $y_1, \cdots, y_n$ (to a finite number of terms with respect to t (or λ)).

† These conditions can be also satisfied by other means (cf. the text in connection with (14.4a) and (15.17)).

‡ For references to J. Horn see (T_1); also see my paper in Compositio Mathematica, loc. cit.

§ Even in the first-order problems occasions arise when the formal solutions diverge. So even for these problems one is forced to use asymptotic methods, unless suitable restrictions are introduced.

‖ Whenever applicable, these methods yield results of greater precision than those obtained by asymptotic methods. On the other hand, as indicated in (T_1), they are applicable only under suitable restrictions.

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ALMOST PERIODIC TRANSFORMATION GROUPS*

BY

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1. In view of the recent work on topological groups it is natural to consider the situation which arises when such groups act as transformation groups on various types of spaces. Such a study is begun here from the point of view of almost periodic transformation groups, the definition of which is suggested by von Neumann's paper on almost periodic functions in a group.† Compact topological transformation groups are a special case of almost periodic transformation groups, at least for a rather wide class of spaces. The paper concerns itself chiefly with the nature of the minimal closed invariant sets of such groups. There are some results for general spaces but the main results are for Euclidean spaces and more particularly for three-dimensional Euclidean spaces. One of the most interesting theorems states that if a compact one-dimensional group acts on three-space in such a way that its orbits are uniformly bounded in diameter, then every point of the space is fixed under the group, so that if such a group is to act in a non-trivial manner the diameters of its orbits must be unbounded. Under some restrictions a similar theorem is proved for one-parameter almost periodic groups. Furthermore it is shown that for this latter class of groups, many of the orbits must actually be simple closed curves if they have one-dimensional closures.

2. The group considered here will be denoted by G. It will be subjected to various conditions as the occasion demands but it will always be Abelian. In case it is the group of real numbers, it will be spoken of as a one-parameter group; in case it is the real numbers reduced modulo one, it will be spoken of as the circle group. The space on which the group acts will be denoted by R. It will be specialized in various ways, but in any case it will always be a locally compact metric space. If x and y are two points of R, the distance between them will be denoted by $d(x, y)$.

DEFINITION 1. *The group G is said to be a transformation group (t.g.) of R if the following conditions are satisfied:*

(1) *for every g in G there is a homeomorphism of R into itself denoted by $x^1 = g(x)$;*

* Presented to the Society, February 20, 1937; received by the editors January 13, 1936.

† These Transactions, vol. 36 (1934), p. 445. Cameron has defined almost periodic transformations. These Transactions, vol. 36 (1934). p. 276.

(2) *if* $g_3=g_1+g_2$, *then* $g_3(x)=g_1[g_2(x)]$.

DEFINITION 2.* *If G is a t.g. of R, G is said to be an almost periodic t.g. (a.p.t.g.) if for every a in R there is a neighborhood $U(a)$ having the following property:*

For any $e>0$ there is a finite set of elements $g_1, g_2, \cdots, g_n$ in G which are such that for any g in G there is an i such that $d[g(x), g_i(x)]<e$ for all x in $G[U(a)]$.†

No use is made of a topology in G in either of these two definitions so that in them G may or may not be topological.

DEFINITION 3. *If G is a topological group and a t.g. of R, then G is said to be a topological t.g. of R provided that $g(x)$ is continuous simultaneously in x and g.*

A t.g. may of course be both topological and a.p. If R is locally compact, a compact topological t.g. is an a.p.t.g.

DEFINITION 4. *The t.g. G is said to be regular*‡ *if for every positive e and every a in R there is a d such that if $d(x, a)<d$, then $d[g(x), g(a)]<e$ for all g in G.*

THEOREM 1. *If G is an a.p.t.g. it is regular.*

Let e be any positive number, and let a be any point of R; let $U(a)$ be the neighborhood of Definition 2, and let $g_1, g_2, \cdots, g_n$ be the finite set of elements associated with $e/3$. Since $g_i(x)$ is a homeomorphism, we may choose $U^1(a)$ within $U(a)$ in such a way that if x is in $U^1(a)$ then

$$d[g_i(a), g_i(x)] < e/3 \qquad (i = 1, 2, \cdots, n).$$

If g is any element of G, there is an i such that

$$d[g(a), g_i(a)] < e/3$$

and for any x in $U^1(a)$

$$d[g(x), g_i(x)] < e/3.$$

From these three inequalities the definition of regularity is seen to be satisfied.

The set $O_a=\sum_g g(a)$ is called the orbit of a.

THEOREM 2. *If G is a regular t.g., then $\overline{O}_a$ is the minimal closed invariant set including a, and if b is in $\overline{O}_a$, then $\overline{O}_a=\overline{O}_b$.*

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* von Neumann, loc. cit.

† Following the usual convention this symbol is used for the set of points in R of the form $g(x)$ when g is in G and x is in $U(a)$.

‡ This concept is due to Kerékjártó. See Acta Litterarum ac Scientiarum, Szeged, vol. 6, p. 235.

The set $\overline{O}_a$ is the minimal closed set including O_a so that for the proof of the first part of the theorem it is only necessary to prove that $\overline{O}_a$ is invariant. Any point p in $\overline{O}_a$ is the limit of a set of points $p_n = g_n(a)$.† But then $g(p)$ is the limit of $g(p_n) = (g+g_n)(a)$ and hence for every g, $g(p)$ is in $\overline{O}_a$ which proves that $\overline{O}_a$ is invariant. It will now be shown that if b is in $\overline{O}_a$ then $\overline{O}_a = \overline{O}_b$. It is clear that $\overline{O}_b$ is in $\overline{O}_a$. To prove that $\overline{O}_a$ is in $\overline{O}_b$ it will be shown that a is in $\overline{O}_b$. Let a positive number e be given, and choose d so that if $d(x, b) < d$, then $d[g(x), g(b)] < e$ for all g. Since b is in $\overline{O}_a$ there is a g such that $d[g(a), b] < d$. Therefore $d[a, -g(b)] < e$; that is, there are points of O_b arbitrarily near a, which is sufficient for the proof.

The following theorem is due in essence to Kerékjártó‡ who stated it for the group of integers and a special R. Only a sketch of the proof will be given.

THEOREM 3. *If G is a regular t.g. of a complete metric space R, then $\overline{O}_x$ for any x is homeomorphic to a topological Abelian group.*

The set $\overline{O}_x$ is a metric space and in order to prove the theorem it must be shown that the points of $\overline{O}_x$ may be considered as the elements of a group in such a way that $a+b$ and $-a$ are continuous operations. It will first be shown how $a+b$ may be defined for any a and b in $\overline{O}_x$. Let A and B be small closed neighborhoods of a and b in $\overline{O}_x$, and let S and T contain those elements s and t of G such that $s(x)$ is in A and $t(x)$ is in B. Now let C be the set of points of the form $(s+t)(x)$, where s is in S and t is in T. It will be shown that the diameter of C may be made arbitrarily small by choosing A and B to have sufficiently small diameters. The point $(s_1 - s_2)(x)$ is near x if A is sufficiently small, for since $s_1(x)$ is near a, $(-s_2+s_1)(x)$ is near $-s_2(a)$ by regularity and $-s_2(a)$ is near x by regularity. Similarly $(t_1 - t_2)(x)$ is near x. From these facts we see, because of regularity, that $(s_1 - s_2 + t_1 - t_2)(x)$ is near x. Hence, again by regularity, points of the type $(s_1+t_1)(x)$ form a set of small diameter (because they are near $(s_2+t_2)(x)$ for fixed s_2 and t_2). As the diameters of A and B approach zero, the diameter of the set C therefore approaches zero and since R is complete, C must shrink toward a limit point which is defined to be $a+b$. This operation is clearly commutative and associative, and the process of defining it shows that it is also continuous simultaneously in a and b. In the above notation let D be all points of the form $-s(x)$. The diameter of D approaches zero with that of A, and the point approached by the shrinking set D is defined to be $-a$. This element satisfies the group postulates for an inverse and is continuous. The theorem is therefore completely demonstrated.

† This is meant to include the case where for all but a finite number of n's, $p_n = p$.

‡ Acta Litterarum ac Scientiarum, Szeged, vol. 7, p. 76.

THEOREM 4. *If G is an a.p.t.g. of a complete metric space then $\overline{O}_a$ is homeomorphic to a compact topological group.*

By Theorem 3, $\overline{O}_a$ is a topological group and it is a compact set because it is closed and conditionally compact.†

If G is a compact topological t.g. of R, then those elements of G, say $H(a)$, which leave a fixed form a compact subgroup of G, and O_a which in this case equals $\overline{O}_a$ is in reality the difference group $G-H(a)$. It is true therefore that the dimension of G is equal to the sum of the dimensions of O_a and $H(a)$.‡

3. If G is an a.p.t.g. of R, let F denote all points of R which are fixed under the action of every element of G, and let M be the set of moving points, that is, the set of points which are moved by some element of G. The set F is closed and hence M is open. Under the action of G, R is divided into a family of non-overlapping sets, the minimal closed invariant sets. This is a continuous family, for from Definition 2 it follows that if $x_n \to x$, then $\overline{O}_{x_n} \to \overline{O}_x$. Consider a set of elements R^* in one-to-one correspondence with the minimal closed invariant sets of R. If x^* is the element of R^* corresponding to $\overline{O}_x$, let $L(x)=x^*$, this being the definition of the function $L(x)$. It is clearly single valued, but in general its inverse is multiple valued. The set R^* may be made a metric space called the auxiliary space by the definition

$$d(x^*, y^*) = d\left[L^{-1}(x^*), L^{-1}(y^*)\right]. \S$$

The function $L(x)$ now is a continuous function. It is an inner transformation of the space R into the space R^*; that is, it takes open sets into open sets.

We shall now be concerned with the case in which R is R_n, Euclidean space of n dimensions. If G is a t.g. of R_n a point may be added to R_n to make it $\overline{R}_n$ the n-sphere, and G may be extended to a t.g. of $\overline{R}_n$ by simply requiring that this point be fixed under all elements of G. If G is topological, it remains so by this definition; whether or not a group retains its a.p. character by this extension need not concern us. In any case the function $L(x)$ remains continuous in the extended case for an a.p.t.g. The space $\overline{R}_n^*$ is compact, connected, and locally connected.

4. We prove the following theorem.

THEOREM 5. *If G is an a.p.t.g. of R_n and if there is a number K such that for every a in R_n* diam $(O_a) < K$, *then $\overline{R}_n^*$ is cyclic.*‖

† See von Neumann, loc. cit.

‡ van Kampen, American Journal of Mathematics, vol. 58 (1936), p. 178.

§ For the definition of the distance between two closed sets see Hausdorff, *Mengenlehre.*

‖ For the definition of cyclic sets see Alexandroff, Mathematische Annalen, vol. 106, p. 218 and p. 223.

Let x be any point of R_n with coordinates $(x_1, \cdots, x_n)$, and let the coordinates of $g(x)$ be denoted by $[g(x)]_i$. For each fixed i and x, $[g(x)]_i$ is a real valued a.p. function defined on the group G. This follows from the definitions of a.p.t.g. and a.p. functions. This function has a mean† which will be denoted by $F_i(x)$. The n functions $F_i(x)$ determine a point denoted by $F(x)$. Hence we have defined a transformation of the space R_n into itself. By the properties of the mean this function is constant on an orbit O_a. The continuity of the function or transformation can be seen in the following way. Let a positive e and a point a be given. Choose d so that if $d(a, x) < d$ then $d[g(a), g(x)] < e$; this last relation may also be written as $|[g(a)]_i - [g(x)]_i| < e(n)^{-1/2}$. But

$$|F_i(a) - F_i(x)| = |M_g[g(a)]_i - M_g[g(x)]_i| \leqq M_g|[g(a)]_i - [g(x)]_i| \leqq M_g e(n)^{-1/2} = e(n)^{-1/2},$$

and therefore $d[F(a), F(x)] < e$. Because $F(x)$ is constant on O_a and continuous it must be constant on $\overline{O}_a$. The properties of the mean also imply that $d[x, F(x)] < K(n)^{1/2}$.

If the point at infinity P_∞ be added to R_n to form the space $\overline{R}_n$, $F(x)$ may be extended by defining $F(P_\infty) = P_\infty$, and the resulting transformation of $\overline{R}_n$ into itself is continuous. The transformation of $\overline{R}_n$ is also essential;‡ this is because the degree of F is one, and we see this from the fact that no point can move to its diametrically opposite point since in R_n $d[x, F(x)] < K(n)^{1/2}$.

We can define a transformation $h(x^*)$ from $\overline{R}_n^*$ to $\overline{R}_n$ in this way: for any point x^* let $h(x^*) = F[L^{-1}(x^*)]$. Since the transformation F takes $L^{-1}(x^*)$ into a single point, $h(x^*)$ is single valued and furthermore it is continuous. Clearly

$$h[L(x)] = F(x).$$

The transformation $h(x^*)$ must be an essential transformation of $\overline{R}_n^*$ into $\overline{R}_n$. If this were not the case there would be a function $h(x^*, t)$ defined for every x^* in $\overline{R}^*$ and for every t in the interval $0 \leqq t \leqq 1$, this function being such that $h(x^*, 0) = h(x^*)$ and $h(x^*, 1)$ is constant. But then

$$k(x, t) = h[L(x), t]$$

is a deformation of $F(x)$ having properties similar to the ones just mentioned for $h(x^*, t)$, and this implies that $F(x)$ is not essential. From this contradiction we conclude that $h(x^*)$ is essential and hence that $\overline{R}_n^*$ is cyclic; that is, it contains an n-dimensional power cycle which is not homologous to zero.

† For the existence and properties of the mean see von Neumann, loc. cit.

‡ See the previously cited paper of Alexandroff.

COROLLARY. *If G is an a.p.t.g. of R_n and if $\overline{R}_n^*$ is not cyclic, then the diameters of the orbits in R_n are unbounded.*

The theorem below can be applied more easily than Theorem 5 to certain special cases which will be considered later. The set $L(M)$ is denoted by M^*, $L(F)$ by F^*.

THEOREM 6. *If G is an a.p.t.g. of R_n whose orbits are uniformly bounded, and if the dimension of M^* is less than n, then every point of R_n is fixed under G.*

The set F^* is homeomorphic to F, and if we assume that the theorem is false and that the set M is not vacuous, then F^* is homeomorphic to a proper part of the sphere $\overline{R}_n$. The set F^* itself therefore cannot be cyclic. Since M is at most $(n-1)$-dimensional, F^*+M^* cannot contain a power cycle homologous to zero. This can be seen from the Phragmén-Brouwer† theorem. Therefore F^*+M^* is not cyclic, but this contradicts Theorem 5, so that we must conclude that M is vacuous and that the theorem is true.

THEOREM 7. *If G is an a.p.t.g. of R_n, then no minimal closed invariant set is n-dimensional and at least one such set is of dimension less than $(n-1)$.*

If a minimal closed invariant set were n-dimensional, it would have to contain an interior point of R_n‡ and the family of minimal closed invariant sets could not be a continuous family in this case.

Suppose now that every minimal closed invariant set is $(n-1)$-dimensional. If $\overline{O}_\alpha$ is $(n-1)$-dimensional, it goes by an ϵ-transformation into a set containing an $(n-1)$-dimensional torus;§ and therefore $\overline{O}_\alpha$ must separate R_n.‖ The point $L(\overline{O}_\alpha)$ must therefore separate $\overline{R}_n^*$. Hence every point except possibly P_∞^* is a cut point of $\overline{R}_n^*$. But since $\overline{R}_n^*$ must have at least two non-cut points,¶ we see that at least one minimal closed invariant set is less than $(n-1)$-dimensional.

5. For the application of these theorems to certain special cases a few preliminary facts are necessary and these will now be discussed. Let G be an a.p.t.g. of R, and let A be a subgroup of G. It can be verified that A is also an a.p.t.g. of R and the minimal closed invariant sets (under A) in R form a continuous family. Denote the space formed by this continuous family of sets by R_A and the continuous transformation taking R into R_A by L_A. If $G-A=B$,

† Alexandroff, loc. cit., p. 186. See also p. 218.

‡ Menger, *Dimensionstheorie*, p. 244.

§ Pontrjagin, Annals of Mathematics, vol. 35 (1934), p. 386.

‖ Alexandroff, Annals of Mathematics, vol. 30 (1929), p. 148.

¶ R. L. Moore, *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13, 1932, p. 45.

then B can be considered in a very natural way as a t.g. of R_A. If $x^* = L_A(x)$ and if $\bar{b}$ is an element of B corresponding to the coset $b+A$, the following definition is made:

$$\bar{b}(x^*) = L_A[b(x)].$$

This is a unique definition and by it the transformations $\bar{b}(x^*)$ are homeomorphisms of R satisfying the conditions for a t.g.

THEOREM 8. *Under the conditions given above B is an a.p.t.g. of R_A.*

Only the a.p. character of B remains to be proved. Let a^* be any point of R_A; then $L_A^{-1}(a^*)$ is a minimal closed invariant set in R which will be denoted by $\overline{O}_a(A)$, the point a being selected as any point of the set. Since G is a.p. there is a neighborhood $U(a)$ having the property that for every positive d there is a finite set $g_1, g_2, \cdots, g_n$ such that for any g in G there is an i such that $d[g(x), g_i(x)] < d$ for all x in $G[U(a)]$. Now assume that d is fixed and that $g_1, g_2, \cdots, g_n$ is the corresponding finite set. Now let $L_A U(a) = U(a^*)$. Let $\bar{g}_i$ be the finite set of elements of B corresponding to the cosets g_i+A. Let $\bar{g}$ be any element of B corresponding to the coset $g+A$, and consider the distance $d[\bar{g}(x^*), \bar{g}_i(x^*)]$, where x^* is any element of $U(a^*)$. Remembering that $g(x^*) = L_A[g(x)]$ and that $\bar{g}_i(x^*) = L_A[g_i(x)]$ as well as the local compactness of R, we see that d may be so chosen that the distance in question is less than any e specified in advance. This is sufficient to prove the theorem.

If K^* is a closed set in an auxiliary space, the set K is said to be a true section of $L^{-1}(K^*)$ provided (1) that it is closed, and (2) that it contains precisely one point on $L^{-1}(x^*)$ for every x^* in K^*.† Bearing in mind the notation of the preceding theorem, we let $L_B(R_A) = R_{AB}$. This space R_{AB} is homeomorphic to R_G which we ordinarily denote by R^*.

LEMMA 1. *Let K^* be a closed set in $R^*(R_{AB})$. If K_A is a true section of $L_B^{-1}(K^*)$, and if K is a true section of $L_A^{-1}(K_A)$, then K is a true section of $L^{-1}(K^*)$.*

This lemma, the proof of which will be left to the reader, provides an opportunity to obtain true sections in complicated cases from true sections for simple cases.

The following lemma due to Zippin will also be useful in this connection.‡

LEMMA 2. *Let $F(t)$ be a continuous function which is defined on a linear*

† Montgomery and Zippin, these Transactions, vol. 40 (1936), p. 24.

‡ Zippin stated this theorem to the author in conversation, but he has not as yet published his proof. The author has found an independent proof. Added in proof. On March 27, 1937, G. T. Whyburn read a paper containing a proof of this theorem. His paper will appear in the Duke Mathematical Journal.

interval $a \leqq t \leqq b$ and whose values are disjoined closed zero-dimensional sets in a compact metric space. Then there is a continuous point valued function $f(t)$ defined on $a \leqq t \leqq b$ in such a way that $f(t)$ is in $F(t)$. This may be chosen so that if c is any point in the interval, $f(c)$ is any desired point of $F(c)$.

This is equivalent to saying that there is a true section of the continuous family of sets which are the functional values of $F(t)$.

THEOREM 9. *If G is a one-parameter topological a.p.t.g. of R whose minimal closed invariant sets are at most one-dimensional, then for any arc a^*b^* in R^* there is a true section ab in $L^{-1}(a^*b^*)$.*

To begin with assume that a^*b^* is in M^*. It will first be shown that if c^* is any point of a^*b^*, then there is a subarc e^*f^* including c^* on its interior which is such that if c is any point of $L^{-1}(c^*)$ there is a true section ef in $L^{-1}(e^*f^*)$ containing c.

In order to prove this, note that the family of curves filling $L^{-1}(ab)$ is a regular family so that Whitney's results on local cross sections† may be applied. Let c be a definite point of $L^{-1}(c^*)$, and let S be a cross section of the family of curves through c (Whitney, loc. cit.). Now $\overline{O}_c \cdot S$ must be zero-dimensional for if it were one-dimensional, then $\overline{O}_c$ would be at least two-dimensional. Hence there is in S an open set O including c which has no points of $\overline{O}_c$ on its boundary. There must be a small arc e^*f^* including c^* in its interior which is such that for any point x^* in e^*f^*, $[L^{-1}(x^*)] \cdot O$ is a closed zero-dimensional set and $L^{-1}(x^*)$ does not intersect the boundary of O. The family of sets $[L^{-1}(x^*)] \cdot O$ for all x^* in e^*f^* is a continuous family of zero-dimensional sets, and hence by Lemma 2 there is an arc ef which is a true section and which goes through c.

By the Heine-Borel theorem there are a finite number of arcs of the same type as e^*f^* which cover a^*b^*. Hence a^*b^* may be divided into a finite number of subarcs $a^*a_1^*$, $a_1^*a_2^*$, $\cdots$, $a_n^*b^*$ which are such that $L^{-1}(a^*a_1^*)$ has a true section through any point of $L^{-1}(a^*)$, $L^{-1}(a_1^*a_2^*)$ has a true section through any point of $L^{-1}(a_1^*)$, and so on. Now choose any true section aa_1 of $L^{-1}(a^*a_1^*)$. Choose a true section of $L^{-1}(a_1^*a_2^*)$ which begins at a_1, call it a_1a_2. Then choose a true section of $L^{-1}(a_2^*a_3^*)$ which begins at a_2, call it a_2a_3. Proceeding in this way there is obtained a finite set of true sections whose sum is a true section of $L^{-1}(a^*b^*)$. The proof of the theorem is now complete for the case where a^*b^* is in M^*.

Suppose now that a^*b^* is anywhere in R^* so that part of it may be in F^*. Let $K^* = a^*b^*$. The set $L^{-1}(K^* \cdot F^*)$ is its own true section. The set $K^* \cdot M^*$

† Annals of Mathematics, vol. 34 (1933), p. 244.

may be represented as the sum of the interiors of a countable set of intervals c^*d^*, where c^* and d^* are in F^*. The interior of any interval c^*d^* may be represented as the sum of a doubly infinite set of adjacent intervals $c_i^*c_{i+1}^*$, where i runs over all positive and negative integers. For each i the set $L^{-1}(c_i^*c_{i+1}^*)$ has a true section c_ic_{i+1}, where either c_i or c_{i+1} can be chosen arbitrarily in $L^{-1}(c_i^*)$ or $L^{-1}(c_{i+1}^*)$ respectively. We may therefore assume, as our notation has anticipated, that the last point of c_ic_{i+1} coincides with the first point of $c_{i+1}c_{i+2}$. Let $L^{-1}(c^*)=c$ and $L^{-1}(d^*)=d$. Then the set $\sum c_ic_{i+1}+c+d$ is a true section of $L^{-1}(c^*d^*)$; because of the a.p. character of G, the sum converges properly toward c and d. Since this same process can be carried out in each interval, it is clear that there is a true section of $L^{-1}(a^*b^*)$ and the proof of the theorem is now complete.

COROLLARY. *Let G be the circle group, and let G be a topological t.g. of R. Then if K^* is an arc in R^*, there is a true section of $L^{-1}(K^*)$.*

This is a special case of the preceding theorem.

THEOREM 10. *Let G be a connected compact one-dimensional t.g. of R. Then if K^* is an arc in R^*, there is a true section K of $L^{-1}(K^*)$.*

In this case G contains a compact zero-dimensional subset A such that $G-A$ is the circle group. Hence the theorem is an immediate consequence of the preceding corollary and Lemmas 1 and 2.

LEMMA 3. *If G is a compact one-dimensional topological group, there is in G an arc C including the zero of G as an interior point and which has a certain group property as follows: There is an open subset V of C including zero and such that if u and v are in V, then $u+v$ is in C.*

This is contained in a result of Pontrjagin.†

THEOREM 11. *Let G be a compact connected one-dimensional t.g. of R. Then if K^* is an arc in M^*, $L^{-1}(K^*)$ is two-dimensional.*

There is an arc ab which is a true section of $L^{-1}(K^*)$ by Theorem 10. Let C be the arc and V the subset of C of the preceding lemma. The "group" C may be assumed to be locally isomorphic to a part of the real number group, and we will speak as if the elements of C were real numbers. If V is chosen sufficiently small, then for every v different from 0 in V, $v(a)\neq a$; for if there were a set of elements v_n with zero as a limit and such that $v_n(a)=a$, then for every element v in V, $v(a)=a$. In this case the set $H(a)$ of elements of G leaving a fixed is one-dimensional, and therefore the orbit is zero-dimensional which means in this case that it is a point. But this is a contradiction, and

† Loc. cit., p. 387.

we conclude that our original statement must be true. A similar statement is true for all elements x in K. There is in fact a V so small that for every v different from zero in V and every x in K, $v(x) \neq x$. If this were not true there would be a set of elements v_n approaching zero and a set of elements x_n in K which may be assumed to approach x_0 such that $v_n(x_n) = x_n$. Assume that V is so chosen that if v is in V, $v(x_0) \neq x_0$. Now let v be any element in V. For any n, $v = k_n v_n + u_n$, where k is an integer and $|u_n| < |v_n|$. We know that $v(x_n)$ approaches $v(x_0)$, but $v(x_n) = (k_n v_n + u_n)(x_n) = u_n(x_n)$. Hence $v(x_0) = x_0$, and we have reached a contradiction from which we conclude that a V of the type stated exists. Assume that V consists of all elements of the type $|v| < r$ where r is a positive number. If $|v_1| < r/2$ and $|v_2| < r/2$ and $v_1 \neq v_2$ then $v_1(x) = v_2(x)$, for if $v_1(x) = v_2(x)$, then $(v_1 - v_2)(x) = x$ and $|v_1 - v_2| < r$.

Consider all elements of the form $v(x)$ for x in K and v in V. This set is homeomorphic to the product of an open interval and a closed interval and is therefore two-dimensional. We have now proved a slightly stronger result than the one stated in the theorem, namely, that the arc ab is actually imbedded in a two-cell which is in $L^{-1}(a^*b^*)$.

6. We have the following theorem:

THEOREM 12. *If G is a one-dimensional connected compact t.g. of R_3, then M^* is at most two-dimensional.*

If M^* were three-dimensional, it would contain a compact three-dimensional set and hence a compact three-dimensional Cantorian manifold.† In fact there would have to be a point a^* in M^* such that an arbitrarily small three-dimensional Cantorian manifold contained it. On the other hand, M^* contains no cut points because the inverse of a point is one-dimensional, and a^* must be an interior point of an arc which is in M^*.‡ This arc separates M^* locally at a^* because the inverse of the arc contains a two-cell and therefore separates R_3 locally. These two facts contradict each other, and it may therefore be concluded that M^* is at most two-dimensional.

THEOREM 13. *If G is a one-dimensional connected compact t.g. of R_3 whose orbits are uniformly bounded, then G leaves every point of R_3 fixed.*

This is an immediate consequence of Theorems 6 and 12.

THEOREM 14. *Let G be a one-parameter topological a.p.t.g. of R_3 whose minimal closed invariant sets are one-dimensional. If the orbits are uniformly bounded, then every point of R_3 is fixed under G.*

It is necessary to prove that M^* is at most two-dimensional. If a^*b^* is

† Alexandroff, Mathematische Annalen, vol. 106, p. 214 and p. 217.

‡ R. L. Moore, loc. cit., p. 148.

an arc in M^* and ab is a true section of $L^{-1}(a^*b^*)$, it can be shown as in Theorem 11 that ab is imbedded in a two-cell in $L^{-1}(a^*b^*)$ and therefore the dimensionality of M^* may be verified in the same way as in Theorem 12.

It would be interesting to remove the restriction on the dimensionality of the minimal closed invariant sets in this theorem.

THEOREM 15. *If G is a one-parameter topological a.p.t.g. of R_3, and if the minimal closed invariant sets are at most one-dimensional, then the points whose orbits are points or simple closed curves are everywhere dense.*

In order to prove the theorem we need only concern ourselves with the moving points, for the fixed points clearly lie on orbits of the type described. Let a^* be any point in M^*, and let e^*f^* be an arc in M^* containing a^* as an interior point. The point a^* may be accessible from $M^*-e^*f^*$, but if not there is in any case a point b^* arbitrarily near a^* which is accessible from $M^*-e^*f^*$. It will be shown that either $L^{-1}(b^*)$ is a simple closed curve or that it has a simple closed curve as an orbit near it. Let ef be a true section of $L^{-1}(e^*f^*)$. Let b^*c^* be an arc in M^* which has only the point b^* in common with e^*f^*, and let bc be a true section of $L^{-1}(b^*c^*)$ with b in the set ef. Let H be the two-cell including ef, the existence of which was proved in Theorem 14.

Our definition of almost periodicity is equivalent in the case under consideration to the more familiar one,† and we can therefore conclude that ef and bc are so chosen that there is a set of real numbers g_n approaching infinity such that for any point x in $H+bc$, $d[g_n(x), x]<1/n$. If $g_n(H)$ intersects H for a large value of n, some point of H is on a periodic orbit, and if this is true no matter how small e^*f^* is, b is arbitrarily near a periodic orbit. This case may therefore be dismissed, and we turn to the case where for all n sufficiently large H and $g_n(H)$ do not intersect. Then a small open set U may be chosen with b as an interior point in such a way that H separates U into two connected parts U_1 and U_2. Assume that bc lies in $\overline{U}_2$. Choose n so large that $g_n(H)$ is very near H and $g_n(bc)$ is very near bc. How $g_n(H)$ must cut either U_1 or U_2, and we may assume that it is U_2; for if it cut U_1 it would only be necessary to interchange the parts played by H and $g_n(H)$. Now $g_n(H)$ must separate b from c in U_2, and hence $g_n(H)$ must intersect bc at an interior point, which contradicts our choice of b^*c^* and bc. This case having led us to a contradiction we conclude that the theorem is true.

† Bochner, Acta Mathematica, vol. 61, p. 154.

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CONSISTENCY OF THE CONDITIONS DETERMINING KOLLEKTIVS*

BY

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1. **Fundamental concepts.** With the establishment of the consistency of the conditions determining collectives,‡ we obtain a sound basis for the theory of probability. We shall see that certain slight modifications of these conditions will be necessary in order to establish consistency but these modifications will destroy none of the intrinsic value of von Mises' theory. In those alternative systems for which consistency has been established, the conditions are so much weaker than those of von Mises that much of the value of his theory has been lost.

The collective is a mathematical idealization of a sequence of occurrences (measurements, tosses of a coin, etc.). A collective K consists of an infinite sequence of elements (points of some space S). Thus§

$$K = e^{(1)}, e^{(2)}, e^{(3)}, \cdots . \tag{1}$$

The points m of S are called labels (Merkmale) and the space S the label space (Merkmalraum).‖

2. **Fields and distribution functions.** Let F be a set such that its elements are subsets of the space S and such that the sum, product, and difference (when defined) of any two of its elements also belong to F. Also let the space S be an element of F. Then F is called a field.¶ Associated with every element A of F, is a probability $\pi(A)$ satisfying the following conditions.

1. If A and B are any two mutually exclusive elements of F (i.e., $A \cdot B$ is the null set 0), then $\pi(A+B)=\pi(A)+\pi(B)$.
2. For every element A of F, $0 \leqq \pi(A)$.

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‡ The concept of collective (Kollektiv) is due to von Mises. See R. von Mises I, II, III. References to the literature are given at the end of this paper.

§ At a later point we shall state the restrictions which must be imposed upon K in order that it may be a collective.

‖ The letters m and e are used to refer respectively to points of S and elements of K. Two e's with different superscripts may be equal to the same element m of S. For example, in the tossing of a coin, the space S consists of two elements m_1 and m_2 (heads and tails) and in general each of these elements is repeated an infinite number of times in the corresponding collective.

¶ Kolmogoroff I.

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3. $\pi(S)=1$.

4. If $A_1, A_2, A_3, \cdots$ is any sequence of elements of S such that $A_1>A_2>A_3>\cdots$ and $\prod_{n=1}^{\infty}A_n=0$, then $\lim_{n\to\infty}\pi(A_n)=0$.

The function $\pi(A)$ satisfying the above conditions is a generalized measure function, but in this paper it will be referred to as a distribution function (Verteilung).

3. **Product spaces and independence.** Let $S_1, S_2, \cdots, S_\alpha$ be α label spaces. We shall denote the product space S by $S_1\times S_2\times\cdots\times S_\alpha$. If $m_1, m_2, \cdots, m_\alpha$ are respectively points of the spaces $S_1, S_2, \cdots, S_\alpha$, then $m=m_1\times m_2\times m_3\times\cdots\times m_\alpha$ is an element of the product space S. The element m may be considered as a point whose coordinates are $m_1, m_2, \cdots, m_\alpha$.

Let K_i be a collective whose label space is S_i $(i=1, 2, \cdots, \alpha)$ and let $e_i^{(n)}$ be the nth term of the collective K_i. We shall form a collective K whose nth element $e^{(n)}$ is given by the equation $e^{(n)}=e_1^{(n)}\times e_2^{(n)}\times e_3^{(n)}\times\cdots\times e_\alpha^{(n)}$. We shall call K the product collective and shall denote it by the expression $K_1\times K_2\times K_3\times\cdots\times K_\alpha$.* von Mises designates as a Verbindung the operation by which the product collective is formed.

Let F_i be a field of the space S_i and let A_i be a set of the field F_i $(i=1, 2, 3, \cdots, \alpha)$. Let A be the product set $A_1\times A_2\times\cdots\times A_\alpha$. Then A shall contain all points $m=m_1\times m_2\times\cdots\times m_\alpha$ such that m_i belongs to A_i. The product field $F=F_1\times F_2\times\cdots\times F_\alpha$ shall contain all product sets which can be formed from the fields $F_1, F_2, \cdots, F_\alpha$. It shall also contain sums, logical products, and differences of such sets. Thus a product field is itself a field.

Let us suppose that the collectives $K_1, K_2, \cdots, K_\alpha$ possess distribution functions $\pi_1(A_1), \pi_2(A_2), \cdots, \pi_\alpha(A_\alpha)$ and that the product collective $K=K_1\times K_2\times\cdots\times K_\alpha$ possesses the distribution function $\pi(A)$. Then the collectives $K_1, K_2, \cdots, K_\alpha$ will be called independent provided

$$\pi(A)=\pi(A_1\times A_2\times\cdots\times A_\alpha)=\pi_1(A_1)\cdot\pi_2(A_2)\cdot\cdots\pi_\alpha(A_\alpha).$$

It will be noted that the order of multiplication is immaterial to the definition of independence. Moreover if the collectives $K_1, K_2, \cdots, K_\alpha$ are independent and if $K_{n_1}, K_{n_2}, \cdots, K_{n_\alpha}$ is any subset of these collectives, then the collectives of this subset are independent. In order to define the independence of an infinite set of collectives, we shall place a restriction upon the product field. Namely if $A_1\times A_2\times\cdots\times A_\alpha\times\cdots$ is any product set of the product field, then only a finite number of the sets A_i will be proper subsets of the corresponding spaces S_i. The remaining sets A_i will be equal to the corresponding spaces S_i. Thus an infinite set of collectives is an independent set if and only if every finite subset is an independent set.

* Dörge I.

4. **Expected value and functions of collectives.** The expected value $p(K)$ of a collective K is defined as follows. Let

$$p_n(K) = \sum_{i=1}^{n} e^{(i)}/n.* \tag{2}$$

Then

$$p(K) = \lim_{n\to\infty} p_n(K).* \tag{3}$$

Functions of collectives are defined in the following manner. Let $f(m)$ be any function defined for all points of S and let $f(K)$ be the sequence

$$f(K) = f(e^{(1)}), f(e^{(2)}), f(e^{(3)}), \cdots. \tag{4}$$

This transforms the collective K into a new collective $f(K)$ and is a generalization of the operation von Mises calls Mischung.† Functions of any number of collectives are automatically defined in terms of the concepts of product spaces and product collectives.

A function which is of particular importance in this theory is the fundamental function of point sets.‡ Let A be any element of F and let $\phi_A(m)$ be the fundamental function of the set A. Then $\phi_A(m)$ has the value 1 when m is a point of A and the value 0 otherwise. In the case of the product space $S=S_1\times S_2\times \cdots \times S_\alpha$, the fundamental function satisfies the equation

$$\phi_{A_1\times A_2\times\cdots\times A_\alpha}(m_1 \times m_2 \times \cdots \times m_\alpha) = \phi_{A_1}(m_1)\cdot\phi_{A_2}(m_2)\cdots\phi_{A_\alpha}(m_\alpha).$$

The expected value $p[\phi_A(K)]$ of the collective $\phi_A(K)$ is the probability with respect to the collective K, of obtaining a point of the set A. That is

$$p[\phi_A(K)] = \pi(A). \tag{5}$$

This may be seen as follows. The summation $\sum_{i=1}^{n}\phi_A(e^{(i)})$ is the number of times that elements of A occur in the first n terms of K. Hence $p_n[\phi_A(K)]$ is the success ratio for elements of A in the first n terms of K. The limit of this success ratio $p[\phi_A(K)]$ is the probability. The collective is said to possess a distribution with respect to a field F provided the limit $\pi(A)$ exists for all sets A of F and provided $\pi(A)$ satisfies condition 4 of §2. It will be observed that conditions 1, 2, and 3 are satisfied automatically.

5. **Selections.** We shall now define an operation called selection (Aus-

* The expressions $p_n(K)$ and $p(K)$ will be used only when the corresponding space S is such that addition and division by a number have a meaning, and $p(K)$ will be used only when the limit exists.

† von Mises reserves the term Mischung for the case where the operation is single-valued and its inverse is multi-valued.

‡ de la Vallée Poussin I.

wahl). Let $n_1, n_2, n_3, \cdots$ be an increasing sequence of positive integers. From any collective K we can form a new collective K' by selecting the n_1st, n_2nd, $\cdots$ terms of K. Thus

$$K' = e^{(n_1)}, e^{(n_2)}, e^{(n_3)}, \cdots .$$

It is convenient to introduce a notation for this operation. Let $x = x^{(1)}, x^{(2)}, x^{(3)}, \cdots$ be a sequence whose terms $x^{(n_1)}, x^{(n_2)}, \cdots$ are all 1's and the rest of whose terms are all 0's. Thus x is a sequence similar to K and the associated label space consists of the elements 1 and 0. We shall denote the operation of selection by the equation $K' = K \subset x$.* It should be observed that $f(K \subset x) = f(K) \subset x$.

A selection x may be regarded as a number written in the binary scale, with digits $x^{(1)}, x^{(2)}, \cdots$ and with the radix point placed immediately before the first digit. Any number in the interval from 0 to 1 (0 excluded) can represent a selection. In the case of the ambiguous representation, the ambiguity is resolved by means of the condition that an infinite number of the terms of the collective must be selected and hence an infinite number of the digits of x must be equal to 1.

Let us operate on the collective K by a selection x_1, obtaining a new collective $K' = K \subset x_1$. Let us operate on the resulting collective K' by a selection x_2, obtaining $K'' = K' \subset x_2$. Let us suppose that x_1 selects the n_1st, n_2nd, $\cdots$ terms of K and that x_2 selects the ν_1st, ν_2nd, $\cdots$ terms of K'. The resultant of the two selections is equivalent to a single operation x which selects the n_{ν_1}st, n_{ν_2}nd, $\cdots$ terms of K. That is $(K \subset x_1) \subset x_2 = K \subset x$.

A set D of selections will be called a fundamental set provided the resultant of any two selections of D is a selection of D and provided D contains the identity selection $1 = 1, 1, 1, \cdots$. A fundamental set fails to be a group since a selection can have no unique inverse. The set consisting only of the identity is an example of a fundamental set. A second example is the set consisting of all possible selections. As a third example we shall construct a fundamental set D_0 which will be used to define independence of terms of a collective. We shall denote the elements of D_0 by the symbol $x_{r,n}$ where r and n are positive integers. The operator $x_{r,n}$ selects the rth, the $(r+n)$th, the $(r+2n)$th, the $(r+3n)$th, $\cdots$ terms of a collective K. The resultant of the operation $x_{r,n}$ followed by the operation $x_{\rho,\nu}$ is equivalent to the operation $x_{R,N}$ where $R = r + (\rho - 1)n$ and $N = n\nu$. The selection $x_{1,1}$ is the identity.

6. **Independence of the terms of a collective.** If a well balanced coin is thrown twice, the probability of obtaining heads on the first throw and tails

* The symbol $\subset$ is the inverted implication sign and is interpreted by the word "if." For this interpretation, see Copeland I.

on the second is 1/4. To verify this experimentally, it would be natural to throw a coin twice and observe whether or not heads were obtained on the first trial and tails on the second. Then repeat the experiment a number of times. When n such experiments have been made, the coin has been thrown $2n$ times. That is, there are determined n groups of two throws each. These throws constitute the first $2n$ terms of a collective K. Let m_1 and m_2 represent respectively the labels heads and tails. Then $\phi_{m_1}(K \subset x_{1,2})$ is a sequence of 1's and 0's such that a 1 occurs when heads is obtained on the first throw of a group of two, and a 0 otherwise. Similarly $\phi_{m_2}(K \subset x_{2,2})$ is a sequence representing tails on the second throw of each group. The product $\phi_{m_1}(K \subset x_{1,2}) \cdot \phi_{m_2}(K \subset x_{2,2})$ is the sequence whose probability we are investigating. We should expect the success ratio $p_n[\phi_{m_1}(K \subset x_{1,2}) \cdot \phi_{m_2}(K \subset x_{2,2})]$ to be approximately 1/4 and the probability $p[\phi_{m_1}(K \subset x_{1,2}) \cdot \phi_{m_2}(K \subset x_{2,2})]$ to be equal to 1/4.

In a like manner we should have

$$p[\phi_{m_2}(K \subset x_{1,3}) \cdot \phi_{m_1}(K \subset x_{2,3}) \cdot \phi_{m_1}(K \subset x_{3,3})] = p[\phi_{m_2}(K \subset x_{1,3})] \cdot p[\phi_{m_1}(K \subset x_{2,3})] \cdot p[\phi_{m_1}(K \subset x_{3,3})] = 1/8,$$

since the probability of tails on the first throw, heads on the second, and heads on the third is 1/8. The same probability should be obtained for the combinations m_1, m_1, m_1; m_1, m_1, m_2; m_1, m_2, m_1; $\cdots$; m_2, m_2, m_2. That is, the sequences $K \subset x_{1,3}$, $K \subset x_{2,3}$, $K \subset x_{3,3}$ should be independent. We shall say that the terms of a collective K are independent provided the sequences $K \subset x_{1,n}$, $K \subset x_{2,n}$, $\cdots$, $K \subset x_{n,n}$ are independent for every positive integer n.

7. **Contractions.** If K is any collective and A is any subset of the corresponding label space, then $\phi_A(K)$ is a sequence of 1's and 0's and hence can be used as a selection operator. The operation $K \subset \phi_A(K)$ is called a contraction (Teilung).* By means of the contraction, the collective K is transformed into a new collective with the contracted label space A.

The operations of selection and contraction can be combined. Thus we may perform a contraction $K \subset \phi_A(K)$ on the collective K and then operate on the resulting collective by a selection x. The resultant of the two operations is denoted as follows, $K \subset \phi_A(K) \cdot \subset \cdot x$. The two dots are used as brackets.† Everything to the left of the first dot is understood to be enclosed in one bracket and the x to the right of the second dot is enclosed in another. Thus the expression $K \subset \phi_A(K) \cdot \subset \cdot x$ means $[K \subset \phi_A(K)] \subset [x]$. That is, the dots indicate that the contraction is performed before the selection. In case

* Contraction is not intended as a translation of the word Teilung but it is descriptive of the operation.

† Lewis and Langford I.

further bracketing is required, two dots may be used to indicate a stronger bracket than one dot. For example, $K \subset x_1 \cdot \subset \cdot \phi_A(K \subset x_1) : \subset : x_2$ means $\{[K \subset x_1] \subset [\phi_A(K \subset x_1)]\} \subset \{x_2\}$. It should be observed that $K \subset \phi_A(K) \cdot \subset \cdot \phi_B[K \subset \phi_A(K)] = K \subset \phi_{A \cdot B}(K)$ for any two sets A and B. That is, the resultant of two contractions is a contraction.

8. **Restrictions on collectives.** von Mises places the following restrictions on the collective.

(M1) The limit $p[\phi_A(K)]$ exists for every element A of the field F.

(M2) If A and B are any two mutually exclusive elements of F such that $p[\phi_A(K)]$ and $p[\phi_B(K)]$ are not both 0 and if x is any selection of the fundamental set D, then the limits $p\{\phi_A[K \subset \phi_{A+B}(K) \cdot \subset \cdot x]\}$ and $p\{\phi_B[K \subset \phi_{A+B}(K) \cdot \subset \cdot x]\}$ exist and

$$p\{\phi_A[K \subset \phi_{A+B}(K) \cdot \subset \cdot x]\} / p\{\phi_B[K \subset \phi_{A+B}(K) \cdot \subset \cdot x]\} = p[\phi_A(K)] / p[\phi_B(K)].$$

(M3) The set D contains all selections which are independent of K.

(M4) The field F contains all possible subsets of the label space F.

The first restriction states that the collective must possess a distribution. The second restriction can be most easily understood by specializing the operators. Let us consider the case in which x is the identity selection. The terms of $K \subset \phi_{A+B}(K)$ belong either to A or to B and hence

$$p\{\phi_A[K \subset \phi_{A+B}(K)]\} + p\{\phi_B[K \subset \phi_{A+B}(K)]\} = 1.$$

From this equation and from condition 2, it follows by simple algebra that

$$p\{\phi_A[K \subset \phi_{A+B}(K)]\} = p[\phi_A(K)] / \{p[\phi_A(K)] + p[\phi_B(K)]\}$$

and

$$p\{\phi_B[K \subset \phi_{A+B}(K)]\} = p[\phi_B(K)] / \{p[\phi_A(K)] + p[\phi_B(K)]\}.$$

That is, the probability that a term of K will lie in A if it lies in A or B is equal to the probability that a term of K will lie in A divided by the probability that it will lie in either A or B. Next let us specialize by letting $A+B=S$. In this case the contraction reduces to the identity. Hence it is easily seen that condition (M2) implies that the distribution of a collective is invariant under the operation of all selections of the set D.

There has been considerable discussion concerning the meaning of condition (M3).* In fact, this seems to be the most serious difficulty in von Mises' theory. Since the objections have been adequately treated in other papers it is not necessary to repeat them here. We shall however give an alternative restriction to which these objections do not apply.

* Kamke I.

It is also necessary to weaken condition (M4). I have proved that conditions (M1) and (M4) cannot both be satisfied if the distribution function is absolutely additive and absolutely continuous.* Thus, for example, these conditions exclude any collective whose probability is distributed in accordance with the normal law.

9. **An alternative set of conditions.** We shall denote by $C[D, \pi(E)]$ the set of all collectives associated with a given fundamental set D and a distribution function $\pi(E)$ defined on a field F. The range of $\pi(E)$ determines the field. We shall demand that the set of collectives belonging to $C[D, \pi(E)]$ satisfy the following conditions.

(C1) $p[\phi_{E_1}(K \subset x_{r_1,n}) \cdot \phi_{E_2}(K \subset x_{r_2,n}) \cdots \phi_{E_h}(K \subset x_{r_h,n})] = \pi(E_1) \cdot \pi(E_2) \cdot \pi(E_3) \cdots \pi(E_h)$ for every set of integers $r_1, r_2, r_3, \cdots, r_h, n$ such that $0 < r_1 < r_2 < \cdots < r_h \leqq n$ and for every set $E_1, E_2, E_3, \cdots, E_h$ of elements of F.

(C2) $K \subset \phi_A(K)$ is an element of $C[D, \pi(E)/\pi(A)]$ for every element A of F such that $0 < \pi(A)$.

(C3) $K \subset x$ is an element of $C[D, \pi(E)]$ for every element x belonging to D.

The collectives satisfying these conditions also satisfy (M1) and (M2). Thus suppose we specialize condition (C1) by letting $h = r_1 = n = 1$. Then since $x_{1,1}$ is the identity selection, (C1) reduces to the condition $p[\phi_E(K)] = \pi(E)$. Next let us observe that if E_1 and E_2 are any two mutually exclusive elements of F such that $\pi(E_1)$ and $\pi(E_2)$ are not both zero, then conditions (C1), (C2), and (C3) imply that

$$p\{\phi_{E_1}[K \subset \phi_{E_1+E_2}(K) \cdot \subset \cdot x]\} = \pi(E_1)/\pi(E_1 + E_2)$$

and

$$p\{\phi_{E_2}[K \subset \phi_{E_1+E_2}(K) \cdot \subset \cdot x]\} = \pi(E_2)/\pi(E_1 + E_2).$$

Conditions (C1), (C2), and (C3) are much stronger than (M1) and (M2). In the first place, (C1) demands that the terms of a collective be independent. Secondly, (C2) and (C3) enable us to operate on a collective by any finite group of selections and contractions, whereas condition (M2) only permits operation first with a contraction, then with a selection. Due to the recursive nature of (C2) and (C3), it is in general true that when one collective of the $C[D, \pi(E)]$ has been obtained, a continuum of such collectives is automatically determined.

It will be recalled that conditions (M3) and (M4) relate to the choice of the fundamental set D and the field F. Whether or not there exist collectives

* Copeland II.

belonging to $C[D, \pi(E)]$ depends upon the choice of D and F. Thus, for example, if D contains all possible selections and if F contains two mutually exclusive elements A and B such that neither $\pi(A)$ nor $\pi(B)$ is equal to zero, then $C[D, \pi(E)]$ is null. In this paper we shall assume that D is denumerable. The restriction which we shall place on F will depend upon the type of label space. Thus if S is finite, F may be unrestricted. If S is euclidean, we shall assume that F consists of all subsets of S whose frontier points are of Lebesgue measure zero.

10. **Special fields.** We shall consider certain special spaces and fields for which the corresponding collectives can be more easily constructed. The collectives thus obtained constitute a basic set in terms of which more general collectives can be formed. We shall construct a space S consisting of points whose coordinates $y_1, y_2, \cdots, y_\alpha$ satisfy the inequalities $0 \leqq y_i < 1$, $i = 1, 2, \cdots, \alpha$. We shall define a field F for S. A set A will belong to F if and only if the frontier points of A constitute a set of Lebesgue measure zero. It can readily be verified that F is a field. We shall let $\pi(A)$ be any absolutely continuous distribution function whose range is F.

Since the collective is a sequence of points and hence denumerable, it is not necessary to use all of the space S. Hence we shall define a space S' consisting of a denumerable everywhere dense set of points in the space S. A subset A' of S' will be an element of the corresponding field F' if and only if there exists an element A of F such that $A \cdot S' = A'$. A distribution function $\pi'(A')$ whose range is F' is defined by the equation $\pi'(A \cdot S') = \pi(A)$.

It is necessary to show that the above distribution function is uniquely defined. In order to do this, we shall first prove that if A' is any element of F' and A_1 and A_2 are elements of F such that $A_1 \cdot S' = A_2 \cdot S' = A'$, then A_1 and A_2 differ by at most a set of points of measure zero. Let f_1 and f_2 be respectively the frontier points of A_1 and A_2 and let $f = f_1 + f_2$. Then the sets $B_1 = A_1 - A_1 \cdot f$ and $B_2 = A_2 - A_2 \cdot f$ are open. Let p be any point of the set B_1. Then it is possible to enclose p in a spherical region σ such that all points of σ belong to A_1. Hence σ must contain points of the set A'. But all points of A' are also points of A_2. Hence p is a limit point of points of A_2. Moreover p is not a point of f_2 since it is not a point of f. Consequently p must belong to A_2. Therefore $B_1 < A_2$ and similarly $B_2 < A_1$. It is now easily seen that the sets $A_1 - A_1 \cdot A_2$ and $A_2 - A_2 \cdot A_1$ are both included in f and hence A_1 and A_2 differ by at most a set of points of measure zero. It follows from the absolute continuity of $\pi(A)$ that $\pi(A_1) = \pi(A_2)$. Hence $\pi'(A')$ is uniquely determined for any set A' of F'.

If K is any collective with respect to the space S' and the distribution function $\pi'(A')$, then K is a collective with respect to the space S and the

distribution function $\pi(A)$, since if $e^{(n)}$ is any term of K, it will belong to the set A if and only if it belongs to the corresponding set A'.

We shall consider fields which are still further restricted. We shall decompose the space S into 2^α mutually exclusive hypercubes $m_{1,1}, m_{1,2}, \cdots, m_{1,2^\alpha}$. The hypercube $m_{1,j}$ is called a mesh and consists of the points whose coordinates $y_1, y_2, \cdots, y_\alpha$ satisfy the inequalities $a_{i,j}/2 \leqq y_i < (a_{i,j}+1)/2$ where each $a_{i,j}$ has the value 1 or 0 and $i = 1, 2, \cdots, \alpha$. Then $m_{1,1}+m_{1,2}+\cdots+m_{1,2^\alpha}=S$. We shall designate as the lattice L_1, the field which consists of the null set and sums of the meshes $m_{1,j}$. We shall form a lattice L_2 by decomposing each of the meshes $m_{1,j}$ into 2^α hypercubes (or meshes) in the same manner in which we decomposed the space S to form the lattice L_1. Thus L_2 contains the meshes $m_{2,1}, m_{2,2}, \cdots, m_{2,2^{2\alpha}}$. In a similar manner we shall form lattices $L_3, L_4, \cdots$. Thus the lattice L_s contains $2^{s\alpha}$ mutually exclusive hypercubes or meshes. We shall assume that the subscripts of the meshes are so chosen that

$$m_{s-1,j+1} = m_{s,j\cdot 2^\alpha+1} + m_{s,j\cdot 2^\alpha+2} + \cdots + m_{s,(j+1)2^\alpha}.$$

We shall let $N = L_1+L_2+\cdots$ and shall call N a net. It is easily seen that N is a field.

The point whose coordinates are $a_{1,j}/2, a_{2,j}/2, \cdots, a_{\alpha,j}/2$ will be called the vertex of the mesh $m_{1,j}$. The vertices of the meshes of the lattices $L_2, L_3, \cdots$ will be located in corresponding positions. Thus for the case $\alpha = 2$, a vertex is located in the lower left-hand corner of its mesh. Let S'' be the space which consists of all the vertices of the meshes of N. Then S'' is denumerable and everywhere dense in S. We shall define a field F'' of the space S'' as follows. A set A'' will be an element of F'' if and only if there exists an element A of the net N such that $A'' = A \cdot S''$. The field F'' may be regarded as an approximation to the field F since if A is a set whose frontier points are of measure zero and if ϵ is any given positive number, there will exist two elements A_1 and A_2 of N such that $A_1 < A < A_2$ and $m(A_2-A_1) < \epsilon$.*

11. **Groups of operations.** We shall obtain a more convenient form for conditions (C1), (C2), and (C3) as applied to the space S' and the field F' defined in §10. Conditions (C2) and (C3) enable us to operate on a collective with any finite group of selections and contractions. The resultant of these operations is a collective and hence must satisfy (C1). Since the resultant of two contractions is a contraction and the resultant of two selections of D is a selection of D and since D contains the identity selection and there exists an identity contraction, there will be no loss of generality in assuming that the selections and contractions of such a group alternate, beginning with

* Copeland II.

a selection and ending with a contraction. Let the selections of such a group be denoted by $x_1, x_2, \cdots, x_\sigma$ and let the contractions be performed with respect to the sets $A_1, A_2, \cdots, A_\sigma$ where $A_\sigma < A_{\sigma-1} < \cdots < A_2 < A_1$. Then the group is ordered as follows: $x_1, A_1, x_2, A_2, \cdots, x_\sigma, A_\sigma$.

Condition (C1) involves a group of selections $x_{r_1,n}, x_{r_2,n}, \cdots, x_{r_h,n}$ of the set D_0 and a group of sets $E_1, E_2, \cdots, E_h$. Let $\mu = \sigma + h$. Then conditions (C1), (C2), (C3) may be stated in terms of certain groups of operations involving μ selections and μ sets where $\mu = 1, 2, 3, \cdots$.

The set $D+D_0$ is denumerable. Hence we can assume the existence of a one-to-one correspondence between the set of all non-negative integers λ and the set of all ordered finite subsets of $D+D_0$. We shall demand that corresponding to each λ, the group of selections contains at least one from the set D_0. The space S' is denumerable and the field F' has the power of the continuum. Hence we can assume that a one-to-one correspondence has been established between the elements of F' and the set of numbers in the interval $0 \leqq \vartheta < 1$. Let t be any non-negative real number and let $\lambda = [t]$ be the largest integer contained in t. Then t determines λ and λ determines a set of μ selections. Hence t determines the integer μ. Let $\vartheta = t - [t]$ be written in the binary scale. This number may be regarded as a sequence and hence can be operated upon by the selection $x_{1,\mu}$.* The sequence $\vartheta \subset x_{1,\mu}$ may in turn be regarded as a number in the interval from 0 to 1. This number determines a set A_1. The numbers $\vartheta \subset x_{2,\mu}, \vartheta \subset x_{3,\mu}, \cdots, \vartheta \subset x_{\sigma,\mu}, \vartheta \subset x_{\sigma+1,\mu}, \cdots, \vartheta \subset x_{\mu,\mu}$ determine respectively the sets $A_2, A_3, \cdots, A_\sigma, E_1, E_2, \cdots, E_h$.

The group of operations thus performed on the collective can now be defined as follows. Let

$$O_{i,t}(K) = O_{i-1,t}(K) \subset x_i \cdot \subset \cdot \phi_{A_i}[O_{i-1,t}(K) \subset x_i],$$

where $0 < i \leqq \sigma$ and $O_{0,t}(K) = K$. Let

$$O_t(K) = \phi_{E_1}[O_{\sigma,t}(K) \subset x_{r_1,n}] \cdot \phi_{E_2}[O_{\sigma,t}(K) \subset x_{r_2,n}] \cdots \phi_{E_h}[O_{\sigma,t}(K) \subset x_{r_h,n}]$$

and

$$P[t, \pi(E)] = \pi(E_1) \cdot \pi(E_2) \cdots \pi(E_h) \cdot \pi^{-h}(A_\sigma).$$

Let Σ' be the set of points t such that $A_1 > A_2 > \cdots > A_\sigma > E_1 + E_2 + \cdots + E_h$ and $\pi(A_\sigma) > 0$. Then conditions (C1), (C2), (C3) are equivalent to the following

(C4) $\qquad p[O_t(K)] = P[t, \pi(E)]$ for every t belonging to Σ'.

In the case of the space S'' and the field F'' defined in §10, a similar set

* The ambiguous representation is resolved in the same manner as in the case of selections except that the number 0 is the sequence 0, 0, 0, $\cdots$.

of conditions can be obtained. Namely let Σ'' be the subset of Σ' for which $A_1, A_2, \cdots, A_\sigma, E_1, E_2, \cdots, E_h$ are elements of the field F''. Then for the field F'', conditions (C1), (C2), (C3) have the form

(C5) $\quad p[O_t(K)] = P[t, \pi(E)]$ for every t belonging to Σ''.

12. **Existence of collectives.** We shall prove the existence of collectives satisfying condition (C5). A one-dimensional label space S will be chosen and the mesh $m_{s,j}$ will be the interval $2^{-s}(j-1) \leqq y < 2^{-s} \cdot j$. The proof is accomplished by the aid of a new number scale. Let $y = \sum_{\nu=1}^{\infty} y^{(\nu)} \cdot 2^{-\nu(\nu+1)/2}$ where $y^{(\nu)} = 0, 1, 2, \cdots, 2^\nu - 1$. Any number y in the closed interval from 0 to 1 can be so developed. This number scale possesses the same type of ambiguity as is found in ordinary scales. Let $K(y)$ be the sequence

$$K(y) = P^{(1)}(y), P^{(2)}(y), P^{(3)}(y), \cdots,$$

where $P^{(\nu)}(y) = y^{(\nu)} \cdot 2^{-\nu}$. We shall show that for almost every y of the interval from 0 to 1, the corresponding sequence $K(y)$ satisfies condition (C5). The proof of this statement is accomplished by means of four lemmas.

The first lemma is concerned with the measure of a set

$$E = E[P^{(n_1)}(y) \epsilon E_1 \cdot P^{(n_2)}(y) \epsilon E_2 \cdots P^{(n_\nu)}(y) \epsilon E_\nu]$$

defined as follows. The expression $P^{(n_i)}(y) \epsilon E_i$ means that the point $P^{(n_i)}(y)$ is an element of the set E_i. The dots joining expressions of this type are conjunctive. The set E consists of all points y which satisfy the conjunction of all conditions within the square brackets. The set E_i is the sum of a_i meshes $m_{s,j}$ where s is independent of i. Hence $m(E_i) = a_i 2^{-s}$.

LEMMA 1. $m\{E[P^{(n_1)}(y) \epsilon E_1 \cdot P^{(n_2)}(y) \epsilon E_2 \cdots P^{(n_\nu)}(y) \epsilon E_\nu] = m(E_1) \cdot m(E_2) \cdots m(E_\nu)$ *if* $s \leqq n_1 < n_2 < \cdots < n_\nu$.

The measure of the set of points y for which the first n_ν digits are prescribed is $2^{-n_\nu(n_\nu+1)/2}$. Thus the proof resolves itself into the problem of counting the number of possible combinations of the first n_ν digits of y for which the conditions within the square brackets are satisfied. The first condition will be satisfied if $P^{(n_1)}(y)$ belongs to one of the a_1 meshes $m_{s,j}$. The point $P^{(n_1)}(y)$ will belong to the mesh $m_{s,j}$ if $2^{-s} \cdot (j-1) \leqq y^{(n_1)} < 2^{-s} \cdot j$. Hence there are 2^{n_1-s} values for the digit $y^{(n_1)}$ consistent with the condition that $P^{(n_1)}(y)$ belongs to $m_{s,j}$. Therefore the number of values for this digit consistent with the first condition within the square brackets is $a_1 \cdot 2^{n_1-s}$. The digits $y^{(n_2)}, y^{(n_3)}, \cdots, y^{(n_\nu)}$ can be similarly treated. There remain certain digits of y which are entirely unrestricted. That is, the first digit of y can be selected in 2^1 ways, the second in 2^2 ways, $\cdots$, the (n_1-1)st in 2^{n_1-1} ways, the (n_1+1)st in 2^{n_1+1} ways, $\cdots$.

Thus these unrestricted digits can be selected in $2^{1+2+\cdots+n_\nu-n_1-n_2-\cdots-n_\nu}$ $=2^{n_\nu(n_\nu+1)/2-n_1-n_2-\cdots-n_\nu}$ ways. It follows that

$$m\{E[P^{(n_1)}(y)\epsilon E_1 \cdot P^{(n_2)}(y)\epsilon E_2 \cdots]\}$$
$$= a_1 2^{n_1-s} a_2 2^{n_2-s} \cdots a_\nu 2^{n_\nu-s} 2^{n_\nu(n_\nu+1)/2-n_1-n_2-\cdots-n_\nu} 2^{-n_\nu(n_\nu+1)/2}$$
$$= m(E_1)\cdot m(E_2)\cdots m(E_\nu).$$

The second lemma will be similar to the first except that it will be applied to the sequence $O_{\sigma,t}[K(y)]$ instead of to $K(y)$. We shall assume that the sets $A_1, A_2, \cdots, A_\sigma$ used in defining the operator $O_{\sigma,t}$ and the sets $E_1, E_2, \cdots, E_\nu$ used in the statement of the lemma are all sums of meshes $m_{s,j}$ with the same subscript s. Let

$$O_{i,t}[K(y)] = P_i^{(1)}(y), P_i^{(2)}(y), \cdots.$$

LEMMA 2. $m\{E[P_\sigma^{(n_1)}(y)\epsilon E_1 \cdot P_\sigma^{(n_2)}(y)\epsilon E_2 \cdots]\} = m(E_1)\cdot m(E_2)\cdots m(E_\nu)$ $\cdot m^{-\nu}(A_\sigma)$ *if* $s \leqq n_1 < n_2 < \cdots < n_\nu$.

We shall prove the lemma inductively. That is, we shall prove that if the equation is true when σ is replaced by $i-1$, it will then be true when σ is replaced by i. If $\sigma=0$, then the above equation reduces to the corresponding equation of Lemma 1, since $m^{-\nu}(S)=1$. The sequence $O_{i,t}[K(y)]$ is obtained from the sequence $O_{i-1,t}[K(y)]$ by the selection x_i and the contraction with respect to the set A_i. Both of these operations throw away certain terms of the sequence $O_{i-1,t}[K(y)]$. The terms thrown away by the selection are entirely unrestricted. However any term thrown away by the contraction must belong to the set A_{i-1} but not to the set A_i. On the basis of the assumption that the lemma is true when σ is replaced by $i-1$, it follows that the measure of the set of points for which the n_νth term of $O_{i,t}[K(y)]$ corresponds to the $(n_\nu+\mu)$th term of the sequence $O_{i-1,t}[K(y)] \subset x_i$ and for which $P^{(n_1)}(y)\epsilon E_1 \cdot P^{(n_2)}(y)\epsilon E_2 \cdots P^{(n_\nu)}(y)\epsilon E_\nu$ is $m(E_1)\cdot m(E_2)\cdots m(E_\nu)$ $\cdot m^{n_\nu-\nu}(A_i)[m(A_{i-1})-m(A_i)]^\mu \cdot m^{-n_\nu-\mu}(A_{i-1})$. That is, a certain ν of the terms of $O_{i-1,t}[K(y)]$ must belong respectively to the sets $E_1, E_2, \cdots, E_\nu$ and $n_\nu-\nu$ of the terms must belong to A_i while the remaining μ terms must belong to A_{i-1} but not to A_i. The $(n_\nu+\mu)$th term must belong to E_ν. Since μ can take on all values from 0 to ∞ and since there are ${}_{n_\nu+\mu-1}C_\mu$ ways in which the μ terms can be selected, then

$$m\{E[P_i^{(n_1)}(y)\epsilon E_1 \cdot P_i^{(n_2)}(y)\epsilon E_2 \cdots]\}$$
$$= m(E_1)\cdot m(E_2)\cdots m(E_\nu)\cdot m^{n_\nu-\nu}(A_i) m^{-n_\nu}(A_{i-1}) \cdot \sum_{\mu=0}^{\infty} {}_{n_\nu+\mu-1}C_\mu$$
$$\cdot [m(A_{i-1}) - m(A_i)]^\mu \cdot m^{-\mu}(A_{i-1})$$

$$= m(E_1)\cdot m(E_2)\cdot\cdots m(E_\nu)\cdot m^{-\nu}(A_i)\cdot m^{n_\nu}(A_i)\cdot m^{-n_\nu}(A_{i-1})$$
$$\cdot\left[1-\frac{m(A_{i-1})-m(A_i)}{m(A_{i-1})}\right]^{-n_\nu}$$
$$= m(E_1)\cdot m(E_2)\cdot\cdots m(E_\nu)\cdot m^{-\nu}(A_i).$$

It follows inductively that this equation still holds when $i=\sigma$.

It will be observed that the operator $x_{s,1}$ selects the sth term and all of the succeeding terms of a sequence K. If the expression $K(y)$ in Lemma 2 is replaced by $K'(y)=K(y)\subset x_{s,1}$, then the condition $s\leqq n_1$ will no longer be necessary.

LEMMA 3. *The function*

$$V'(y,t)=\limsup_{\mu\to\infty}\left|\,p_\mu\{O_t[K'(y)]\}-P[t,m(E)]\,\right|$$

is equal to 0 *almost everywhere in the interval* $\Delta: 0<y<1$.

If we let $E[V'(y,t)>\epsilon]$ be the set of points for which $V'(y,t)>\epsilon$, then

$$E(V'>0)=E(V'>1/2)+E(V'>1/3)+E(V'>1/4)+\cdots.$$

Hence the lemma will be established if we can prove that $m[E(V'>\epsilon)]=0$ for every positive number ϵ. We have the equation

$$E(V'>\epsilon)=\lim_{\mu_0\to\infty}\sum_{\mu=\mu_0}^{\infty}E(\left|\,p_\mu\{O_t[K'(y)]\}-P[t,m(E)]\,\right|>\epsilon),$$

and hence

$$m[E(V'>\epsilon)]\leqq\sum_{\mu=\mu_0}^{\infty}m[E(\left|\,p_\mu\{O_t[K'(y)]\}-P[t,m(E)]\,\right|>\epsilon)]$$

for every positive integer μ_0. The remainder of the proof consists in establishing the convergence of this series. From Lemma 2 it follows that

$$m[E(\left|\,p_\mu\{O_t[K'(y)]\}-P[t,m(E)]\,\right|>\epsilon)]=\sum_{|s/\mu-P[t,m(E)]|>\epsilon}{}_\mu C_s$$
$$\cdot P^s[t,m(E)]\{1-P[t,m(E)]\}^{\mu-s},$$

where the expression below the summation sign indicates that s is summed over all values consistent with this inequality. Borel has proved the convergence of all series of the form*

$$\sum_{\mu=1}^{\infty}\sum_{|s/\mu-p|>\epsilon}{}_\mu C_s\cdot p^s\cdot q^{\mu-s},$$

* Borel I.

where p and q are positive numbers and $p+q=1$. Hence Lemma 3 is established.

The purpose of Lemma 4 is to show that Lemma 3 will still be true if $K'(y)$ is replaced by $K(y)$. We shall first construct a set D' defined as follows. If x is any selection of D and if s is any positive integer, then there will exist a selection x' of D' such that

$$K \subset x \cdot \subset \cdot x_{s',1} = K \subset x_{s,1} \cdot \subset \cdot x',$$

where s' is such that either $n_{s'-1} < s \leqq n_{s'}$ or $s \leqq n_1$, and where $n_1, n_2, n_3, \cdots$ are the ordinal numbers of the terms selected by x. Thus $n_{s'}-s+1$, $n_{s'+1}-s+1, \cdots$ are the ordinal numbers of the terms selected by x'. Since the selection $x_{1,1}$ is the identity, it follows that $D<D'$. Moreover D' is denumerable. In Lemma 4 it will be assumed that the set Σ'' is defined in terms of D'.

LEMMA 4. *If t is any number belonging to the set Σ'' and if s is any positive integer, then there exists a number t' belonging to Σ'' and a positive integer r such that*

$$O_t(K) \subset x_{r,1} = O_{t'}(K \subset x_{s,1}).$$

Given the selection x and the positive integer s, there exist a selection x' and a positive integer s' such that $K \subset x \cdot \subset \cdot x_{s',1} = K \subset x_{s,1} \cdot \subset \cdot x'$. There exists a second positive integer s'' such that $K \subset x \cdot \subset \cdot x_{s',1} : \subset : \phi_E(K \subset x \cdot \subset \cdot x_{s',1}) = K \subset x \cdot \subset \cdot \phi_E(K \subset x) : \subset : x_{s'',1}$. From this it follows inductively that there exist a positive integer r and a number t' belonging to Σ'' such that

$$O_{\sigma,t}(K) \subset x_{r,1} = O_{\sigma,t'}(K \subset x_{s,1}).$$

The number t' differs from t because the selections involved in the operator $O_{\sigma,t'}$ are different from those involved in the operator $O_{\sigma,t}$. The contractions are the same for both operators. Since

$$\phi_E[O_{\sigma,t}(K) \subset x_{r,1}] = \phi_E[O_{\sigma,t}(K)] \subset x_{r,1},$$

we have the equation

$$O_t(K) \subset x_{r,1} = O_{t'}(K \subset x_{s,1}).$$

With the aid of Lemmas 1, 2, 3, 4, we can prove the following theorem.

THEOREM 1. *For almost every y of the interval $\Delta: 0<y<1$, the corresponding sequence $K(y)$ satisfies condition* (C5).

Let

$$V'(y, t') = \limsup_{\mu \to \infty} \left| \, p_\mu \{O_{t'}[K(y) \subset x_{s,1}]\} - P[t, m(E)] \, \right|,$$

$$V(y, t) = \limsup_{\mu \to \infty} \left| p_\mu\{O_t[K(y)]\} - P[t, m(E)] \right|.$$

The sets involved in the operator $O_{t'}$ are sums of meshes $m_{s,j}$. Of course for any given operator $O_{t'}$, it is only necessary to let s be the subscript of the finest mesh occurring in the sets corresponding to this operator. Since the contractions involved in $O_{t'}$ are the same as those involved in O_t, s depends directly upon t. The number t' depends upon both t and y. Let M be the set of points y for which $V'(y, t') = 0$ for every t' of the set Σ''. Since Σ'' is denumerable, it follows from Lemma 3 that $m(M) = 1$. Let y be any point of M and t any point of Σ''. Then by Lemma 4 there exists a pair of numbers r and t' such that

$$p\{O_{t'}[K(y) \subset x_{s,1}]\} = p\{O_t[K(y)] \subset x_{r,1}\} = p\{O_t[K(y)]\}.$$

And hence $V(y, t) = V'(y, t') = 0$. Therefore for almost every y, the corresponding sequence $K(y)$ satisfies condition (C5).

13. **Segments of collectives.** We shall now consider the problem of constructing collectives satisfying condition (C4). The method will appear considerably different from that of the previous problem but will in fact be guided by it. A study of the differences between these two problems will be of advantage in the work which we shall do. We now have to deal with sets whose frontier points are of measure zero instead of with sets consisting of finite sums of meshes. The collective $K(y)$ is so constructed that each term is capable of adapting itself to a finer mesh than is the preceding term. That is, the point $P^{(s)}(y)$ can belong to any one of the meshes of the lattice L_s whereas the point $P^{(s-1)}(y)$ cannot. Thus after a certain term in this collective all points are capable of adapting themselves to any given set consisting of a finite sum of meshes. However if we have a set whose frontier points are of measure zero, then in general the adaptation is never complete. Moreover the tendency toward adaptation is so rapid that it is difficult to follow the variation of the success ratio. Our first point of difference from the preceding method will be for the purpose of slowing up this adaptation.

We shall generalize the method of formation of the sequence $K(y)$. Let $N_1, N_2, N_3, \cdots$ be a sequence of positive integers which will be determined at a later point. Let $\nu_s = N_1 + N_2 + \cdots + N_{s-1}$ if $1 < s$ and $\nu_1 = 0$. Let $P^{(k)}(y) = [y^{(k)} \cdot 2^{s-k}] \cdot 2^{-s}$ where s is determined so that $\nu_s < k \leqq \nu_{s+1}$. It will be observed that if $N_1 = N_2 = \cdots = 1$, then $k = s$ and this definition of $K(y)$ is the same as the previous one. We shall say that the points $P^{(\nu_s+1)}(y), P^{(\nu_s+2)}(y), \cdots, P^{(\nu_{s+1})}(y)$ constitute the sth segment of $K(y)$. These points are adapted to the meshes $m_{s,j}$. The sequences $O_{i,t}[K(y)]$ admit of a similar decomposition into segments. Let

$$\mu(s, i, y, t) = \nu(s, i-1, y, t) \cdot p_{\nu(s,i-1,y,t)}(x_i),$$
$$\nu(s, i, y, t) = \mu(s, i, y, t) \cdot p_{\mu(s,i,y,t)}(\phi_{A_i}\{O_{i-1,t}[K(y)] \subset x_i\}),$$

where $\nu(s, 0, y, t) = \nu_s$. The points of $O_{i,t}[K(y)]$ whose superscripts range from $\nu(s, i, y, t)+1$ to $\nu(s+1, i, y, t)$ constitute the sth segment of $O_{i,t}[K(y)]$. If $N(s, i, y, t) = \nu(s+1, i, y, t) - \nu(s, i, y, t)$, then there are $N(s, i, y, t)$ points in the sth segment of this sequence. The points are selected from the sth segment of $K(y)$. Let

$$\nu(s, \sigma, y, t) \cdot p_{\nu(s,\sigma,y,t)}(x_{r_h,n}) = \mu(s, y, t).$$

Then

$$r_h + n \cdot [\mu(s, y, t) - 1] \leqq \nu(s, \sigma, y, t) < r_h + n \cdot \mu(s, y, t).$$

The points of $O_t[K(y)]$ whose superscripts range from $\mu(s, y, t)+1$ to $\mu(s+1, y, t)$ constitute the sth segment of this sequence. All except the first of the points of this segment correspond to points of the sth segment of $K(y)$. The first point of the sth segment of $O_t[K(y)]$ corresponds to n points which may be selected from both the $(s-1)$st and the sth segments of $K(y)$. Let $J(s, y, t) = \mu(s+1, y, t) - \mu(s, y, t)$. Then there are $J(s, y, t)$ points in the sth segment of $O_t[K(y)]$. It will be observed that $[N(s, \sigma, y, t)/n] \leqq J(s, y, t)$. We shall investigate the success ratio of the sequence $O_t[K(y)]$ within its sth segment, that is, the expression $p_N\{O_t[K(y)] \subset x_{\mu,1}\}$ where $\mu = \mu(s, y, t)$ and $0 < N \leqq J(s, y, t)$. Our problem is to choose the integers N_s so that this success ratio can be made to approximate $P[t, m(E)]$ for a set of points y whose measure approximates 1.

14. **Lengths of segments.** The sth segment of $O_t[K(y)]$ must contain a sufficient number of terms to permit the success ratio within this sequence to approximate the desired value. In order to accomplish this, there must also be lower bounds to the lengths of the sth segments of the sequences $O_{i,t}[K(y)]$. We shall appraise the measures of certain sets of points y for which these segments have the desired lengths. Thus the lower bounds should be numbers which are independent of y. Let

$$M(s, i, t) = \min N(s, i-1, t) \cdot p_{N(s,i-1,t)}(x_i \subset x_{\nu,1}),$$

where $\nu = \nu(s, i-1, y, t)$ and where the minimum is taken with respect to y;

$$N(s, i, t) = [M(s, i, t) \cdot \{m(A_{s,i})/m(A_{s,i-1}) - 2^{-s-1}\}],$$

where the square brackets signify the largest integer contained in the number within these brackets;

$$N(s, 0, t) = N_s.$$

The numbers $N(s, i, t)$ thus inductively defined will be taken as the lower bounds of the lengths $N(s, i, y, t)$ of the sth segments of the sequences $O_{i,t}[K(y)]$. We shall let $G(s, i, t) = E[N(s, i, t) \leqq N(s, i, y, t)]$, where $0 < i \leqq \sigma$, denote the set of points y for which this segment has the desired length. Since the integers $\nu = \nu(s, i-1, y, t)$ depend only on the first ν_s digits of y, it is easily seen that $\lim_{N_s \to \infty} N(s, i, t) = \infty$, and hence $N(s, i, t)$ can be made arbitrarily large by a proper choice of N_s. Let

$$J(s, t) = [N(s, \sigma, t)/n].$$

Then the length $J(s, y, t)$ of the sth segment of $O_t[K(y)]$ will be at least as great as $J(s, t)$ for all values of y for which $N(s, \sigma, t) \leqq N(s, \sigma, y, t)$. Let

$$H(s, t) = [J(s, t) \cdot 2^{-s-1} - \mu(s, t) - \nu_s],$$

where $\mu(s, t) = J(1, t) + J(2, t) + \cdots + J(s-1, t)$. Then $\lim_{N_s \to \infty} H(s, t) = \infty$. Let

$$E_{s,i} = \sum_{2^{-s}(j-1) \in E_i} m_{s,j} \quad \text{and} \quad A_{s,i} = \sum_{2^{-s}(j-1) \in A_i} m_{s,j}.$$

Thus $E_{s,j}$ is a sum of meshes of the lattice L_s. It consists of all such meshes whose vertices lie in E_j. The set $A_{s,j}$ is similarly defined. Let

$$P[s, t, m(E)] = m(E_{s,1}) \cdot m(E_{s,2}) \cdots m(E_{s,h}) \cdot m^{-h}(A_{s,\sigma}),$$

$$G(s, t) = E(|\, p_N\{O_t[K(y)] \subset x_{\mu,1}\} - P[s, t, m(E)]\,| < 2^{-s-1} \cdot \subset \cdot H(s-1, t) \leqq N \leqq J(s, y)),$$

where $\mu = \mu(s, y, t)$. The symbol $\subset$ means "if." The set $G(s, t)$ consists of those points y for which the inequality to the left of the symbol $\subset$ holds for all values of N satisfying the inequalities to the right of this symbol. Thus if y is a member of this set, then a restriction is placed upon the success ratio of $O_t[K(y)]$ within its sth segment.

The integers $N(s, i, t)$, $J(s, t)$, $H(s, t)$ depend upon the numbers N_s. While the determination of the numbers N_s will not be given until a later point, it is necessary to note here that the numbers $N(s, i, t)$, $J(s, t)$, $H(s, t)$ must all be positive. It then follows from the definition of $H(s, t)$ that $\{H(s, t) + \mu(s, t)\} \cdot 2^{s+1} \leqq J(s, t)$, $J(s-1, t) < J(s, t)$ and $H(s-1, t) \leqq H(s, t) < J(s, t)$. Hence $H(s-1, t) < J(s, y, t)$ whenever $N(s, \sigma, t) \leqq N(s, \sigma, y, t)$.

15. **Appraisals of certain set measures.** We shall prove that

$$m\{C[G(s, 1, t) \cdot G(s, 2, t) \cdots G(s, \sigma, t)]\} \leqq \alpha(s, 1, t) + \alpha(s, 2, t) + \cdots + \alpha(s, \sigma, t) \tag{6}$$

and

$$m\{C[G(s,1,t)\cdot G(s,2,t)\cdots G(s,\sigma,t)\cdot G(s,t)]\} \leqq \alpha(s,1,t)+\alpha(s,2,t)+\cdots+\alpha(s,\sigma,t)+\alpha(s,t), \tag{7}$$

where

$$\alpha(s,i,t)=\sum_{R/M(s,i,t)<m(A_{s,i})/m(A_{s,i-1})-2^{-s-1}} {}^{M(s,i,t)}C_R \cdot\left(\frac{m(A_{s,i})}{m(A_{s,i-1})}\right)^{R}\left(1-\frac{m(A_{s,i})}{m(A_{s,i-1})}\right)^{M(s,i,t)-R}, \tag{8}$$

and

$$\alpha(s,t)=\sum_{N=H(s-1,t)}^{\infty}\ \sum_{|R/N-P[s,t,m(E)]|\geqq 2^{-s-1}-1/N} \cdot P^{R}[s,t,m(E)]\{1-P[s,t,m(E)]\}^{N-R}. \tag{9}$$

The proof will be accomplished by means of the following lemmas.

LEMMA 1. *$m\{E[P^{(n_1)}(y)\in E_1\cdot P^{(n_2)}(y)\in E_2\cdots P^{(n_h)}(y)\in E_h]\}=m(E_1)\cdot m(E_2)\cdots m(E_h)$, where $\nu_s<n_1<n_2<\cdots<n_h$ and where the sets $E_1, E_2, \cdots, E_h$ are elements of L_s.*

LEMMA 2. *$m\{E[P_i^{(n_1)}(y)\in E_1\cdot P_i^{(n_2)}(y)\in E_2\cdots]\}\leqq m(E_1)\cdot m(E_2)\cdots m(E_h)\cdot m^{-h}(A_{s,i})$, where $n_1=\nu(s,\ i,\ y,\ t)+\lambda_1<n_2=\nu(s,\ i,\ y,\ t)+\lambda_2<\cdots<n_h=\nu(s,i,y,t)+\lambda_h\leqq\nu(s+1,i,y,t)$, where $\lambda_1,\lambda_2,\cdots,\lambda_h$ are positive integers independent of y and $O_{i,t}[K(y)]=P_i^{(1)}(y),\ P_i^{(2)}(y),\ P_i^{(3)}(y),\cdots$ and the sets $E_1, E_2, \cdots, E_h$ are elements of L_s.*

The first lemma is proved in a manner similar to that of Lemma 1, §12. In order to prove Lemma 2, let us first consider a special case in which t is restricted in such a manner that $A_1=A_{s,1}$, $A_2=A_{s,2}$, $\cdots$, $A_i=A_{s,i}$. For such a choice of t, it follows that the expressions to the left and to the right of the sign $\leqq$ would actually be equal if the restriction $n_h\leqq\nu(s+1,i,y,t)$ were removed. This fact can be proved in a manner similar to that of Lemma 2, §12. The restriction $n_h\leqq\nu(s+1,i,y,t)$ makes the above specialization of t unnecessary. Moreover this restriction in general decreases the set whose measure we are appraising. Hence Lemma 2 is established.

We have the equation

$$\begin{aligned} m\{C[G(s,1,t)\cdot G(s,2,t)\cdots G(s,i,t)]\} &= m\{C[G(s,1,t)\cdots G(s,i-1,t)]\} \\ &\quad + m\{G(s,1,t)\cdots G(s,i-1,t)C[G(s,i,t)]\}. \end{aligned}$$

We shall show that the measure of the set $G(s,1,t)\cdots G(s,i-1,t)C[G(s,i,t)]$ is less than or equal to $\alpha(s,i,t)$. If y is a member of this set, then there must

be at least $N(s, i-1, t)$ points in the sth segment of $O_{i-1,t}[K(y)]$. Hence there must be at least $M(s, i, t)$ points in the sth segment of $O_{i-1,t}[K(y)] \subset x_i$. Suppose that R of the first $N(s, i, t)$ points of the sth segment of $O_{i-1,t}[K(y)] \subset x_i$ belong to the set $A_{s,i}$. Then it must at least be true that $R < N(s, i, t)$ and hence $R/M(s, i, t) < m(A_{s,i})/m(A_{s,i-1}) - 2^{-s-1}$. It follows from Lemma 2 that $m\{G(s, 1, t) \cdot G(s, 2, t) \cdots G(s, i-1, t) \cdot C[G(s, i, t)]\} \leqq \alpha(s, i, t)$. Thus inequality (6) is established. In a like manner it is easily seen that inequality (7) is also true.

We are now in a position to determine the integers N_s. It will be observed that there are only a finite number of sets of the sets $A_{s',i}$ and $E_{s',i}$ such that $s' \leqq s$ and $t \leqq s$, and only a finite number of sets of selections x_i and $x_{r,n}$ such that $t \leqq s$. There correspond a finite set of values of t, the particular choice of the set being unimportant. By a proper selection of the integer N_s, the integers $M(s, i, t)$ can be made so large that the numbers $\alpha(s, i, t)$ can be made arbitrarily small for all values of t. Moreover $H(s-1, t)$ can be made so large that $\alpha(s, t)$ can be made arbitrarily small. We shall choose N_s so that

$$\sum_{(t)} \{\alpha(s, 1, t) + \alpha(s, 2, t) + \cdots + \alpha(s, \sigma, t)\} \leqq 2^{-s-1},$$

where the summation is extended over all values of t of the above mentioned finite set. Then

$$m\left\{C\left(\prod_{(t)} G(s, 1, t) \cdot G(s, 2, t) \cdots G(s, \sigma, t)\right)\right\} \leqq 2^{-s-1},$$

where the symbol $\prod_{(t)}$ means that the conditions to the right of this symbol must be satisfied simultaneously for all values of t. Let

$$\prod_{s=s_0}^{\infty} \prod_{(t)} G(s, 1, t) \cdot G(s, 2, t) \cdots G(s, \sigma, t) \cdot G(s, t) = G_{s_0}.$$

Then $m[C(G_{s_0})] \leqq 2^{-s_0}$. Let

$$G = \sum_{s_0=1}^{\infty} G_{s_0}.$$

Then $m(G) = 1$.

We shall show that if y is any member of the set G, then $K(y)$ satisfies condition (C4). This will be done with the aid of the following lemma.*

LEMMA 3. *If $K = P^{(1)}, P^{(2)}, P^{(3)}, \cdots$ is a sequence whose label space consists of the elements* 0 *and* 1 *and if there exist a decreasing sequence of positive*

* This lemma was proved by Regan (see Regan I). A much simpler lemma whose function is similar has been proved by von Mises (von Mises VI). However I have been unable to adapt von Mises' lemma to this problem.

numbers $\epsilon_1, \epsilon_2, \epsilon_3, \cdots$ *such that* $\lim_{s\to\infty}\epsilon_s=0$, *and two sets of positive integers* $J_1, J_2, J_3, \cdots$ *and* $H_1, H_2, H_3, \cdots$ *such that for* $s_0 \leqq s$

(a) $$| p_N(K \subset x_{\mu,1}) - p | < \epsilon_s/3$$

whenever $H_s \leqq N \leqq J_{s+1}$, *where* $\mu=\mu_{s+1}=J_1+J_2+\cdots+J_s$,

(b) $$| p_{J_s}(K \subset x_{\mu,1}) - p | + (\mu_s + H_s)/J_s < \epsilon_s/3,$$

where $\mu=\mu_s$, *then*

(c) $$| p_N(K) - p | < \epsilon_s$$

if $\mu_{s+1} \leqq N \leqq \mu_{s+2}$ *and* $s_0 \leqq s$.

Since $\lim_{s\to\infty}P[s, t, m(E)]=P[t, m(E)]$, it follows that there exists a decreasing sequence of positive numbers $\epsilon_1, \epsilon_2, \epsilon_3, \cdots$ such that

(10) $$| P[s, t, m(E)] - P[t, m(E)] | + 2^{-s} < \epsilon_s/3$$

and $\lim_{s\to\infty}\epsilon_s=0$. From the definition of $H(s, t)$ it follows that $H(s, t)+\nu_s \leqq J(s, t)\cdot 2^{-s-1}$. Hence

(11) $$[H(s, t) + \mu(s, y, t)]/J(s, y, t) \leqq [H(s, t) + \nu_s]/J(s, t) \leqq 2^{-s-1}.$$

Let y be a member of $G(s, t)$. If the inequality imposed by this condition be combined with (10) and (11), we get

(12) $$| p_{J(s,y,t)}\{O_t[K(y)] \subset x_{\mu,1}\} - P[t, m(E)] | + [H(s, t) + \mu]/J(s, t) \leqq \epsilon_s/3,$$

where $\mu=\mu(s, y, t)$. In a like manner it can be proved that

(13) $$| p_N\{O_t[K(y)] \subset x_{\mu,1}\} - P[t, m(E)] | < \epsilon_s/3$$

if $H(s, t) \leqq N \leqq J(s+1, y, t)$ and $\mu=\mu(s+1, y, t)$ and $y \epsilon G(s+1, t)$. Hence from Lemma 3 we conclude that

(14) $$| p_N\{O_t[K(y)]\} - P[t, m(E)] | < \epsilon_s$$

if $\mu(s+1, y, t) \leqq N \leqq \mu(s+2, y, t)$ and if y is an element of $G(s, t)\cdot G(s+1, t)$. If y is an element of G, then there will exist a number s_0 such that condition (14) will hold whenever $s_0 \leqq s$. This gives us the following theorem.

THEOREM 2. *For almost every* y *the corresponding sequence* $K(y)$ *satisfies condition* (C4).

16. **Generalizations.** We shall consider the case in which the space S is the region $0 \leqq y_i < 1$, where $i=1, 2, 3, \cdots, \alpha$. Let $\pi(E)$ be an absolutely additive, absolutely continuous function defined in S and such that $0<\pi(E)$ whenever $0<M(E)$. We shall construct a collective $K=P^{(1)}, P^{(2)}, P^{(3)}, \cdots$ which satisfies (C4). In order to do this, we shall set up a correspondence

between the points of S and the points of the interval I: $0 \leqq y < 1$. Let S be decomposed into a system of meshes $m_{s,j}$ and let us define a corresponding set of intervals $I_{s,j}$. Let

$$I_{s,j}: \quad \pi(m_{s,1} + m_{s,2} + \cdots + m_{s,j-1}) \leqq y < \pi(m_{s,1} + \cdots + m_{s,j}),$$

where $i < j$,

$$I_{s,1}: \quad 0 \leqq y < \pi(m_{s,1}).$$

Corresponding to any y of I and to any positive integer s there exists a positive integer j_s such that y is an element of I_{s,j_s}. Let

$$P_y = \prod_{s=1}^{\infty} \bar{m}_{s,j_s},$$

where $\bar{m}_{s,j_s}$ is the set consisting of the mesh m_{s,j_s} plus the frontier points of this mesh. Thus corresponding to any point y of the interval I there is determined uniquely by this enclosure sequence, a point P_y of the region $\bar{S}$, where $\bar{S}$ is equal to S plus the frontier points of S. If y is any point of an interval $I_{s,j}$, then P_y is a point of the corresponding set $\bar{m}_{s,j}$. The inverse of this correspondence is not necessarily unique.

Let $P_{s,j}$ be the point of S which corresponds to the point $2^{-s}(j-1)$ of the interval I. Let A be any subset of S having frontier points of measure zero. Let s be any positive integer and $j_1, j_2, \cdots, j_n$ be the set of all positive integers such that the corresponding points P_{s,j_i} are elements of the set A. It may of course happen that the points $P_{s,j_1}, P_{s,j_2}, \cdots, P_{s,j_n}$ of the set A are not all distinct. Let $A_s' = m'_{s,j_1} + m'_{s,j_2} + \cdots + m'_{s,j_n}$, where the sets $m'_{s,j}$ are meshes of the interval I.

We shall prove that $\lim_{s\to\infty} m(A_s') = \pi(A)$. We shall prove this fact first for the case in which the set A is one of the meshes $m_{\sigma,i}$. Given any positive number ϵ there exists a positive number σ' and a corresponding element $B_{\sigma'}$ of $L_{\sigma'}$ such that all of the points which belong to $m_{\sigma,i}$ are interior points of $B_{\sigma'}$, and $\pi(B_{\sigma'}) < \pi(m_{\sigma,i}) + \epsilon$. Corresponding to the meshes $m_{\sigma',j}$ of the set $B_{\sigma'}$ is a set of intervals $I_{\sigma',j}$. We shall denote this set of intervals by $b_{\sigma'}$. We shall show that if P is any point of the mesh $m_{\sigma,i}$, then all of the corresponding points y belong to $b_{\sigma'}$. That is, if y belongs to $Cb_{\sigma'}$, then P_y belongs to $Cm_{\sigma,i}$. Let y be any point of $Cb_{\sigma'}$. Then y belongs to some interval $I_{\sigma',j}$ and the corresponding point P_y belongs to the corresponding set $\bar{m}_{\sigma',j}$. The set $\bar{m}_{\sigma',j}$ can be neither interior nor adjacent to $m_{\sigma,i}$. Hence P_y belongs to $Cm_{\sigma,i}$. Therefore if s is sufficiently large, then $m(A_s') < \pi(m_{\sigma,i}) + \epsilon$. Since a similar inequality holds for every mesh of the lattice L_σ, it follows that if s is sufficiently large, then $\pi(m_{\sigma,i}) - \epsilon < m(A_s')$. That is, $\lim_{s\to\infty} m(A_s') = \pi(m_{\sigma,i})$. The extension to

the case in which A consists of a finite sum of meshes is immediate. Next let A be an arbitrary set whose frontier points are of measure zero. Given any positive number ϵ, there exist two elements B_1 and B_2 of N such that $B_1 < A < B_2$ and $\pi(B_2 - B_1) < \epsilon$. Thus it is easily seen that $\lim_{s\to\infty} m(A_s') = \pi(A)$.

The points $P_{s,j}$ are everywhere dense in the space S and hence constitute a space S' defined in §10. There correspond a field F' and a distribution function $\pi'(A)$. Let A be any set of the field F' and let

$$A_s' = \sum_{P_{s,j} \in A} m_{s,j}. \tag{15}$$

Then $\lim_{s\to\infty} m(A_s') = \pi'(A)$. Let $A_1, A_2, \cdots, A_\sigma, E_1, E_2, \cdots, E_h$ be the sets involved in the operator O_t. There correspond sets $A_{s,1}, A_{s,2}, \cdots, A_{s,\sigma}$, $E_{s,1}, E_{s,2}, \cdots, E_{s,h}$ defined in accordance with equation (15). Let

$$P[s, t, m(E)] = m(E_{s,1}) \cdot m(E_{s,2}) \cdots m(E_{s,h}) \cdot m^{-h}(A_{s,\sigma}).$$

Then $\lim_{s\to\infty} P[s, t, m(E)] = P[t, m(E)]$. If y is any member of the set G defined in §15, then there is determined a sequence $K(y)$. There is also determined a corresponding sequence K whose terms are elements of the space S'. There exists an integer s_0 such that

$$\left| p_N[O_t(K \subset x_{\mu,1})] - P[s, t, m(E)] \right| < 2^{-s-1}$$

whenever $s_0 \leqq s$, $H(s-1, t) \leqq N \leqq J(s, y)$ and $\mu = \mu(s, y, t)$. Hence by reasoning similar to that of Theorem 2, §15, it follows that $p[O_t(K)] = P[t, m(E)]$. That is, K satisfies condition (C4) and hence is a collective with respect to the space S' and the field F'. It is also a collective with respect to the space S and the field F.

We shall now remove the restriction that $\pi(A)$ must be greater than zero whenever $m(A)$ is greater than zero. Let $\pi(A)$ be any absolutely additive, absolutely continuous distribution function such that there exists a set B with the following properties.

1. The frontier points of B are of measure zero, i.e., B belongs to the field F.
2. $\pi(B) = 1$.
3. If A is any set of the field F, then $0 < \pi(A \cdot B)$ whenever $0 < m(A \cdot B)$.

We shall define a new distribution function $\pi'(A)$ by the equation

$$\pi'(A) = [\pi(A \cdot B) + m(A \cdot CB)]/2.$$

Then there exist collectives having $\pi'(A)$ as their distribution function. Let K be such a collective and let us perform a contraction on K with respect

to the set B. The resulting sequence $K \subset \phi_B(K)$ is a collective having the distribution function

$$\pi'(A \cdot B)/\pi'(B) = \pi(A).$$

Next if S_1 is the whole of the α-dimensional euclidean space, then there exist collectives with respect to this space. To show this, it is only necessary to construct a transformation which is continuous and has a continuous inverse and which carries S into S_1.* Such a transformation carries sets of measure zero into sets of measure zero.

Suppose we have a space S which consists of a denumerable infinity of elements $m_1, m_2, m_3, \cdots$. Let the field F contain all possible subsets of S. We shall define a corresponding set of intervals $I_1, I_2, I_3, \cdots$ as follows.

$$I_j\colon \quad \pi(m_1 + m_2 + \cdots + m_{j-1}) \leqq y < \pi(m_1 + m_2 + \cdots + m_j) \text{ if } 1 < j,$$
$$I_1\colon \quad 0 \leqq y < \pi(m_1).$$

Then $\sum_{j=1}^{\infty} I_j = I$ where I is the interval $0 \leqq y < 1$. If A is any element of F, then the corresponding sum of intervals I_j constitutes a set whose frontier points are of measure zero. Let $K' = P^{(1)}, P^{(2)}, \cdots$ be any member of the set $C[D, m(E)]$ and let the corresponding label space be the interval I. We shall define a corresponding collective $K = e^{(1)}, e^{(2)}, e^{(3)}, \cdots$ having the label space S. If $P^{(n)}$ belongs to the interval I_j, then the corresponding term $e^{(n)}$ shall be the element m_j. Then K is easily seen to be a collective. Any space consisting of a finite number of elements can be similarly treated.

We shall show the existence of collectives whose associated label space S has infinitely many dimensions. We can express S as the following infinite product.

$$S = I_1 \times I_2 \times \cdots \times I_\alpha \times I_{\alpha+1} \times \cdots,$$

where I_α is the interval $0 \leqq y_\alpha < 1$. The field F is defined as follows. A set A belongs to F if and only if there exists a number α such that $A = A_\alpha \times I_{\alpha+1} \times I_{\alpha+2} \times \cdots$ and the frontier points of A_α are of measure zero.

For each interval I_α we shall construct a net N_α consisting of lattices $L_{s,\alpha}$. We shall also construct for the space S a net N consisting of lattices L_s. Let $L_s = L_{s,1} \times L_{s,2} \times \cdots \times L_{s,s}$. Then there are 2^{s^2} meshes $m_{s,1}, m_{s,2}, \cdots, m_{s,j}, \cdots, m_{s,2^{s^2}}$ in L_s. We shall assume that the subscripts of the meshes are so chosen that

$$m_{s-1,j+1} = \sum_{i=j\cdot 2^{2s-1}+1}^{(j+1)2^{2s-1}} m_{s,i}.$$

* Copeland II.

From this point on, the existence proof for collectives in this space is the same as that for collectives in the space of α dimensions.

Let $K=P^{(1)}, P^{(2)}, P^{(3)}, \cdots$ be a collective having the label space S. We shall express K as a product collective. Let k_α be the sequence whose nth term $P_\alpha^{(n)}$ is the αth coordinate of the point $P^{(n)}$. Then $K=k_1\times k_2\times k_3\times \cdots$. Let $\alpha_1, \alpha_2, \alpha_3, \cdots$ be a sequence of positive integers and let $\beta_n=\alpha_1+\alpha_2+\cdots+\alpha_{n-1}$. Let $S_n=S_{\beta_n+1}\times S_{\beta_n+2}\times\cdots\times S_{\beta_{n+1}}$, $K_n=k_{\beta_n+1}\times k_{\beta_n+2}\times\cdots\times k_{\beta_{n+1}}$. Then K_n is a collective whose label space is S_n. Moreover $K_{n_1}\times K_{n_2}\times\cdots\times K_{n_\nu}$ is a collective having the label space $S_{n_1}\times S_{n_2}\times\cdots\times S_{n_\nu}$. In the nomenclature of von Mises, collectives which can be combined in this manner to form new collectives are said to be "verbindbar." The spaces $S_1, S_2, \cdots$ and the collectives $K_1, K_2, \cdots$ can be transformed by one or another of the methods described earlier in this section. We shall denote the transformed spaces and collectives respectively by $S_1', S_2', S_3', \cdots$ and $K_1', K_2', K_3', \cdots$. The space S and the field F become S' and F'. Products of the collectives $K_1', K_2', \cdots$ are collectives having the corresponding product label spaces.

17. **Alternative systems.** It is readily verified that $K_i' \subset x$ and $K_i' \subset \phi_{A_j}(K_j')$ are collectives if x is an element of D and A_j is a subset of S_j' and $I_1'\times I_2'\times\cdots\times I_{j-1}'\times A_j'\times I_{j+1}'\times\cdots$ is an element of F'. Operations of the form $K_i' \subset \phi_{A_j}(K_j')\cdot \subset \cdot\phi_{A_m}(K_m')$ and $K_i' \subset x_1\cdot \subset \cdot\phi_{A_j}(K_j' \subset x_2)$ do not necessarily produce collectives. However it may be true that K can be so chosen that all such operations and groups of these operations produce collectives. In fact in the system of Dörge, all of these operations are included.* Dörge considers a set of selections S which is equivalent to the set D given in this paper. He then constructs an extended set $\hat{S}$ which contains all the operators which can be formed by combining selections and contractions in the manner just described. Thus $\hat{S}$ includes all the operators of the form $O_{\sigma,t}$. In the present paper the operators O_t play a rôle similar to those operators of the set $\hat{S}$ which are not of the form $O_{\sigma,t}$. In Dörge's system the spaces $S_1', S_2', \cdots$ consist of finite numbers of elements and the collectives $K_1', K_2', \cdots$ are independent. No general existence proof has been given for this system.

In the system of Reichenbach,† a fundamental element is an expression of the form $(i)(x_i\epsilon 0\exists_p y_i\epsilon P)$. These elements admit of several interpretations. One of the most important interpretations is the following. Let

$$x = x_1, x_2, x_3, \cdots, x_i, \cdots,$$

$$y = y_1, y_2, y_3, \cdots, y_i, \cdots$$

be two collectives with corresponding label spaces S and T and let $K=x\times y$ be

* Dörge I.

† Reichenbach I.

a collective with the label space $S \times T$. Then the expression $(i)(x_i \epsilon 0 \exists_p y_i \epsilon P)$ is equivalent to the equation $p\{\phi_{0 \times P}[K \subset \phi_{0 \times S}(K)]\} = p$. Another interpretation is obtained as follows. Let y be a collective, x be a selection of the set D, and 0 be the set consisting of the number 1. Then $x = \phi_0(x)$. In this case the expression $(i)(x_i \epsilon 0 \exists_p y_i \epsilon P)$ means $p[\phi_P(K \subset x)] = p$.

If the collectives K_1', K_2', $\cdots$ are independent, then K' constitutes a Tornier matrix of a very general nature.* The ith row of this matrix is the collective K_i'. In the definition of such a matrix, conditions (C2) and (C3) are not required. Condition (C1) is specialized by setting $h = r_1 = n = 1$.

References

E. Borel.

I. *Traité du Calcul des Probabilités.* Edited by P. Dubreil. Paris, Gauthier-Villars, 1926.

A. H. Copeland.

I. *The theory of probability from the point of view of admissible numbers,* Annals of Mathematical Statistics, vol. 3 (1932), pp. 143–156.

II. *Admissible numbers in the theory of geometrical probability,* American Journal of Mathematics, vol. 53 (1931), pp. 153–162.

K. Dörge.

I. *Zu der von R. von Mises gegebenen Begründung der Wahrscheinlichkeitsrechnung,* Mathematische Zeitschrift, vol. 32 (1930), pp. 232–258.

E. Kamke.

I. *Über neuere Begründungen der Wahrscheinlichkeitsrechnung,* Jahresbericht der Deutschen Mathematiker Vereinigung, vol. 42 (1932), pp. 14–27.

II. *Einführung in die Wahrscheinlichkeitstheorie,* Leipzig, Hirzel, 1932.

A. Kolmogoroff.

I. *Grundbegriffe der Wahrscheinlichkeitsrechnung,* Ergebnisse der Mathematik, vol. 2, No. 3.

C. de la Vallée Poussin.

I. *Sur l'intégral de Lebesgue,* these Transactions, vol. 16 (1915), pp. 435–501.

C. I. Lewis and C. H. Langford.

I. *Symbolic Logic.* New York, Century Publishing Co., 1934.

R. von Mises.

I. *Grundlagen der Wahrscheinlichkeitsrechnung,* Mathematische Zeitschrift, vol. 5 (1919), pp. 52–99.

II. *Wahrscheinlichkeit, Statistik und Wahrheit.* Berlin, Springer, 1928.

III. *Wahrscheinlichkeitsrechnung und ihrer Anwendung in der Statistik und theoretischen Physik.* Leipzig and Vienna, Deuticke, 1931.

IV. *Über Zahlenfolgen, die ein Kollektiv-ähnliches Verhalten zeigen,* Mathematische Annalen, vol. 108 (1933), pp. 757–772.

F. Regan.

I. *Applications of the theory of admissible numbers to time series with constant probability,* these Transactions, vol. 36 (1934), pp. 511–529.

H. Reichenbach.

I. *Axiomatik der Wahrscheinlichkeitsrechnung,* Mathematische Zeitschrift, vol. 34 (1931–32), pp. 568–619.

E. Tornier.

I. *Die Axiome der Wahrscheinlichkeitsrechnung,* Crelle's Journal, vol. 163 (1930), pp. 45–64.

* Tornier I.

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ON EFFECTIVE SETS OF POINTS IN RELATION TO INTEGRAL FUNCTIONS*

BY

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1. **Introduction.** Let $f(z)$ be an integral function and let $M(r, f) = \max_{|z| \leqq r} |f(z)|$. The order ρ and the type $\kappa(f)$ of $f(z)$ are defined by the relations

$$\rho = \limsup_{r \to \infty} \frac{\log \log M(r, f)}{\log r}; \qquad \kappa(f) = \limsup_{r \to \infty} \frac{\log M(r, f)}{r^{\rho}}.$$

Let $[z_n]$, $z_n = r_n e^{i\theta_n}$ be a distinct sequence of complex numbers such that

$$0 < r_1 \leqq r_2 \leqq \cdots \leqq r_n \to \infty$$

as $n \to \infty$. Let $\rho_1 > 0$ be any number. The type $\kappa(f, \rho_1, [z_n])$ of $f(z)$ over the set $[z_n]$ is defined by the relation

$$\kappa(f, \rho_1, [z_n]) = \limsup_{n \to \infty} \frac{\log |f(z_n)|}{|z_n|^{\rho_1}}.$$

If $f(z)$ is of order ρ, it is evident that $\kappa(f, \rho_1, [z_n]) \leqq 0$ when $\rho_1 > \rho$. If $\rho_1 \leqq \rho$, the value of $\kappa(f, \rho_1, [z_n])$ can vary from $-\infty$ to ∞.

1.1. DEFINITION. Let $f(z)$ be a function of order ρ; we shall say that $[z_n]$ is an effective set, or briefly an E-set, for $f(z)$ when $\kappa(f, \rho, [z_n]) = \kappa(f)$.

1.2. It is easy to see that any given function $f(z)$ always possesses an E-set; for, on $|z| = r$, there is at least one point $z(r)$ such that $M(r, f) = |f(z(r))|$; also, a sequence $[r_n]$, $r_1 < r_2 < \cdots < r_n \to \infty$ as $n \to \infty$, exists for which

$$\kappa(f) = \lim_{n \to \infty} \frac{\log M(r_n, f)}{r_n^{\rho}};$$

hence $[z(r_n)]$ is an effective set for $f(z)$. A more interesting question is to ascertain whether all functions of a given class specified by some simple property possess an *E-set in common*. In this paper an attempt is made to answer this question.

1.3. We denote $C(\rho, d)$ the class of all functions of order ρ and type less than d where ρ and d are any two given positive numbers. We regard all functions of order less than ρ as of order ρ and minimal type, that is $\kappa(f) = 0$, un-

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less $f(z)\equiv 0$ in which case $\kappa(f)=\kappa(f, \rho, [z_n])=-\infty$ for all ρ and $[z_n]$. These are included in $C(\rho, d)$ for the purposes of this paper.

2. We shall, first, discuss a few general properties of an E-set for a given class $C(\rho, d)$.

THEOREM 1. *In order that a set $[z_n]$ may be an E-set for a class $C(\rho, d)$ it is necessary that*

(i) *the exponent of convergence (which we shall speak of as the order) of $[z_n]$ cannot be less than ρ;*

(ii) *if the order of $[z_n]$ be ρ, any function with zeros at $z=z_n$ must be of order ρ and type not less than d unless such a function is identically zero;*

(iii) *the set $[\theta_n]$ of amplitudes of $[z_n]$ must be everywhere dense in $0\leqq\theta\leqq 2\pi$.*

Proof. If $[z_n]$ were of order $\rho'<\rho$, the canonical product $\sigma(z)$ with simple zeros at $[z_n]$ is of order ρ' and therefore is of order ρ and minimal type so that, by the definition of $C(\rho, d)$

$$\kappa(\sigma)=\kappa(\sigma, \rho, [z_n])=0.$$

But $\sigma(z_n)=0$ so that

$$\kappa(\sigma, \rho, [z_n])=-\infty.$$

This contradiction shows that the order of $[z_n]$ cannot be less than ρ. A similar argument proves (ii). To prove (iii), suppose that $[\theta_n]$ is not everywhere dense in $(0, 2\pi)$. Then there is a θ_0 such that $\theta_0-\delta\leqq\theta\leqq\theta_0+\delta$ contains no θ_n, $\delta>0$ being sufficiently small. We can suppose without loss of generality that $\theta_0=0$ so that the angle $|\theta|\leqq\delta$ does not contain any point of $[z_n]$. Now, let $H_\rho(z)$ be defined by*

$$H_\rho(z)=\begin{cases}\displaystyle\prod_{n=1}^{\infty}\left(1+\frac{z}{n^{1/\rho}}\right), & 0<\rho\leqq\tfrac{1}{2},\\[2ex] \displaystyle\sum_{n=0}^{\infty}\frac{z^n}{\Gamma\left(\frac{n}{\rho}+1\right)}, & \rho>\tfrac{1}{2}.\end{cases}$$

It is known that, if z is outside $|\theta|\leqq\delta$,

$$\limsup_{|z|\to\infty}\frac{\log|H_\rho(z)|}{|z|^\rho}<\kappa(H_\rho).$$

By considering a function of the form $H_\rho(\eta z)$ where η is such that $\eta^\rho\kappa(H_\rho)<d$, we conclude that $[z_n]$ cannot be an E-set for $H_\rho(\eta z)$ which obviously belongs to $C(\rho, d)$. Hence $[\theta_n]$ is everywhere dense in $(0, 2\pi)$. It may be noted that

* For $0<\rho\leqq\frac{1}{2}$, see Paley and Wiener, *Fourier Transforms in the Complex Domain*, p. 79; for $\rho>\frac{1}{2}$, $H_\rho(z)$ are Mittag-Leffler's functions, Acta Mathematica, vol. 29 (1905), pp. 101–181.

since an E-set remains an E-set when any other set is added to it, we cannot expect to improve upon the result (i) of Theorem 1.

2.1. We shall now give a general criterion for a set $[z_n]$ to form an E-set for a class $C(\rho, d)$. Let $A_n(h)$ denote the circle with center z_n and radius $|z_n|^{-h}$; and let $A(h)$ denote the system of circles $A_n(h)$, $n=1, 2, \cdots$. We prove

THEOREM 2. *The set $[z_n]$ of order ρ will form an E-set for $C(\rho, d)$ provided there exists a function $g(z)$ with simple zeros at $z=z_n$ and $h>\rho$ such that the following relations hold:*

$$\text{(i)} \qquad \lim_{n\to\infty} \frac{\log |g'(z_n)|}{|z_n|^{\rho}} = d; \qquad \text{(ii)} \qquad \lim \frac{\log |g(z)|}{|z|^{\rho}} = d$$

as $|z|\to\infty$ outside the circles $A(h)$.

2.2. We shall establish two lemmas in the first place.

LEMMA 1. *There exists a sequence $[R_n]$, $R_1<R_2<\cdots<R_n\to\infty$ as $n\to\infty$, $R_{n+1}\leqq aR_n$, $a>1$ being given, such that no circle $|z|=R_n$, $n=1, 2, \cdots$, cuts any circle of $A(h)$.*

Proof. Let $b>1$. Consider the ring $r\leqq|z|\leqq br$. The sum of the diameters of those circles of $A(h)$ whose centers lie in the ring cannot exceed $\sum_{r\leqq|z_n|\leqq br}|z_n|^{-h}$ which is less than a fixed positive constant since $\sum|z_n|^{-h}$ converges when $h>\rho$. Therefore if r is sufficiently large, there is at least one circle $|z|=R$ in the ring $r\leqq|z|\leqq br$ which does not cut any circle of $A(h)$. Hence, there is an n_0 such that for all $n\geqq n_0$, the ring $b^n\leqq|z|\leqq b^{n+1}$ contains a circle of the type required. Taking $b=a^{1/2}$ we get the required result.

LEMMA 2. *Any function $g(z)$ satisfying the condition* (ii) *of Theorem* 2 *is of order ρ and type d.*

Proof. Let $a>1$ be given and let $[R_n]$ be the sequence of Lemma 1. On $|z|=R_n$, we have, by (ii)

$$M(R_n, g) \leqq \exp\left[(d+\epsilon)R_n^{\rho}\right]$$

for $n\geqq n_0=n_0(\epsilon)$. Since $M(r, g)$ is an increasing function of r and $R_{n+1}\leqq aR_n$, we get for all $r\geqq r_0=r_0(\epsilon)$

$$M(r, g) \leqq \exp\left[a^{\rho}(d+\epsilon)r^{\rho}\right],$$

so that $\kappa(g)\leqq a^{\rho}d$ and since a is any number greater than one, we get $\kappa(g)\leqq d$. But obviously $d\leqq\kappa(g)$. Hence $\kappa(g)=d$.

2.3. **Proof of Theorem** 2. Let $f(z)$ be any function of $C(\rho, d)$ and $[R_r]$ the sequence of Lemma 1 for some $a>1$. Let

$$I_\nu = \frac{1}{2\pi i}\int_{|x|=R_\nu} \frac{x^m f(x)}{g(x)} \frac{dx}{x-z},$$

where $m \geqq 0$ is an integer. By (ii) we find that

$$I_\nu \to 0 \tag{1}$$

as $\nu \to \infty$ uniformly in any fixed circle $|z| \leqq R$. But

$$I_\nu = \frac{z^m f(z)}{g(z)} - \sum_{|z_n|<R_\nu} \frac{z_n^m f(z_n)}{g'(z_n)} \frac{1}{z-z_n}, \tag{2}$$

while by (i) the series

$$\sum_{n=1}^{\infty} \left| \frac{z_n^m f(z_n)}{g'(z_n)} \frac{1}{z-z_n} \right|$$

converges uniformly except at the points $z=z_n$. Therefore (1) and (2) give

$$z^m \frac{f(z)}{g(z)} = \sum_{n=1}^{\infty} \frac{z_n^m f(z_n)}{g'(z_n)} \frac{1}{z-z_n}. \tag{3}$$

Let $\kappa(f, \rho, [z_n]) = \beta \leqq \kappa(f) < d$. Choose η so that $0 < \eta < d-\beta$ and λ so that $d\lambda^\rho = d-\beta-\eta$. Let $\chi(z) = c_0 + c_1 z + c_2 z^2 + \cdots$ be any integral function of order ρ and type not exceeding $d\lambda^\rho$. Then by (i) the double series

$$\sum_{(m,n)} \left| \frac{c_m z_n^m f(z_n)}{g'(z_n)} \frac{1}{z-z_n} \right|$$

converges uniformly except at $z=z_n$, so that (3) gives

$$\frac{f(z)\chi(z)}{g(z)} = \sum_{n=1}^{\infty} \frac{f(z_n)\chi(z_n)}{g'(z_n)} \frac{1}{z-z_n}. \tag{4}$$

In (4) we can take $\chi(z) = g(\lambda z)$ since in this case $\kappa(\chi) = d\lambda^\rho$ by Lemma 2. So (4) gives

$$f(z) = \frac{g(z)}{g(\lambda z)} \sum_{n=1}^{\infty} \frac{f(z_n) g(\lambda z_n)}{g'(z_n)} \frac{1}{z-z_n}. \tag{5}$$

Let $A_\lambda(h)$ denote the circles around the zeros of $g(\lambda z)$ similar to $A(h)$. Then, given $a > 1$, we can, just as in Lemma 1, choose a sequence $[R_n]$, $R_{n+1} \leqq aR_n$, such that the circles $|z| = R_n$ do not cut any circle of either $A(h)$ or $A_\lambda(h)$. Using (i), (ii) and the choice of λ, we get from (5),

$$M(R_n, f) \leqq \exp\left[(d - d\lambda^\rho + \epsilon) R_n^\rho\right] \tag{6}$$

for $n \geqq n_0 = n_0(\epsilon)$. Starting from (6), an argument of the type used in Lemma 2

shows that $k(f) \leqq d - d\lambda^\rho = \beta + \eta$ and since η is subject to the sole restriction $0 < \eta < d - \beta$ we get $\kappa(f) \leqq \beta$. Since $\beta \leqq \kappa(f)$, we get $\kappa(f) = \beta$ which is the result required.

2.4. In some cases it is possible to conclude that the relation (i) of Theorem 2 follows from (ii). The circles of $A(h)$ determine a sequence of non-overlapping domains $D_1, D_2, \cdots, D_n, \cdots$. Let p_n denote the number of points of $[z_n]$ lying in D_n. We shall prove

LEMMA 3. *If $p_n \leqq P$, a fixed positive number, then* (ii) *of Theorem 2 involves* (i).

Proof. Let z_n be contained in D_{q_n}. Let

$$P_n(z) = \prod_{z_\nu \subset D_{q_n}} \left(1 - \frac{z}{z_\nu}\right),$$

and

$$g(z) = P_n(z) Q_n(z).$$

The greatest and the least distances of the boundary of D_{q_n} from the origin lie in the interval $(|z_n| - H, |z_n| + H)$ where $H = \sum_{n=1}^{\infty} |z_n|^{-h}$. Since the degree of $P_n(z)$ does not exceed P, we have

$$\lim \frac{\log |P_n(z)|}{|z|^\rho} = 0,$$

as $|z| \to \infty$ outside the domains D_ν, uniformly in n. Therefore on the boundary of D_{q_n}, we have by (ii)

$$(7) \quad \exp\left[(d - \epsilon)(|z_n| - H)^\rho\right] \leqq |Q_n(z)| \leqq \exp\left[(d + \epsilon)(|z_n| + H)^\rho\right].$$

Since $Q_n(z)$ does not vanish in D_{q_n}, (7) holds in the interior of D_{q_n}, in particular, at $z = z_n$. Hence

$$\lim_{n \to \infty} \frac{\log |Q_n(z_n)|}{|z_n|^\rho} = d.$$

Moreover

$$g'(z_n) = P_n'(z_n) Q_n(z_n),$$

and arguing as before we get

$$\lim_{n \to \infty} \frac{\log |g'(z_n)|}{|z_n|^\rho} = \lim_{n \to \infty} \frac{\log |Q_n(z_n)|}{|z_n|^\rho} = d.$$

So the lemma is proved.

3. Using Theorem 2, we shall set up an E-set for a given class $C(\rho, d)$. We first establish the following

LEMMA 4. *Let* $\rho=2/\alpha$ *and*

$$\sigma_\rho(z)=\prod_{n=1}^{\infty}\left(1-\frac{z^n}{n^{\alpha n}}\right). \tag{8}$$

Then $\sigma_\rho(z)$ *is an integral function of order* ρ *satisfying* (i) *and* (ii) *of Theorem* 2, *with* $d=\alpha/4=1/(2\rho)$.

Proof. If we prove (ii) with some $h>\rho$, (i) will follow from Lemma 3 since the circles of $A(h)$ are obviously non-overlapping after a certain stage. Let $p>0$ be any integer and let z lie in the ring $p^\alpha\leqq|z|\leqq(p+1)^\alpha$.

$$\sigma_\rho(z)=\left(1-\frac{z^p}{p^{\alpha p}}\right)\left(1-\frac{z^p}{(p+1)^{\alpha(p+1)}}\right)F_p(z), \tag{9}$$

where

$$F_p(z)=\prod_{n=1}^{p-1}\left(1-\frac{z^n}{n^{\alpha n}}\right)\prod_{n=p+2}^{\infty}\left(1-\frac{z^n}{n^{\alpha n}}\right)=S_1\times S_2, \tag{10}$$

say. We have

$$\begin{aligned}\log|S_1| &= \tfrac{1}{2}p(p-1)\log|z|-\alpha\sum_{n=1}^{p-1}n\log n+\log\prod_{n=1}^{p-1}\left|1-\frac{n^{\alpha n}}{z^n}\right| \\ &= \frac{\alpha}{4}p^2+O(p\log p)+\log\prod_{n=1}^{p-1}\left|1-\frac{n^{\alpha n}}{z^n}\right| \qquad (11)\\ &= \frac{\alpha}{4}|z|^{2/\alpha}+O(|z|^{1/\alpha}\log|z|)+\log\prod_{n=1}^{p-1}\left|1-\frac{n^{\alpha n}}{z^n}\right|.\end{aligned}$$

Now, since $p^\alpha\leqq|z|\leqq(p+1)^\alpha$, we have

$$\prod_{n=1}^{p-1}\left(1-\frac{n^{\alpha n}}{p^{\alpha n}}\right)\leqq\prod_{n=1}^{p-1}\left|1-\frac{n^{\alpha n}}{z^n}\right|\leqq\prod_{n=1}^{p-1}\left(1+\frac{n^{\alpha n}}{p^{\alpha n}}\right). \tag{12}$$

Using the fact that $(1+1/x)^x$ steadily increases and $(1-1/x)^{-x}$ steadily decreases to e as x varies in $0<x<\infty$, we get

$$\left(\frac{n}{p}\right)^{\alpha n}=\left\{\left(1-\frac{p-n}{p}\right)^{p/(p-n)}\right\}^{\alpha n(p-n)/p}\leqq\exp\left[-\frac{\alpha n(p-n)}{p}\right]$$

$$\leqq\exp\left[-\frac{\alpha}{2}\cdot\min(n,p-n)\right].$$

Therefore (12) gives

$$0<a=\left\{\prod_{n=1}^{\infty}(1-e^{-\alpha n/2})\right\}^2\leqq\prod_{n=1}^{p-1}\left|1-\frac{n^{\alpha n}}{z^{\alpha n}}\right|\leqq\left\{\prod_{n=1}^{\infty}(1+e^{-\alpha n/2})\right\}^2=b,$$

so that by (11),

$$\log |S_1| = \frac{\alpha}{4}|z|^{2/\alpha} + O(|z|^{1/\alpha}\log|z|). \tag{13}$$

A similar argument shows that

$$\log |S_2| = O(1),$$

so that (9), (10), and (13) give

$$\log|\sigma_\rho(z)| = \frac{\alpha}{4}|z|^{2/\alpha} + O(|z|^{1/\alpha}\log|z|) + \log\left|1-\frac{z^p}{p^{\alpha p}}\right|\left|1-\frac{z^{p+1}}{(p+1)^{\alpha(p+1)}}\right|, \tag{14}$$

where $p^\alpha \leqq |z| \leqq (p+1)^\alpha$. Taking $h>\rho=2/\alpha$, we find from (14) that when z is outside the circles of $A(h)$ but inside the ring $p^\alpha \leqq |z| \leqq (p+1)^\alpha$,

$$\log|\sigma_\rho(z)| = \frac{\alpha}{4}|z|^{2/\alpha} + O(|z|^{1/\alpha}\log|z|), \tag{15}$$

and since p is any integer, we get

$$\lim \frac{\log|\sigma_\rho(z)|}{|z|^\rho} = \frac{\alpha}{4} = \frac{1}{2\rho}$$

as $|z|\to\infty$ outside the circles of $A(h)$. So the lemma is proved.

3.1. It is easy to see that if $[z_n]$ is an E-set for $C(\rho, d)$ then $[\eta z_n]$ is an E-set for $C(\rho, d/\eta^\rho)$. Hence Lemma 4 enables us, in conjunction with Theorem 2, to state

THEOREM 3. *Let $\rho>0$, $d>0$ be given. The set of points*

$$[(2\rho d)^{-1/\rho}n^{2/\rho}e^{2\nu\pi i/n}], \qquad n = 1, 2, 3, \cdots; \nu = 0, 1, 2, \cdots, n-1,$$

forms an E-set for the class $C(\rho, d)$. In other words, if $f(z)$ is any function of order ρ and type less than d, then

$$\limsup_{r\to\infty}\frac{\log M(r,f)}{r^\rho} = \kappa(f) = \limsup_{n\to\infty}\frac{\log|f[(2\rho d)^{-1/\rho}n^{2/\rho}e^{2\nu\pi i/n}]|}{(2\rho d)^{-1}n^2}.$$

3.2. As a corollary we get, taking $d=1/(2\rho)$, in Theorem 3,

THEOREM 4. *If $f(z)$ is any integral function of order less than ρ or of order ρ and minimal type ($\kappa(f)=0$), then*

$$\limsup_{n\to\infty}\frac{\log|f(n^{2/\rho}e^{2\nu\pi i/n})|}{n^2} = 0.$$

3.3. If $\sigma(z)$ is the canonical product with simple zeros at the lattice points $z=m+in$, $m, n=0, \pm 1, \pm 2, \cdots$, it is known from the pseudo-periodic properties of $\sigma(z)$ that (i) and (ii) of Theorem 2 hold for $\sigma(z)$ with $\rho=2$ and $d=\pi/2$. Hence the class $C(2, \pi/2)$ has the peculiarly simple E-set $z=m+in$, $m, n=0, \pm 1, \pm 2, \cdots$.

3.4. I have shown elsewhere* that a function of $C(2, \pi/2)$ bounded at the lattice-points must be a constant. The question may be asked whether the same is true of an E-set for $C(\rho, d)$ for which the conditions of Theorem 2 hold. That this is in fact the case can be shown by using exactly the same method followed in the case of the lattice points.† So we can state

THEOREM 5. *Let $[z_n]$ be a set of points satisfying the conditions* (i) *and* (ii) *of Theorem* 2. *Then any function of order ρ and type less than d bounded at the points $[z_n]$ must reduce to a constant.*

3.5. As a particular case of Theorem 5 we get

THEOREM 6. *An integral function of order ρ and type less than $1/(2\rho)$ bounded at the points*

$$n^{2/\rho}e^{2\nu\pi i/n}, \qquad n = 1, 2, 3, \cdots; \nu = 0, 1, 2, \cdots, n-1$$

reduces to a constant.

3.6. It may be noted that an E-set for a class $C(\rho, d)$ is a fixture to that class and is independent of the individual functions of the class. Theorems 2–4 throw a good deal of light on the peculiar behaviour of the functions of $C(2, \pi/2)$ at the lattice points. These latter were, in fact, the starting point of the investigations of this paper. It is very probable that conditions closely allied to those of Theorem 2 are also necessary for an E-set although I have not succeeded in discovering exactly what these conditions are. The question whether (ii) of Theorem 2 always involves (i) is also unsolved.

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* Journal of the London Mathematical Society, vol. 11 (1936), pp. 247–250.

† Since $f(z_n)=0$, $\kappa(f)\leqq 0$; if $\kappa(f)<0$, $f(z)\equiv 0$; if $\kappa(f)=0$, then also $\kappa(f^p)=0$, so that formula (3) holds with $m=0$ and $f^p/p!$, $p=0, 1, 2, \cdots$, in place of $f(z)$. An addition and an argument, as in Lemma 2, will show that $e^{f(z)}$ is of finite order, that is, $f(z)$ is a polynomial which must be a constant since $f(z_n)=O(1)$.

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FOURIER SERIES AND MEAN MODULI OF CONTINUITY*

BY

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1. **Introduction.** Let $f(\theta)$ be a real-valued function of the real variable θ, of period 2π, and let $f(\theta)\in L$.† We consider the formal Fourier series of $f(\theta)$ and write

$$f(\theta) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\theta} \sim a_0/2 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta),$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\tau)e^{-in\tau}d\tau = \tfrac{1}{2}(a_n - ib_n),\ c_{-n} = \overline{c_n},\ n = 0, 1, 2, \cdots. \tag{1.1}$$

We denote by Lip (α) the class of functions satisfying the condition

$$\text{Lip } (\alpha) \qquad |f(\theta + t) - f(\theta)| < K|t|^{\alpha},$$

where $\alpha>0$ and $K=K_f$ is a constant depending on f.

In 1914, S. Bernstein [1, 2]‡ proved the following result:

If $f(\theta)\in\text{Lip }(\alpha)$, $\alpha>1/2$, then $\sum|c_n|<\infty$; for $\alpha<1/2$ this is no longer true.

This result was generalized by the writer [12] in two directions:

1. The class Lip (α) was replaced by the more general class

$$\text{Lip}_2\ (\alpha;\ 2) \qquad \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(\theta + 2t) + f(\theta - 2t) - 2f(\theta)]^2 d\theta\right\}^{1/2} < Kt^{\alpha},\quad t > 0.$$

2. The exponent of convergence of the sequence of Fourier coefficients $\{c_n\}$ of a function of class $\text{Lip}_2\ (\alpha;\ 2)$ was determined, that is, the smallest constant k such that $\sum|c_n|^{k+\epsilon}<\infty$ for all $\epsilon>0$. It was proved that $k=2/(2\alpha+1)$ for $0<\alpha\leqq 1$ and that there exist functions in $\text{Lip}_2\ (\alpha;\ 2)$ such that $\sum|c_n|^{2/(2\alpha+1)}=\infty$.

In a later paper [14] these results were extended to more general classes,

$$\text{Lip}_2\ (\alpha;\ p) \qquad \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta + 2t) + f(\theta - 2t) - 2f(\theta)|^p d\theta\right\}^{1/p} < Kt^{\alpha},$$

$$t > 0,\ 0 < \alpha \leqq 1,\ 1 < p \leqq 2.$$

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† That is, $f(\theta)$ is integrable in the sense of Lebesgue. In general, by L^p we denote the class of functions measurable and such that $|f(\theta)|^p\in L$.

‡ Numbers in brackets refer to the bibliography at the end of this paper.

For functions of this class, the exponent of convergence of the sequence $\{c_n\}$ was found to be $k=p/(\alpha p+p-1)$ and it was proved that there exist functions in $\mathrm{Lip}_2(\alpha; p)$ such that $\sum|c_n|^k=\infty$.

In an earlier paper Hardy and Littlewood [5] found the same value for k for the class

$$\mathrm{Lip}_1(\alpha; p) \quad \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta+t)-f(\theta-t)|^p d\theta\right\}^{1/p} < Kt^\alpha, \quad t>0,$$

under the restrictive assumption $\alpha p>1$.*

Let $f(\theta)$ be continuous and of period 2π. The modulus of continuity of $f(\theta)$ is defined by

$$(1.2) \qquad \xi(f;\delta)\equiv\xi(\delta)=\max_{|t|\leqq\delta}\max_{\theta}|f(\theta+t)-f(\theta)|.$$

In 1934, S. Bernstein [3] generalized his result as follows:

If $\sum_1^\infty n^{-1/2}\xi(1/n)<\infty$ then $\sum|c_n|<\infty$. Conversely let $\eta(\delta)$ be a positive function such that $\eta(\delta)\delta^{-\alpha}$ is increasing in δ for sufficiently small positive values of α, and decreasing for some $\alpha<1$. If in addition $\sum_1^\infty n^{-1/2}\eta(1/n)=\infty$ then there exists a continuous function $f(\theta)$ such that $\xi(f;\delta)<\eta(\delta)$ while $\sum|c_n|=\infty$.

In what follows we shall prove some theorems which include all the results mentioned above as special cases and which also extend certain results of Hardy and Littlewood concerning the classes $\mathrm{Lip}_1(\alpha; p)$.†

The following facts will be frequently used in the sequel.

(i) *Young-Hausdorff inequalities.* Let $1<p\leqq 2$, $g(\theta)\sim\sum_{-\infty}^{\infty}\gamma_n e^{in\theta}$. Then

$$\{\sum|\gamma_n|^{p'}\}^{1/p'}\leqq\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}|g(\tau)|^p d\tau\right\}^{1/p},\ddagger$$

$$\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}|g(\tau)|^{p'}d\tau\right\}^{1/p'}\leqq\{\sum|\gamma_n|^p\}^{1/p}.$$

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* This paper of Hardy and Littlewood contains many important results concerning the class $\mathrm{Lip}_1(\alpha; p)$; cf. also [4]. It should be observed that the class $\mathrm{Lip}_2(\alpha; p)$ includes the class $\mathrm{Lip}_1(\alpha; p)$.

† Results analogous to the theorem above of S. Bernstein were obtained for the convergence of Fourier integrals by Titchmarsh [15] and by Hille and Tamarkin [8]. Results of the present paper can also be extended to the cases of Fourier integrals and of almost periodic functions. These extensions will be treated in a subsequent paper. The main results of the present paper were communicated to the Harvard Mathematical Colloquium on May 23, 1935.

‡ By $p', q', \cdots$ we denote quantities which are related to $p, q, \cdots$ by $1/p+1/p'=1/q+1/q'=\cdots=1$.

It should be observed that Young-Hausdorff inequalities are trivially satisfied even in the case $p=1$, if we agree to replace

$$\{\sum|\gamma_n|^{p'}\}^{1/p'}, \qquad \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}|g(\tau)|^{p'}d\tau\right\}^{1/p'}$$

by $\sup_n|\gamma_n|$, $\sup_\tau|g(\tau)|$ respectively, where $\sup_\tau|g(\tau)|$ denotes in this case the "effective upper bound of $|g(\tau)|$" [7, p. 135].

(ii) ***Hölder inequality.*** Assuming $q>1$, we have

$$\sum_{\nu=1}^{\infty}|a_\nu b_\nu| \leqq \left(\sum_{\nu=1}^{\infty}|a_\nu|^q\right)^{1/q}\left(\sum_{\nu=1}^{\infty}|b_\nu|^{q'}\right)^{1/q'},$$

$$\int_a^b|f(t)g(t)|\,dt \leqq \left(\int_a^b|f(t)|^q dt\right)^{1/q}\left(\int_a^b|g(t)|^{q'}dt\right)^{1/q'},$$

with the same agreement as in (i) in case $q=1$.

(iii) ***Minkowski inequalities.*** If $q\geqq 1$,

$$\left\{\int_a^b dt\left(\int_\alpha^\beta|\phi(t,\tau)|\,d\tau\right)^q\right\}^{1/q} \leqq \int_\alpha^\beta d\tau\left(\int_a^b|\phi(t,\tau)|^q dt\right)^{1/q},$$

$$\left\{\int_a^b\left|\sum_{\nu=1}^{n}\phi_\nu(x)\right|^q dx\right\}^{1/q} \leqq \sum_{\nu=1}^{n}\left\{\int_a^b|\phi_\nu(x)|^q dx\right\}^{1/q}.$$

(iv) If we put

$$\frac{1}{b-a}\left(\int_a^b|\phi(t)|^p dt\right)^{1/p} = M_p(\phi;a,b)\equiv M_p(\phi)\equiv M_p, \qquad p>0,$$

then M_p is an increasing* function of p and $\log(M_p)^p$ is a convex function of p, that is

$$(1.3)\quad (M_p)^p \leqq (M_{p_1})^{\rho_1 p_1}(M_{p_2})^{\rho_2 p_2}, \quad p=\rho_1 p_1+\rho_2 p_2, \quad \rho_1+\rho_2=1, \quad \rho_1\geqq 0, \quad \rho_2\geqq 0.$$

Furthermore, on denoting by $M(\phi;a,b)\equiv M(\phi)\equiv M$ the effective upper bound of $|\phi|$ in (a,b) we have

$$(1.4)\qquad \lim_{p\to\infty}M_p(\phi)\equiv M_\infty(\phi)=M(\phi).$$

Finally if we allow $p_2\to\infty$ in (1.3) and observe that then

$$\rho_1=\frac{p_2-p}{p_2-p_1}\to 1, \qquad \rho_2 p_2=\frac{p-p_1}{p_2-p_1}p_2\to p-p_1,$$

* By increasing (decreasing) we mean non-decreasing (non-increasing), unless explicitly stated to the contrary.

we have

$$(1.5) \qquad (M_p)^p \leqq (M_{p_1})^{p_1} M^{p-p_1}, \qquad 0 < p_1 \leqq p.^*$$

The notation $\eta(t)$ will be used to denote a positive decreasing function of t such that $\eta(t) \downarrow 0$ as $t \downarrow 0$. By γ will be meant an arbitrary positive constant. The symbol $O(\phi(t, \gamma))$ will have its usual meaning with the understanding that O refers to ϕ as a function of t, for γ fixed, and that O need not hold uniformly for all γ.

2. **Auxiliary propositions.** In order not to interrupt the subsequent discussion we shall prove in this section some inequalities which will be of fundamental importance in the sequel.

LEMMA 2.1. *Let $0 \leqq A_1 \leqq A_2 \leqq \cdots \leqq A_n \leqq \cdots$ be any non-decreasing sequence of positive numbers, and $\rho > 0$. Then all statements (for all γ)*

$$(2.1, \gamma) \qquad A_n = O(n^\rho \eta(\gamma/n))$$

are equivalent.

It is clear that if $\gamma < \gamma_1$, then $(2.1, \gamma)$ implies $(2.1, \gamma_1)$. If $\gamma > \gamma_1$, let k be an integer such that $\gamma/\gamma_1 \leqq k$. On replacing n by kn in $(2.1, \gamma)$ we have

$$A_n \leqq A_{kn} = O((kn)^\rho \eta(\gamma/kn)) = O(n^\rho \eta(\gamma_1/n)),$$

so that again $(2.1, \gamma)$ implies $(2.1, \gamma_1)$.

LEMMA 2.2. *All statements*

$$(2.2, \gamma) \qquad \sum_{\nu=1}^{\infty} \eta(\gamma/\nu) < \infty,$$

$$(2.3, \gamma) \qquad \sum_{\lambda=0}^{\infty} 2^\lambda \eta(\gamma 2^{-\lambda}) < \infty,$$

$$(2.4) \qquad \int_1^\infty \eta(1/t)\,dt < \infty$$

are equivalent. It is understood that in $(2.2, \gamma)$, $(2.3, \gamma)$, γ may assume independently any value >0, not necessarily the same in these two formulas.

First assume that γ has the same value in $(2.2, \gamma), (2.3, \gamma)$. Since $\eta(\gamma/n) \downarrow 0$ as $n \uparrow \infty$ we have

$$n\eta(\gamma/2n) \leqq \sum_{\nu=n+1}^{2n} \eta(\gamma/\nu) \leqq n\eta(\gamma/n).$$

On putting here $n = 2^\lambda$ and summing over $\lambda = 0, 1, 2, \cdots$ we see at once that

* We refer to [7] to the proofs of (i–iv).

$(2.2, \gamma)$ is equivalent to $(2.3, \gamma)$. That $(2.2, \gamma)$ is equivalent to (2.4) follows immediately from the Cauchy integral convergence test. Since however (2.4) does not depend on γ, we may give γ independently any values in $(2.2, \gamma)$ and in $(2.3, \gamma)$, and all resulting statements will be equivalent.

LEMMA 2.3. *All statements*

$$(2.4, \gamma) \qquad \sum_{\lambda=1}^{\infty} \eta(2^{-\lambda}\gamma/n) = O(\eta(\gamma/n)) \text{ as } n \to \infty,$$

$$(2.5, \gamma) \qquad \sum_{\nu=n}^{\infty} \nu^{-1}\eta(\gamma/\nu) = O(\eta(\gamma/n)) \text{ as } n \to \infty,$$

$$(2.6) \qquad \int_u^{\infty} x^{-1}\eta(1/x)dx = O(\eta(1/u)) \text{ as } u \to \infty$$

are equivalent, in the same sense as in Lemma 2.2.

Again we first assume γ to be the same in $(2.4, \gamma)$ and $(2.5, \gamma)$. The equivalence of these two statements follows then immediately from the inequality

$$\tfrac{1}{2}\eta(\gamma/2n) \leqq \sum_{\nu=n+1}^{2n} \nu^{-1}\eta(\gamma/\nu) \leqq \eta(\gamma/n).$$

Since $\nu^{-1}\eta(\gamma/\nu)$ decreases when ν increases we may again apply the Cauchy test, with the result

$$\sum_{\nu=n}^{\infty} \nu^{-1}\eta(\gamma/\nu) < n^{-1}\eta(\gamma/n) + \int_{n/\gamma}^{\infty} x^{-1}\eta(1/x)dx,$$

$$\int_u^{\infty} x^{-1}\eta(1/x)dx < \gamma^{-1}u^{-1}\eta(1/u) + \int_{n/\gamma}^{\infty} x^{-1}\eta(1/x)dx$$

$$< \gamma^{-1}u^{-1}\eta(1/u) + \sum_{\nu=n}^{\infty} \nu^{-1}\eta(\gamma/\nu),$$

where, in the second formula, $n = [\gamma u]+1$. The first of these formulas shows that (2.6) implies (2.5, γ), and the second that (2.5, γ) implies (2.6). Since (2.6) does not depend on γ, we complete the proof as in Lemma 2.2.

LEMMA 2.4. *Let* $\{a_n\}$ *be any sequence of constants* $a_n \geqq 0$, *and* $\rho > 0$, *then the statements*

$$(2.7) \qquad \sum_{\nu=1}^{\infty} |\sin \nu t|^{\rho} a_\nu = O(\eta(t)) \text{ as } t \to 0$$

and

$$n^{-\rho}\sum_{\nu=1}^{n}\nu^{\rho}a_{\nu}+\sum_{\nu=n+1}^{\infty}a_{\nu}=O(\eta(\pi/2n))\ \text{as}\ n\to\infty \tag{2.8}$$

are equivalent.

We first prove that (2.8) implies (2.7). Let (2.8) be satisfied and $t>0$ be given. Then on taking

$$n=[\pi/2t],\qquad n\leqq\pi/2t<n+1,\qquad t>\pi/2(n+1),$$

we have

$$\sum_{\nu=1}^{\infty}|\sin\nu t|^{\rho}a_{\nu}\leqq t^{\rho}\sum_{\nu=1}^{n+1}\nu^{\rho}a_{\nu}+\sum_{\nu=n+2}^{\infty}a_{\nu}=O\left(\eta\left(\frac{\pi}{2(n+1)}\right)\right)=O(\eta(t)).$$

Conversely, let (2.7) be satisfied and a sufficiently large integer n be given. Let $\delta>0$ be such that

$$n\leqq\frac{\pi}{2\delta}<n+1,\qquad\frac{\pi}{2(n+1)}<\delta\leqq\frac{\pi}{2n}. \tag{2.9}$$

Since $\eta(t)$ increases when t increases, an integration of (2.7) over $(0,\delta)$ yields

$$\sum_{\nu=1}^{\infty}a_{\nu}\int_{0}^{\delta}|\sin\nu t|^{\rho}dt=O(\delta\eta(\delta)).$$

On the other hand, in view of our choice of δ,

$$\int_{0}^{\delta}|\sin\nu t|^{\rho}dt>(2\nu/\pi)^{\rho}\int_{0}^{\delta}t^{\rho}dt>(\nu/2)^{\rho}\delta^{\rho+1}/(\rho+1),\qquad\nu=1,2,\cdots,n,$$

and

$$\int_{0}^{\delta}|\sin\nu t|^{\rho}dt=\nu^{-1}\int_{0}^{\nu\delta}|\sin t|^{\rho}dt\geqq\nu_{\delta}/\nu\int_{0}^{\pi/2}(\sin t)^{\rho}dt$$

$$>\nu_{\delta}/\nu\int_{0}^{\pi/2}(\sin t)^{\rho}\cos t\,dt=\frac{\nu_{\delta}}{\nu(\rho+1)}\geqq\frac{\delta}{\pi(\rho+1)},\qquad\nu=n+1,\cdots,$$

where

$$\nu_{\delta}=\left[\frac{2\nu\delta}{\pi}\right],\quad\nu_{\delta}\leqq\frac{2\nu\delta}{\pi}<\nu_{\delta}+1,\quad\nu_{\delta}\geqq1>\frac{2\nu\delta}{\pi}-\nu_{\delta},\quad\nu_{\delta}>\frac{\nu\delta}{\pi}.$$

This gives

$$\sum_{\nu=1}^{n}(\nu\delta)^{\rho}a_{\nu}+\sum_{\nu=n+1}^{\infty}a_{\nu}=O(\eta(\delta)),$$

which is equivalent to (2.8) in view of (2.9).

LEMMA 2.5. *Under the assumptions $\rho>0$ and*

$$(2.4,\gamma)\qquad \sum_{\lambda=1}^{\infty}\eta(2^{-\lambda}\gamma/n)=O(\eta(\gamma/n))\ \text{as}\ n\to\infty,\quad \gamma>0.$$

all the statements

$$(2.10,\gamma)\qquad n^{-\rho}\sum_{\nu=1}^{n}\nu^{\rho}a_{\nu}+\sum_{\nu=n+1}^{\infty}a_{\nu}=O(\eta(\gamma/n))\ \text{as}\ n\to\infty,$$

$$(2.11,\gamma)\qquad n^{-\rho}\sum_{\nu=1}^{n}\nu^{\rho}a_{\nu}=O(\eta(\gamma/n))\ \text{as}\ n\to\infty,$$

$$(2.12)\qquad \sum_{\nu=1}^{\infty}|\sin\nu t|^{\rho}a_{\nu}=O(\eta(t))\qquad \text{as}\ t\to 0$$

are equivalent.

To prove that (2.10, γ) and (2.11, γ) are equivalent it suffices to prove that (2.11, γ) implies (2.10, γ), the converse being trivial. Put

$$u_0=0,\qquad u_n=\sum_{\nu=1}^{n}\nu^{\rho}a_{\nu},\qquad n=1,2,\cdots.$$

Then, if (2.11, γ) is satisfied, we have

$$\begin{aligned}\sum_{\nu=n+1}^{2n}a_{\nu}&=\sum_{\nu=n+1}^{2n}(u_{\nu}-u_{\nu-1})\nu^{-\rho}\\&=-u_n(n+1)^{-\rho}+\sum_{\nu=n+1}^{2n-1}u_{\nu}(\nu^{-\rho}-(\nu+1)^{-\rho})+u_{2n}(2n)^{-\rho}\\&=O(\eta(\gamma/n))+\sum_{\nu=n+1}^{2n-1}O(\nu^{-1}\eta(\gamma/\nu))+O(\eta(\gamma/2n))=O(\eta(\gamma/n)).\end{aligned}$$

Consequently,

$$\sum_{\nu=n+1}^{\infty}a_{\nu}=O\left(\sum_{\lambda=0}^{\infty}\eta(2^{-\lambda}\gamma/n)\right)=O(\eta(\gamma/n)),$$

by (2.4, γ). Thus (2.10, γ) and (2.11, γ) are equivalent. The proof of Lemma 2.5 is now readily completed by using Lemmas 2.1, 2.3, and 2.4.

LEMMA 2.6. *Let $a(u)$, $b(u)$ be functions satisfying conditions*

$$(2.13)\qquad 0<\gamma\leqq a(u)\leqq 1,\quad 0<\gamma\leqq b(u)\leqq 1,\quad a(1)=b(1)=1,$$

neither of them being identically equal to 1. *Then the following statements are equivalent:*

$$\int_1^u \eta(1/x)dx = O(u\eta(1/u)) \quad as \quad u \to \infty, \tag{2.14}$$

$$\eta(1/u) = \eta(1)a(u)u^{-1} \exp\left\{\int_1^u a(t)t^{-1}dt\right\}, \tag{2.15}$$

$$\int_1^u x\eta(1/x)^2dx = O(u^2\eta(1/u)^2) \quad as \quad u \to \infty, \tag{2.16}$$

$$\eta(1/u) = \eta(1)b(u)u^{-1} \exp\left\{\int_1^u b^2(t)t^{-1}dt\right\}. \tag{2.17}$$

We first prove the equivalence of (2.14) and (2.15). On putting $\phi(x) = \eta(1/x)$ for $x \geqq 1$ and $\phi(x) = \phi(1)$ for $0 < x < 1$, we see that $\phi(x) \downarrow 0$ as $x \uparrow \infty$. Now if (2.14) is satisfied, we have

$$u\phi(u) \leqq \int_0^u \phi(x)dx \leqq \gamma^{-1}u\phi(u),$$

where necessarily $0 < \gamma < 1$. Put

$$a(u) = \frac{u\phi(u)}{\phi_1(u)}, \qquad \phi_1(u) = \int_0^u \phi(x)dx, \qquad \phi_1(1) = \phi(1) = \eta(1).$$

Clearly $a(u)$ satisfies (2.13) while

$$\frac{\phi_1'(u)}{\phi_1(u)} = \frac{a(u)}{u}, \qquad \phi_1(u) = \phi_1(1) \exp\left\{\int_1^u a(t)t^{-1}dt\right\}.$$

On substituting this expression of $\phi_1(u)$ into $\phi(u) = a(u)u^{-1}\phi_1(u)$ we get (2.15). Thus (2.14) implies (2.15). The converse is readily proved by an easy computation by using (2.13). To prove the equivalence of (2.16) and (2.17) we apply an analogous argument starting from the inequalities

$$\tfrac{1}{2}u^2\phi^2(u) \leqq \int_0^u x\phi^2(x)dx \leqq (2\gamma)^{-1}u^2\phi^2(u)$$

(implied by (2.16)) and putting presently

$$b^2(u) = \frac{u^2\phi^2(u)}{2\phi_2(u)}, \qquad \phi_2(u) = \int_0^u x\phi^2(x)dx,$$

$$\phi_2'(u) = u\phi^2(u), \qquad 2b^2(u)\phi_2(u) = u\phi_2'(u),$$

$$\phi_2'(u) = u\phi^2(u) = \phi^2(1)b^2(u)u^{-1} \exp\left\{2\int_1^u b^2(x)x^{-1}dx\right\}.$$

To complete the proof of our lemma it remains to show that (2.14) is equivalent to (2.16). On assuming (2.14) to be satisfied we have (2.15), whence

$$\int_1^u x\eta(1/x)^2dx < \eta(1)^2 \int_1^u a(x)x^{-1} \exp\left\{2\int_1^x a(t)t^{-1}dt\right\}dx$$

$$= \tfrac{1}{2}\eta(1)^2\left[\exp\left\{2\int_1^u a(t)t^{-1}dt\right\}\right] = O(u^2\eta(1/u)^2),$$

which is (2.16). Conversely on assuming (2.16) we have (2.17) so that

$$\eta(1/u) > \eta(1)b^2(u)u^{-1} \exp\left\{\int_1^u b^2(x)x^{-1}dx\right\} \equiv \eta_1(1/u),$$

$$\eta(1/u) < \eta(1)\gamma^{-1}b^2(u)u^{-1} \exp\left\{\int_1^u b^2(x)x^{-1}dx\right\} = \gamma^{-1}\eta_1(1/u).$$

The function $\eta_1(1/u)$ clearly is of the type (2.15), hence satisfies (2.14), so that

$$\int_1^u \eta(1/x)dx < \gamma^{-1}\int_1^u \eta_1(1/x)dx = O(u\eta_1(1/u)) = O(u\eta(1/u)).$$

The proof of Lemma 2.6 is now complete.

REMARK 2.1. *Condition* (2.14) *of Lemma* 2.6 *is satisfied if, for a positive* $\alpha<1$, $\eta(t)t^{-\alpha}\downarrow$ *when* $t\uparrow$. *By Lemma* 2.6, *conditions* (2.15), (2.16), (2.17) *will be also satisfied in this case.*

Indeed our assumption implies that $\eta(1/x)x^{\alpha}\uparrow$ when $x\uparrow$, whence

$$\int_1^u \eta(1/x)dx \leqq u^{\alpha}\eta(1/u)\int_1^u x^{-\alpha}dx = O(u\eta(1/u)).$$

REMARK 2.2. *If in* (2.15)

$$0 < \gamma \leqq a(u) \leqq \gamma_1 < 1, \qquad u > 1, \tag{2.18}$$

then

$$\int_1^{\infty} x^{-1}\eta(1/x)dx < \infty \tag{2.19}$$

and

$$\int_{t^{-1}}^{\infty} \tau^{-1}\eta(\tau)d\tau = O(\eta(t)) \text{ as } t\to 0. \tag{2.20}$$

Conditions (2.19), (2.20) *are also satisfied if for some positive* $\alpha<1$, $\eta(t)t^{-\alpha}\downarrow$ *when* $t\downarrow 0$.

Indeed if (2.18) is satisfied then

$$\eta(1/u) = O\left(u^{-1}\exp\left\{\gamma_1\int_1^u t^{-1}dt\right\}\right) = O(u^{\gamma_1-1}) \text{ as } u\to\infty,$$

so that (2.19) holds. Again using the same notation as in the proof of Lemma 2.6 and assuming for simplicity $\eta(1)=1$ we have

$$\begin{aligned}\int_{t^{-1}}^{\infty} x^{-1}\eta(1/x)dx &= x^{-1}\phi_1(x)\Big|_{t^{-1}}^{\infty} + \int_{t^{-1}}^{\infty} x^{-2}\phi_1(x)dx \\ &= -\frac{\eta(t)}{a(t^{-1})} + \int_{t^{-1}}^{\infty}\eta(1/x)\frac{dx}{xa(x)}.\end{aligned}$$

Hence

$$\int_{t^{-1}}^{\infty} x^{-1}\eta(1/x)\left(\frac{1}{a(x)}-1\right)dx = \frac{\eta(t)}{a(t^{-1})} = O(\eta(t)) \text{ as } t\to 0.$$

Since, however,

$$\frac{1}{a(x)} - 1 \geqq \gamma_1^{-1} - 1 > 0,$$

we also have

$$\int_{t^{-1}}^{\infty} x^{-1}\eta(1/x)dx = O(\eta(t))$$

which is (2.20). The last two statements of Remark 2.2 are readily proved if we observe that $\eta(1/x)x^{\alpha}\downarrow$ as $x\uparrow$ implies (2.19) immediately while (2.20) follows then from

$$\int_{t^{-1}}^{\infty} x^{-1}\eta(1/x)dx < t^{-\alpha}\eta(t)\int_{t^{-1}}^{\infty} x^{-\alpha-1}dx = O(\eta(t)) \text{ as } t\to 0.$$

3. **Mean moduli of continuity.** For any function $f(\theta)$ we shall use the notation

$$\begin{aligned}\Delta_0 f(\theta;t) &= f(\theta),\\ \Delta_1 f(\theta;t) &= f(\theta+t) - f(\theta-t),\\ \Delta_2 f(\theta;t) &= \Delta_1\Delta_1 f(\theta;t) = f(\theta+2t) + f(\theta-2t) - 2f(\theta),\end{aligned}$$

$$\Delta_m f(\theta;t) = \Delta_1\Delta_{m-1}f(\theta;t) = \sum_{\nu=0}^{m}(-1)^{\nu}C_{m,\nu}f(\theta+(m-2\nu)t), \quad m = 1, 2, 3, \cdots. \tag{3.1}$$

If $f(\theta)\in L^p$, $p>0$ we define the expression

$$\omega_p^{(m)}(f;\delta) = \sup_{0\leqq t\leqq\delta} M_p(\Delta_m f(\theta;t);-\pi,\pi) \tag{3.2}$$

as the mean modulus of continuity of order m of $f(\theta)$ in L^p. We shall also write $\omega_p(\delta)$ or ω_p instead of $\omega_p^{(m)}(f;\delta)$ if no confusion results. In view of §1, (iv), ω_p is an increasing function of p. It is clear that $\omega_p(\delta)$ is also an increasing function of δ. Moreover it is well known that $M_p(\Delta_1 f(\theta;t))\to 0$ as $t\to 0$ ("continuity in L^p" of a function of L^p). The same is true then of $M_p(\Delta_m f(\theta;t))$ so that $\omega_p^{(m)}(f;\delta)\downarrow 0$ as $\delta\downarrow 0$. We also note that when $f(\theta)$ is continuous

$$\lim_{p\to\infty}\omega_p^{(1)}(f;\delta)\equiv\omega_\infty^{(1)}(f;\delta)=\xi(f;2\delta) \tag{3.3}$$

(cf. §1, (1.2)). Furthermore if

$$f(\theta)\sim\sum_{\nu=-\infty}^{\infty}c_\nu e^{i\nu\theta},$$

then

$$\Delta_m f(\theta;t)\sim\sum_{\nu=-\infty}^{\infty}c_\nu(e^{i\nu t}-e^{-i\nu t})^m e^{i\nu\theta}. \tag{3.4}$$

With this notation we prove

LEMMA 3.1. *If $p\geqq 1$,* then*

$$2^m|c_n|\leqq\omega_1^{(m)}(\pi/2n)\leqq\omega_p^{(m)}(\pi/2n),\qquad n=0,1,2,\cdots. \tag{3.5}$$

We have

$$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\theta)e^{-in\theta}d\theta=\frac{(-i)^m}{2\pi}\int_{-\pi}^{\pi}f(\theta+(m-2\nu)\pi/2n)e^{-in\theta}(-1)^\nu d\theta,$$

whence, by (3.1),

$$2^m|c_n|\leqq\frac{1}{2\pi}\int_{-\pi}^{\pi}|\Delta_m f(\theta;\pi/2n)|\,d\theta\leqq\omega_1^{(m)}(\pi/2n)\leqq\omega_p^{(m)}(\pi/2n).$$

THEOREM 3.1. *Let $f(\theta)\in L^p$, $1\leqq p\leqq 2$, $k>0$, and*

$$\sum_{n=1}^{\infty}n^{-k/p'}(\omega_p^{(m)}(\pi/2n))^k<\infty.\dagger \tag{3.6, p}$$

---

* For the case $m=1$, cf. [19, p. 18].

† When $p=1$ we write

$$\sum_{n=1}^{\infty}(\omega_1^{(m)}(\pi/2n))^k<\infty. \tag{3.6, 1}$$

The same remark refers to analogous situations in the sequel, without being stated explicity.

Under these conditions

$$\sum_{n=0}^{\infty} |c_n|^k < \infty .$$

For $p=1$ the theorem follows directly from (3.5). Now assume $p>1$. If $k=p'$, the statement of the theorem is an immediate consequence of the Young-Hausdorff inequality, without using assumption (3.6, p). Hence the same will be true when $k>p'$. Finally, let $0<k<p'$. The Young-Hausdorff inequality, being applied to (3.4), gives

$$2^{mp'} \sum_{\nu=-\infty}^{\infty} |\sin \nu t|^{mp'} |c_\nu|^{p'} = 2^{mp'+1} \sum_{\nu=1}^{\infty} |\sin \nu t|^{mp'} |c_\nu|^{p'}$$
$$\leqq M_p(\Delta_m f(\theta; t))^{p'} \leqq \omega_p^{(m)}(f; t)^{p'}.$$

Hence, by Lemma 2.4,

$$n^{-mp'} \sum_{\nu=1}^{n} \nu^{mp'} |c_\nu|^{p'} + \sum_{\nu=n+1}^{\infty} |c_\nu|^{p'} = O(\omega_p^{(m)}(\pi/2n)^{p'}) \tag{3.7}$$

and, a fortiori,

$$n^{-mp'} \sum_{\nu=1}^{n} \nu^{mp'} |c_\nu|^{p'} = O(\omega_p^{(m)}(\pi/2n)^{p'}). \tag{3.8}$$

On replacing n by $2n$ we have

$$\sum_{\nu=n+1}^{2n} |c_\nu|^{p'} = O(\omega_p^{(m)}(\pi/4n)^{p'}). \tag{3.9}$$

Now, by the Hölder inequality,

$$\sum_{\nu=n+1}^{2n} |c_\nu|^k \leqq n^{1/q} \left(\sum_{\nu=n+1}^{2n} |c_\nu|^{kq'} \right)^{1/q'}, \qquad q > 1.$$

If we choose here

$$kq' = p', \qquad q = p'/(p' - k)$$

and combine with (3.9), we get

$$\sum_{\nu=n+1}^{2n} |c_\nu|^k = O(n^{1-k/p'} \omega_p^{(m)}(\pi/4n)^k) = O\left(\sum_{\nu=n+1}^{2n} \nu^{-k/p'} \omega_p^{(m)}(\pi/2\nu)^k \right).$$

To complete the proof of Theorem 3.1 it remains to put $n=2^\lambda$, $\lambda=0, 1, \cdots$ and take summation over λ. In view of (3.6, p) this yields

$$\sum_{\nu=2}^{\infty} |c_\nu|^k = O\left(\sum_{\nu=2}^{\infty} \nu^{-k/p'} \omega_p^{(m)}(\pi/2\nu)^k\right) < \infty .$$

REMARK 3.1. *From Lemma 2.2 we conclude that assumption* (3.6, p) *of Theorem* 3.1 *can be replaced by any one of the equivalent assumptions*

(3.6, p, γ) $$\sum_{n=1}^{\infty} n^{-k/p'} \omega_p^{(m)}(\gamma/n)^k < \infty ,$$

(3.10, p) $$\sum_{\lambda=0}^{\infty} 2^{\lambda(1-k/p')} \omega_p^{(m)}(\gamma 2^{-\lambda})^k < \infty ,$$

(3.11, p) $$\int_1^{\infty} x^{-k/p'} \omega_p^{(m)}(1/x)^k dx < \infty .$$

REMARK 3.2. *Note that a result of the preceding discussion is that any one of the conditions of Remark* 3.1 *imply* (3.8).

The following theorem is a generalization of Theorem 3.1.

THEOREM 3.2. *Let*

$$1 \leqq p \leqq 2,\ 0 < p_1 \leqq p \leqq p_2,\ p = \rho_1 p_1 + \rho_2 p_2,\ \rho_1 + \rho_2 = 1,\ \rho_1 \geqq 0,\ \rho_2 \geqq 0.$$

If $f(\theta) \in L^{p_1}$ *and if*

(3.12) $$\sum_{n=1}^{\infty} n^{-k/p'} \left[\omega_{p_1}^{(m)}(\gamma/n)^{\rho_1 p_1} \omega_{p_2}^{(m)}(\gamma/n)^{\rho_2 p_2}\right]^{k/p} < \infty , \qquad 0 < k \leqq p',$$

then

$$\sum_{n=0}^{\infty} |c_n|^k < \infty .$$

By the convexity property

$$\omega_p^p \leqq \omega_{p_1}^{\rho_1 p_1} \omega_{p_2}^{\rho_2 p_2},$$

and our theorem follows immediately from Theorem 3.1. In the case where $\omega_{p_1}^{(m)}(\delta) < K\delta^{\alpha_1}$, $\omega_{p_2}^{(m)}(\delta) < K\delta^{\alpha_2}$, $\alpha_1 > 0$, $\alpha_2 > 0$, [6, p. 252] condition (3.12) is satisfied whenever

$$\alpha_1 \rho_1 p_1 + \alpha_2 \rho_2 p_2 > 1 + \left(\frac{1}{k} - 1\right)p, \qquad k \leqq p'.$$

With the notation $\rho_1 = \rho$, $\rho_2 = 1-\rho$, $0 \leqq \rho \leqq 1$, $p = \rho p_1 + (1-\rho)p_2$ we get

(3.13) $$\alpha_1 \rho p_1 + \alpha_2 (1-\rho) p_2 > 1 + \left(\frac{1}{k} - 1\right)p, \qquad k \leqq p'.$$

In particular, for $k=1$,

$$\rho\alpha_1 p_1 + (1-\rho)\alpha_2 p_2 > 1.$$

This condition for the case $m=1$ was given in a slightly different form by Hardy and Littlewood [6, p. 252]. For $p=2$, $p_1 \leqq 2 \leqq p_2$, this condition reduces to

$$(p_2-2)\alpha_1 p_1 + p_1 > p_2 - (2-p_1)\alpha_2 p_2.$$

Similarly, (3.13) reduces to

$$(p_2-2)\alpha_1 p_1 + p_1 > p_2 - (2-p_1)\alpha_2 p_2 + 2\left(\frac{1}{k}-1\right)(p_2-p_1).$$

If $\alpha_1 p_1 = 1$ this becomes

$$k > 2(p_2-p_1)/\{2p_2 + (2-p_1)\alpha_2 p_2 - p_1 - 2\},$$

which is less than 1 if $\alpha_2 p_2 > 1$.

In the special case $\alpha_1 = p_1 = 1$ we obtain the condition

$$k > 2(p_2-1)/\{(2+\alpha_2)p_2 - 3\}.$$

Note that the class $\mathrm{Lip}_1(1,1)$ coincides with the class of functions of bounded variation [4, p. 599; 5, p. 619]. If, in addition, $f(\theta) \varepsilon \mathrm{Lip}(\alpha_2)$, p_2 can be taken arbitrarily large, and so, on allowing $p_2 \to \infty$, we get

$$k > 2/(2+\alpha_2).$$

This result, in the case $m=1$, is due to Waraszkiewicz [16], while Zygmund [18, pp. 591–598; 19, p. 138] gave an example of divergence of $\sum |\rho_n|^k$ for $k=2/(2+\alpha_2)$. To summarize we state the

COROLLARY 3.1. *Assume that* $p_1 \leqq 2 \leqq p_2$ *and that*

$$\mathrm{Lip}_m(\alpha; p) \qquad \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |\Delta_m f(\theta; t)|^p d\theta\right\}^{1/p} < Kt^{\alpha}, \qquad t > 0,$$

for $\alpha=\alpha_1$, $p=p_1$ *and* $\alpha=\alpha_2$, $p=p_2$. *The series* $\sum_0^{\infty} |c_n|^k$ *converges provided* k *satisfies* (3.13).

The basic inequality in the proof of Theorem 3.1 was (3.8). A partial converse is represented by the

THEOREM 3.3. *Let* $1 \leqq p \leqq 2$. *Let* $\eta(t)$, *in addition to the general assumption* $\eta(t) \downarrow 0$ *as* $t \downarrow 0$, *be such that*

$$\sum_{\lambda=1}^{\infty} \eta\left(\frac{\pi}{n} 2^{-\lambda}\right)^p = O(\eta(\pi/n)^p) \text{ as } n \to \infty. \tag{3.14}$$

If the Fourier coefficients of $f(\theta)$ satisfy the condition

$$\sum_{\nu=1}^{n} \nu^{mp} |c_\nu|^p = O(n^{mp}\eta(\pi/n)^p) \tag{3.15}$$

then

$$\omega_{p'}^{(m)}(f;t) = O(\eta(t)) \text{ as } t \to 0.$$

For a given $t<\pi$ we have by the Young-Hausdorff inequality

$$M_{p'}(\Delta_m f(\theta;t))^p \leqq 2^{pm} \sum_{\nu=-\infty}^{\infty} |\sin \nu t|^{mp} |c_\nu|^p$$

$$\leqq 2^{pm+1}\left\{ \sum_{\nu=1}^{n} |\sin \nu t|^{mp} |c_\nu|^p + \sum_{\nu=n+1}^{\infty} |c_\nu|^p \right\}$$

(cf. 1, (i)), where

$$n = [\pi/t] + 1 > 1, \qquad \pi/n < t \leqq \pi/(n-1).$$

Let $S_1(t)$, $S_2(t)$ denote the first and the second term in $\{\ \}$, respectively. Then, by (3.15),

$$S_1(t) \leqq t^{mp} \sum_{\nu=1}^{n} \nu^{mp} |c_\nu|^p = O(\eta(\pi/n)^p) = O(\eta(t)^p).$$

Furthermore, if we replace n by $2n$ in (3.15) we readily find

$$\sum_{\nu=n+1}^{2n} |c_\nu|^p = O(\eta(\pi/n)^p),$$

whence, by (3.14),

$$S_2(t) = \sum_{\nu=n+1}^{\infty} |c_\nu|^p = O\left(\sum_{\lambda=1}^{\infty} \eta\left(\frac{\pi}{n} 2^{-\lambda}\right)^p \right) = O(\eta(\pi/n)^p) = O(\eta(t)^p).$$

Thus

$$M_{p'}(\Delta_m f(\theta;t)) = O(\eta(t)),$$

which finally implies

$$\omega_{p'}^{(m)}(f;t) = O(\eta(t)).$$

REMARK 3.3. *From Lemmas 2.3 and 2.1 we conclude that assumption* (3.14) *of Theorem 3.3 can be replaced by any one of the equivalent assumptions* ($\gamma>0$)

$$\sum_{\lambda=1}^{\infty} \eta(2^{-\lambda}\gamma/n)^p = O(\eta(\gamma/n)^p) \text{ as } n \to \infty. \tag{3.14, γ}$$

$$(3.16,\gamma)\qquad \sum_{\nu=n}^{\infty} \nu^{-1}\eta(\gamma/\nu)^p = O(\eta(\gamma/n)^p) \text{ as } n\to\infty,$$

$$(3.17)\qquad \int_u^{\infty} x^{-1}\eta(1/x)^p dx = O(\eta(1/u)^p) \text{ as } u\to\infty,$$

and the assumption (3.15) *by the equivalent assumption*

$$(3.15,\gamma)\qquad \sum_{\nu=1}^{n} \nu^{mp}|c_\nu|^p = O(n^{mp}\eta(\gamma/n)^p), \qquad \gamma>0 \text{ arbitrary.}$$

In the special case $\eta(t)=Kt^\alpha \log (1/t)^\beta$, $0<t<1$, (3.14, γ) with $\gamma=1$ becomes

$$\sum_{\lambda=1}^{\infty} n^{-\alpha}2^{-\alpha\lambda}(\log n+\lambda\log 2)^\beta = O(n^{-\alpha}(\log n)^\beta)$$

which is satisfied whenever $\alpha>0$, β arbitrary. In the case $\alpha=1/p$, $p>1$, $\beta=0$, $m=1$ this reduces to a theorem of Hardy and Littlewood [5, p. 620].

If $p=p'=2$, the preceding discussion yields

THEOREM 3.4. *Let $\eta(t)$, in addition to the property $\eta(t)\downarrow 0$ as $t\downarrow 0$, be such that*

$$(3.18)\qquad \sum_{\lambda=1}^{\infty} \eta\left(2^{-\lambda}\frac{\gamma}{n}\right)^2 = O(\eta(\gamma/n)^2) \text{ as } n\to\infty.$$

Then condition

$$(3.19)\qquad \omega_2^{(m)}(f;t) = O(\eta(t)) \text{ as } t\to 0$$

is necessary and sufficient in order that

$$(3.20)\qquad \sum_{\nu=1}^{n} \nu^{2m}|c_\nu|^2 = O(n^{2m}\eta(\gamma'/n)^2) \text{ as } n\to\infty,\ \gamma'>0.$$

4. **Conjugate series.*** The series

$$(4.1)\qquad \sum_{n=1}^{\infty}(a_n\sin n\theta - b_n\cos n\theta)$$

is called the conjugate of

$$a_0/2+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta).$$

* We refer to Zygmund [19] for various facts we use here of the theory of conjugate series and conjugate functions.

If the latter is the Fourier series of a function $f(\theta)\varepsilon L^p$, $p>1$, then (4.1) is also the Fourier series of a function $\tilde{f}(\theta)\varepsilon L^p$, which is defined almost everywhere as the Cauchy principal of the integral

$$\tilde{f}(\theta) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\cot\tfrac{1}{2}(t-\theta)dt. \tag{4.2}$$

THEOREM 4.1. *If $p>1$ and*

$$\omega_p^{(m)}(f;t) = O(\eta(t)) \text{ as } t\to 0,$$

then also

$$\omega_p^{(m)}(\tilde{f};t) = O(\eta(t)).$$

The formulas

$$\tilde{f}(\theta) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta+\phi)\cot\phi/2\,d\phi,$$

$$\Delta_m\tilde{f}(\theta;t) = -\frac{1}{2\pi}\int_{-\pi}^{\pi}\Delta_m f(\theta+\phi;t)\cot\phi/2\,d\phi$$

show that $\Delta_m\tilde{f}(\theta;t)$ is conjugate to $\Delta_m f(\theta;t)$. Hence by a theorem of M. Riesz [19], if A_p denotes a constant depending only on p,

$$M_p(\Delta_m\tilde{f}(\theta;t)) \leqq A_p M_p(\Delta_m f(\theta;t)) = O(\eta(t)).$$

This implies

$$\omega_p^{(m)}(\tilde{f};t) = O(\eta(t)).$$

5. **Inequalities for power series and polynomials.** In the present section we state and prove some inequalities associated with power series and polynomials, which will be useful in constructing various "Gegenbeispiele" in subsequent sections.

Let

$$F(z) = \sum_{n=0}^{\infty} c_n z^n, \qquad z = re^{i\theta},$$

be a power series convergent for $0\leqq r<1$. We write

$$M_p(F;r) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|F(re^{i\theta})|^p d\theta\right)^{1/p}, \qquad p>0.$$

It is well known that $M_p(F;r)$ is an increasing function of r and that

$$\lim_{p\to\infty} M_p(F;r) \equiv M_\infty(F;r) = M(F;r) \equiv \max_{|z|\leqq r} |F(z)|.$$

LEMMA 5.1. *Let $\chi(t)$ be a positive function,*

$$\chi(t/2) = O(\chi(t)) \text{ as } t \to 0, \tag{5.1}$$

and $F(z)$ such that

$$M_p(F;r) = O((1-r)^{-\beta}\chi(1-r)) \text{ as } r \to 1,\ \beta \geqq 0,\ p \geqq 1. \tag{5.2}$$

Then

$$M_q(F;r) = O((1-r)^{-\beta-1/p+1/q}\chi(1-r)), \qquad p < q \leqq \infty. \tag{5.3}$$

We first prove our lemma in the case $q=\infty$, when it reduces to

$$M(F;r) = O((1-r)^{-\beta-1/p}\chi(1-r)). \tag{5.4}$$

We have

$$F(z) = F(re^{i\theta}) = \frac{1}{2\pi}\int_{|\zeta|=\rho} \frac{F(\zeta)d\zeta}{\zeta - z}, \qquad \zeta = \rho e^{i\tau},$$

$$\rho = \frac{1+r}{2}, \qquad 1-\rho = \rho - r = \frac{1-r}{2},$$

whence

$$|F(z)| \leqq \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|F(\zeta)|d\tau}{|\rho e^{i(\theta-\tau)} - r|} \leqq M_1(F;\rho)O((1-r)^{-1}).$$

Combining this with (5.2) for $p=1$ we see that

$$M(F;r) = O((1-r)^{-\beta-1}\chi(1-r)),$$

which proves (5.4) in case $p=1$. If $p>1$, we have by the Hölder inequality

$$\begin{aligned} M(F;r) &\leqq M_p(F;\rho)\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |\rho e^{i\tau} - r|^{-p'}d\tau\right\}^{1/p'} \\ &= M_p(F;\rho)O\left\{\int_0^\infty [(\rho-r)^2 + \tau^2]^{-p'/2}d\tau\right\}^{1/p'} \\ &= M_p(F;\rho)O((1-r)^{-1/p}) = O((1-r)^{-\beta-1/p}\chi(1-r)). \end{aligned}$$

The result in the case $p<q<\infty$ is now readily obtained from the convexity property combined with (5.4). Indeed, by (1.5),

$$\begin{aligned} M_q{}^q &\leqq M_p{}^p M^{q-p} = O((1-r)^{-\beta p}\chi(1-r)^p)O((1-r)^{-(q-p)(\beta+1/p)}\chi(1-r)^{q-p}) \\ &= O((1-r)^{-\beta q+1-q/p}\chi(1-r)^q). \end{aligned}$$

In the remaining lemmas of this section we shall use the following notation.

Let

$$F(z) = \sum_{\nu=0}^{\infty} c_\nu z^\nu \tag{5.5}$$

converge for $r<1$. Let

$$\sigma_n \equiv \sigma_n(z) \equiv \sigma_n(F; z) = n^{-1}\{nc_0 + (n-1)c_1 z + \cdots + c_{n-1}z^{n-1}\}$$

be the arithmetic mean of the n first partial sums of the series (5.5). Let $\Omega(u)$ be a non-negative and convex function of $u \geqq 0$, $\Omega(0)=0$, so that $\Omega(u)$ is increasing. It is well known [19, pp. 83–84] that

$$M_1[\Omega(|\sigma_n|)] \leqq M_1[\Omega(|F(z)|)]. \tag{5.6}$$

In particular

$$M_p(\sigma_n(z)) \leqq M_p(F(z)), \qquad p \geqq 1.* \tag{5.7}$$

We shall use these facts to prove

LEMMA 5.2. *If a polynomial*

$$P_n(z) = c_0 + c_1 z + \cdots + c_n z^n, \qquad z = e^{i\theta}$$

satisfies the inequality

$$M_1[\Omega(|P_n(z)|)] \leqq 1, \tag{5.8}$$

then its derivative $P_n'(z)$ *satisfies*

$$M_1[\Omega(n^{-1}|P_n'(z)|)] \leqq 1.\dagger \tag{5.9}$$

On writing

$$P_n^*(z) = c_n + c_{n-1}z + \cdots + c_0 z^n = z^n P_n(\bar{z}), \qquad \bar{z} = e^{-i\theta}$$

we have

$$n\sigma_n(P_n^*; z) = nc_n + (n-1)c_{n-1}z + \cdots + c_1 z^{n-1} = z^{n-1}P_n'(\bar{z}),$$

whence, by (5.6),

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \Omega[n^{-1}|P_n'(z)|]d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Omega[|\sigma_n(P_n^*; z)|]d\theta$$

$$\leqq \frac{1}{2\pi}\int_{-\pi}^{\pi} \Omega[|P_n^*(z)|]d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Omega[|P_n(z)|]d\theta$$

* Cf. [9, p. 333].

† Lemma 5.2 is a special case of a theorem of Zygmund [17, p. 394] which states the same result for any trigonometric polynomial instead of a polynomial in $e^{i\theta}$. We give a separate proof because of its simplicity.

which is the desired result.

COROLLARY 5.1. *If*

$$M_p(P_n; r) \leqq 1, \qquad 0 \leqq r = |z| \leqq 1, \qquad p \geqq 1, \tag{5.10}$$

then

$$M_p(P_n'; r) \leqq n. \tag{5.11}$$

For $r=1$ this is a special case of Lemma 5.2, corresponding to $\Omega(u)=u^p$. Since $M_p(P_n; r)$, $M_p(P_n'; r)$ are increasing functions of r, the result follows for any $r<1$. The limiting case of Corollary 5.1, as $p\to\infty$ is the classical theorem of M. Riesz [11]: If $|P_n(z)| \leqq 1$, $|z| \leqq 1$, then $|P_n'(z)| \leqq n$. The proof above can be readily modified to yield this result also [13].

COROLLARY 5.2. *Let* $\zeta=\rho e^{it}$, $0<\rho\leqq 1$, $0<t\leqq\pi/2$ *and*

$$Q(z) = \frac{P_n(z\zeta) - P_n(z\bar{\zeta})}{z(\zeta - \bar{\zeta})}, \qquad z = re^{i\theta}, \qquad 0 < r \leqq 1.$$

If (5.10) *is satisfied, then*

$$M_p(Q; r) \leqq nt/\sin t \leqq \frac{\pi}{2} n, \qquad p \geqq 1. \tag{5.12}$$

We have

$$P_n(z\zeta) - P_n(z\bar{\zeta}) = z\int_{\bar{\zeta}}^{\zeta} P_n'(zu)du = i\rho z \int_{-t}^{t} P_n'(z\rho e^{i\tau})e^{i\tau}d\tau,$$

so that

$$Q(z) = \frac{1}{2\sin t}\int_{-t}^{t} P_n'(z\rho e^{i\tau})e^{i\tau}d\tau = \frac{t}{\sin t} I(z),$$

where

$$I(z) = \frac{1}{2t}\int_{-t}^{t} P_n'(z\rho e^{i\tau})e^{i\tau}d\tau.$$

By the Minkowski inequality

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |I(z)|^p d\theta\right)^{1/p} \leqq \frac{1}{2t}\int_{-t}^{t} d\tau \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} |P_n'(z\rho e^{i\tau})|^p d\theta\right]^{1/p},$$

whence

$$M_p(Q; r) \leqq \frac{t}{\sin t} M_p(P_n'; r\rho) \leqq \frac{t}{\sin t} n \leqq \frac{\pi}{2} n.$$

6. **Converse theorems.** In the present section we shall prove that various results of §3 are the "best possible," in a certain sense. We first prove a lemma which will be needed in the sequel.

LEMMA 6.1. *If* $\alpha_\nu>0$, $\beta_\nu>0$, $\alpha_\nu \downarrow 0$ *as* $\nu \uparrow \infty$, *and* $\sum_{\nu=0}^{\infty}\alpha_\nu\beta_\nu=\infty$, *then*

$$\sum_{\nu=0}^{n}(\alpha_\nu-\alpha_{n+1})\beta_\nu \to \infty \quad as \quad n\to\infty .$$

For a given positive integer m we can choose n so large that $\alpha_m>2\alpha_{n+1}$. Then $\alpha_\nu-\alpha_{n+1}>\frac{1}{2}\alpha_\nu$, $\nu=0, 1, \cdots, m$, and

$$\sum_{\nu=0}^{n}(\alpha_\nu-\alpha_{n+1})\beta_\nu > \sum_{\nu=0}^{m}(\alpha_\nu-\alpha_{n+1})\beta_\nu > \frac{1}{2}\sum_{\nu=0}^{m}\alpha_\nu\beta_\nu \to \infty \text{ as } m\to\infty .$$

Now to show that the result of Theorem 3.1 in the case $k=1$ is "the best possible" we shall prove the following:

THEOREM 6.1. *Let* $\eta(t)$, *in addition to the property* $\eta(t)\downarrow 0$ *as* $t\downarrow 0$, *be such that*

$$\int_1^u \eta(1/x)dx = O(u\eta(1/u)) \quad as \quad u\to\infty , \tag{6.1}$$

while

$$\sum_{n=1}^{\infty} n^{-1/p'}\eta(1/n) = \infty , \qquad 1\leqq p\leqq 2. \tag{6.2}$$

Then there exists a function $f(\theta)\in L^p$ *such that*

$$\omega_p^{(m)}(f;\delta)<\eta(\delta) \text{ but } \sum_{n=0}^{\infty}|c_n|=\infty .$$

We introduce a sequence of polynomials

$$g_n(z) = 2^{-n(1+1/p')}\left(\sum_{\nu=0}^{2^n} z^\nu\right)^2, \qquad n=1, 2, \cdots , \tag{6.3}$$

a sequence of numbers $\epsilon_n=\eta(2^{-n})$ and a sequence of positive integers $\{\lambda_n\}$ such that

$$\lambda_0=0, \quad \lambda_n=2^{n+1}+n-2, \quad \lambda_{n+1}>\lambda_n+2^{n+1}, \qquad n=0, 1, \cdots . \tag{6.4}$$

Construct the power series

$$G(z) = \sum_{\nu=1}^{\infty}(\epsilon_\nu-\epsilon_{\nu+1})g_\nu(z)z^{\lambda_\nu} \equiv \sum_{\nu=0}^{\infty}\gamma_\nu z^\nu . \tag{6.5}$$

We shall establish the following properties of $G(z)$.

(i) $\gamma_n>0$, $n=0, 1, 2, \cdots$ and $\sum_{n=0}^{\infty}\gamma_n=\infty$.

(ii) $G(z)$ converges for $|z|<1$, and as $r\to 1$, $G(re^{i\theta})$ converges for almost all θ to a limit function

$$G(e^{i\theta}) \equiv g(\theta)\epsilon L^p, \qquad g(\theta)\sim\sum_{n=0}^{\infty}\gamma_n e^{in\theta}.$$

(iii) $\omega_p^{(1)}(g;\delta)=O(\eta(\delta))$ as $\delta\to 0$.

These properties clearly imply the truth of Theorem 6.1 if one observes that $\omega_p^{(m)}(g;\delta)=O(\omega_p^{(1)}(g;\delta))$ as $\delta\to 0$. The fact that $\gamma_n>0$ follows immediately from the positiveness of the coefficients of $g_n(z)$. Furthermore

$$\sum_{\nu=0}^{\infty}\gamma_\nu > \sum_{\nu=0}^{\infty}(\epsilon_\nu-\epsilon_{\nu+1})(1+2+3+\cdots+2^\nu)2^{-\nu(1+1/p')}$$

$$> \frac{1}{2}\sum_{\nu=0}^{\infty}(\epsilon_\nu-\epsilon_{\nu+1})2^{\nu(1-1/p')}.$$

But

$$(6.6)\qquad \sum_{\nu=1}^{n}(\epsilon_\nu-\epsilon_{\nu+1})2^{\nu(1-1/p')} = \sum_{\nu=1}^{n}\epsilon_\nu\beta_\nu - \epsilon_{n+1}2^{n(1-1/p')} = \sum_{\nu=1}^{n}(\epsilon_\nu-\epsilon_{n+1})\beta_\nu,$$

where

$$\beta_1 = 2^{1-1/p'}, \qquad \beta_\nu = 2^{\nu(1-1/p')} - 2^{(\nu-1)(1-1/p')}, \qquad \nu = 2, 3, \cdots, n,$$

and

$$\sum_{\nu=1}^{n}\beta_\nu = 2^{n(1-1/p')}.$$

By Lemma 2.2, (6.2) is equivalent to the divergence of the series $\sum_{\nu=1}^{\infty}\epsilon_\nu 2^{\nu(1-1/p')}$, so that the series

$$\sum_{\nu=2}^{\infty}\epsilon_\nu\beta_\nu = (1-2^{-1+1/p'})\sum_{\nu=2}^{\infty}\epsilon_\nu 2^{\nu(1-1/p')} = \infty.$$

By Lemma 6.1, $\sum_{\nu=1}^{n}(\epsilon_\nu-\epsilon_{n+1})\beta_\nu\to\infty$ as $n\to\infty$, whence $\sum_{n=0}^{\infty}\gamma_n=\infty$, and property (i) is proved.

To prove (ii) denote by $u_n(z)$ the general term of the sum (6.5) defining $G(z)$,

$$u_n(z) = (\epsilon_n-\epsilon_{n+1})g_n(z)z^{\lambda_n}.$$

We have

$$M_p(u_n(e^{i\theta})) = (\epsilon_n - \epsilon_{n+1})2^{-n(1+1/p')}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{\nu=0}^{2^n} e^{i\nu\theta}\right|^{2p} d\theta\right)^{1/p}$$

$$= (\epsilon_n - \epsilon_{n+1})2^{-n(1+1/p')}\left(\frac{1}{\pi}\int_0^{\pi}\left|\frac{\sin(2^n+1)\theta/2}{\sin\theta/2}\right|^{2p} d\theta\right)^{1/p}$$

$$< (\epsilon_n - \epsilon_{n+1})2^{-n(1+1/p')}\pi^{2-1/p}\left(\int_0^{\pi}\left|\frac{\sin(2^n+1)\theta/2}{\theta}\right|^{2p} d\theta\right)^{1/p}$$

$$< (\epsilon_n - \epsilon_{n+1})2^{-n(1+1/p')}\pi^2\left(\int_0^{\infty}\left|\frac{\sin x}{x}\right|^{2p} dx\right)^{1/p}\left(\frac{1+2^n}{2}\right)^{2-1/p}.$$

Hence

$$M_p(u_n(z)) \leqq M_p(u_n(e^{i\theta})) < A(\epsilon_n - \epsilon_{n+1}), \qquad z = re^{i\theta}, \tag{6.7}$$

where

$$A = \pi^2\left(\int_0^{\infty}\left|\frac{\sin x}{x}\right|^{2p} dx\right)^{1/p} \leqq \pi^2\left(\int_0^{\infty}\left(\frac{\sin x}{x}\right)^4 dx\right)^{1/2}.$$

Thus

$$|u_n(z)| < B(\epsilon_n - \epsilon_{n+1})(1-r)^{-1/p},$$

where B is a constant which does not depend on n and the series defining $G(z)$ converges in $|z|<1$. Moreover, by the Minkowski inequality,

$$M_p(G(z)) \leqq \sum_{n=1}^{\infty} M_p(u_n(z)) \leqq A\epsilon_1.$$

This ensures the properties of $G(z)$ stated in (ii) (cf. F. Riesz [10]).

We now pass on to estimating $M_p(\Delta_1 G(z;t))$. On putting $\zeta = e^{it}$ we have for $r<1$,

$$\Delta_1 G(z;t) = G(re^{i(\theta+t)}) - G(re^{i(\theta-t)}) = \sum_{\nu=1}^{\infty}\{u_\nu(z\zeta) - u_\nu(z\bar{\zeta})\} = \sum_{\nu=1}^{\infty}\Delta_1 u_\nu.$$

Hence

$$M_p(\Delta_1 G(z;t)) \leqq \sum_{\nu=1}^{\infty} M_p(\Delta_1 u_\nu) = \sum_{\nu=1}^{n} + \sum_{\nu=n+1}^{\infty} \equiv S_1 + S_2. \tag{6.8}$$

From (6.7) we get immediately

$$S_2 \leqq 2A\epsilon_{n+1}.$$

The general term of S_1 can be estimated by means of Corollary 5.2 and (6.7), with the result

$$S_1 \leqq 2At \sum_{\nu=1}^{n} (2^{\nu+1} + \lambda_\nu)(\epsilon_\nu - \epsilon_{\nu+1}) \leqq 10At \sum_{\nu=1}^{n} 2^\nu(\epsilon_\nu - \epsilon_{\nu+1})$$

$$\leqq 10At \sum_{\nu=1}^{n} (2^{\nu+1} - 2^\nu)\epsilon_\nu \leqq 20At \int_1^{2^n} \eta(1/x)dx = O(t2^n\eta(2^{-n})),$$

by (6.1). Now assume $0<t<1$. If we choose n so that $2^{n-1}<t^{-1}\leqq 2^n$, $n\geqq 1$, then

$$S_1 = O(\eta(t)), \qquad S_2 = O(\eta(t)),$$

whence

$$\omega_p^{(1)}(g; \delta) = O(\eta(\delta)),$$

and the proof of Theorem 6.1 is complete.

In the case $p=2$ it is possible to extend the result of Theorem 6.1 to hold for any $k>0$ as is shown by

THEOREM 6.2. *Let $\eta(t)$, in addition to the property $\eta(t) \downarrow 0$ as $t \downarrow 0$ be such that*

$$\int_1^u x\eta(1/x)^2dx = O(u^2\eta(1/u)^2) \text{ as } u \to \infty, \tag{6.9}$$

while, for a given k, $0<k<2$,

$$\sum_{n=1}^{\infty} n^{-k/2}\eta(1/n)^k = \infty. \tag{6.10}$$

Then there exists a function $f(\theta)\epsilon L^2$ such that

$$\omega_2^{(m)}(f; \delta) < \eta(\delta) \text{ but } \sum_{n=0}^{\infty} |c_n|^k = \infty.$$

The proof in many respects is similar to that of Theorem 6.1, so that we shall discuss in detail only the points where these two proofs differ from each other. We introduce the same sequence of polynomials $g_n(z)$ with p' replaced by 2,

$$g_n(z) = 2^{-3n/2}\left(\sum_{\nu=0}^{2^n} z^\nu\right)^2, \qquad n = 1, 2, \cdots,$$

and changing slightly the notation of Theorem 6.1, construct the power series

$$G(z) = \sum_{\nu=1}^{\infty} \alpha_\nu g_\nu(z)z^{\lambda_\nu} \equiv \sum_{\nu=0}^{\infty} \gamma_\nu z^\nu, \qquad \alpha_\nu = (\epsilon_\nu^k - \epsilon_{\nu+1}^k)^{1/k}, \tag{6.11}$$

which reduces to the power series $G(z)$ of Theorem 6.1 for $k=1$. We shall

establish the same properties (ii), (iii) which were proved for the $G(z)$ of Theorem 6.1, with p replaced by 2 and the property

$$(\mathrm{I}, k) \qquad \gamma_n > 0, \qquad \sum_{n=0}^{\infty} \gamma_n^k = \infty,$$

instead of (i). As in the case of Theorem 6.1, Theorem 6.2 will follow from these properties.

To prove (I, k) we note again that $\gamma_n > 0$, $n = 0, 1, 2, \cdots$, while

$$\sum_{\nu=0}^{\infty} \gamma_\nu^k > \sum_{\nu=1}^{\infty} \alpha_\nu^k (1 + 2^k + 3^k + \cdots + 2^{\nu k}) 2^{-3\nu k/2} > (k+1)^{-1} \sum_{\nu=1}^{n} \alpha_\nu^k 2^{\nu(1-k/2)}.$$

Property (I, k) will be proved if we show that

$$(6.12) \qquad \sum_{\nu=1}^{\infty} \alpha_\nu^k 2^{\nu(1-k/2)} = \sum_{\nu=1}^{\infty} (\epsilon_\nu^k - \epsilon_{\nu+1}^k) 2^{\nu(1-k/2)} = \infty.$$

Since $\alpha_\nu^k = \epsilon_\nu^k - \epsilon_{\nu+1}^k$ this can be done in the same manner as in the proof of Theorem 6.1, if we replace there η by η^k, ϵ_ν by ϵ_ν^k, p' by 2, and β_ν by $2^{\nu(1-k/2)} - 2^{(\nu-1)(1-k/2)}$ for $\nu > 1$, and by $2^{1-k/2}$ for $\nu = 1$.

To prove property (ii), with p replaced by 2, we put

$$u_n(z) = \alpha_n g_n(z) z^{\lambda_n}$$

and prove, in the same manner as in the proof of Theorem 6.1, that

$$(6.13) \qquad M_2(u_n(z))^2 < A^2 \alpha_n^2.$$

Since

$$(6.14) \qquad \begin{aligned} \alpha_n^2 = (\epsilon_n^k - \epsilon_{n+1}^k)^{2/k} &= (\epsilon_n^k - \epsilon_{n+1}^k)(\epsilon_n^k - \epsilon_{n+1}^k)^{2/k-1} \\ &< (\epsilon_n^k - \epsilon_{n+1}^k)\epsilon_n^{2-k} < \epsilon_n^2 - \epsilon_{n+1}^2, \end{aligned}$$

we have $\sum_{\nu=1}^{\infty} \alpha_\nu^2 < \infty$, and the remaining argument will be the same as in Theorem 6.1.

Finally, to prove property (iii) we now have

$$M_2^2(\Delta_1 G(z; t)) = \sum_{\nu=1}^{\infty} M_2^2(\Delta_1(u_\nu)) = \sum_{\nu=1}^{n} + \sum_{\nu=n+1}^{\infty} \equiv S_1^{(2)} + S_2^{(2)}.$$

In view of (6.13) and (6.14) we have

$$S_2^{(2)} = O(\epsilon_{n+1}^2),$$

while

$$S_1^{(2)} = t^2 O\left(\sum_{\nu=1}^{n} (\epsilon_\nu^2 - \epsilon_{\nu+1}^2) 2^{2\nu} \right) = t^2 O\left(\sum_{\nu=1}^{n} \epsilon_\nu^2 (2^{2\nu} - 2^{2\nu-2}) \right)$$

$$= t^2 O\left(\sum_{\nu=1}^{n} 2^\nu \epsilon_j^2 (2^\nu - 2^{\nu-1})\right) = t^2 O\left(\int_1^{2^n} x\eta(1/x)^2 dx\right).$$

On assuming $0<t<1/2$ and choosing n so that

$$2^n < t^{-1} \leqq 2^{n+1}, \qquad n \geqq 1,$$

we now have, in view of (6.9),

$$S_1^{(2)} = O\left(t^2 \int_1^{t^{-1}} x\eta(1/x)^2 dx\right) = O(\eta(t)^2),$$

$$S_2^{(2)} = O(\eta(t)^2),$$

from which property (iii) is readily obtained.

REMARK 6.1. *In view of Lemma* 2.6 *the conditions* (6.1) *of Theorem* 6.1 *and* (6.9) *of Theorem* 6.2 *are equivalent.*

In the case $0<k\leqq 1$ the result of Theorem 6.2 can be improved as shows the following:

THEOREM 6.3. *If conditions of Theorem* 6.2 *are satisfied with* $0<k\leqq 1$, *then there exists a continuous function* $f(\theta)$ *such that* $\xi(f;\delta)<\eta(\delta)$ *but* $\sum_{\nu=0}^{\infty}|c_\nu|^k=\infty$. *The same result holds even when* $1<k<2$, *provided we make additional assumptions*

$$\int_1^\infty x^{-1}\eta(1/x)dx = \int_0^1 \tau^{-1}\eta(\tau)d\tau < \infty \tag{6.15}$$

and

$$\int_{t-1}^\infty x^{-1}\eta(1/x)dx = \int_0^t \tau^{-1}\eta(\tau)d\tau = O(\eta(t)) \text{ as } t\to 0. \tag{6.16}$$

The proof is based on a result of one of our earlier papers [12]. We introduce a sequence of primes $\{q_\nu\}$, $q_\nu\equiv 1 \pmod 4$ and such that for ν sufficiently large, $\nu>\nu_0$ say, each interval $(2^{\nu-1}, 2^\nu)$ would contain just one q_ν. We set

$$n_\nu = q_1 + q_2 + \cdots + q_\nu \tag{6.17}$$

and observe that

$$n_\nu = O(2^\nu) = O(q_\nu) \text{ as } \nu\to\infty. \tag{6.18}$$

For each $q_\nu\equiv q$ we put

$$a_j^{(q)} = 2q^{-3/2}(q-j)\left(\frac{j}{q}\right), \qquad j = 1, 2, \cdots, q-1,$$

where (j/q) is the Legendre symbol, and consider the polynomial

$$g_\nu(z) = \tfrac{1}{2}(a_{q-1}^{(q)} + a_{q-2}^{(q)} z + \cdots + a_1^{(q)} z^{q-2} + a_1^{(q)} z^{q} + \cdots + a_{q-1}^{(q)} z^{2(q-1)}).$$

It is known (cf. [12]) that

$$|g_\nu(z)| \leqq 1 \quad \text{for} \quad |z| \leqq 1.$$

Finally, let

$$\begin{aligned} u_\nu(z) &= \alpha_\nu q_\nu^{-1/2} z^{2n^{\nu-1}-\nu+1} g_\nu(z), \\ \alpha_\nu &= (\epsilon_\nu^k - \epsilon_{\nu+1}^k)^{1/k} 2^{\nu/2}, \qquad \epsilon_\nu = \eta(2^{-\nu}). \end{aligned} \tag{6.19}$$

Note that for $0<k\leqq 1$,

$$(\epsilon_\nu^k - \epsilon_{\nu+1}^k)^{1/k} = (\epsilon_\nu^k - \epsilon_{\nu+1}^k)(\epsilon_\nu^k - \epsilon_{\nu+1}^k)^{1/k-1} \leqq (\epsilon_\nu^k - \epsilon_{\nu+1}^k)\epsilon_\nu^{1-k} \leqq \epsilon_\nu - \epsilon_{\nu+1},$$

and for $1<k<2$,

$$(\epsilon_\nu^k - \epsilon_{\nu+1}^k)^{1/k} \leqq \epsilon_\nu.$$

Thus

$$\begin{aligned} \alpha_\nu &\leqq 2^{\nu/2}(\epsilon_\nu - \epsilon_{\nu+1}), \qquad 0 < k \leqq 1, \\ \alpha_\nu &\leqq 2^{\nu/2}\epsilon_\nu, \qquad 1 < k < 2, \end{aligned}$$

whence

$$\sum_{\nu=1}^{n} 2^{\nu/2}\alpha_\nu \leqq \begin{cases} \displaystyle\sum_{\nu=1}^{n} 2^\nu(\epsilon_\nu - \epsilon_{\nu+1}) \leqq \sum_{\nu=1}^{n} (2^{\nu+1} - 2^\nu)\epsilon_\nu \leqq 2\int_1^{2^n} \eta(1/x)dx, & 0 < k \leqq 1, \\ \displaystyle\sum_{\nu=1}^{n} 2^\nu\epsilon_\nu = \sum_{\nu=1}^{n} (2^{\nu+1} - 2^\nu)\epsilon_\nu \leqq 2\int_1^{2^n} \eta(1/x)dx, & 1 < k < 2. \end{cases} \tag{6.20}$$

Similarly, for $0<k\leqq 1$,

$$\sum_{\nu=n+1}^{\infty} 2^{-\nu/2}\alpha_\nu \leqq \sum_{\nu=n+1}^{\infty} (\epsilon_\nu - \epsilon_{\nu+1}) = \epsilon_{n+1}, \tag{6.21}$$

while for $1<k<2$,

$$\sum_{\nu=n+1}^{\infty} 2^{-\nu/2}\alpha_\nu \leqq \sum_{\nu=n+1}^{\infty} \epsilon_\nu = 2\sum_{\nu=n+1}^{\infty} (2^\nu - 2^{\nu-1})2^{-\nu}\epsilon_\nu < 2\int_{2^n}^{\infty} x^{-1}\eta(1/x)dx. \tag{6.22}$$

Since $1/q_\nu < 2^{-\nu+1}$ for $\nu > \nu_0$, these formulas show that under assumptions of our theorem,

$$\sum_{\nu=1}^{\infty} \alpha_\nu q_\nu^{-1/2} < \infty, \qquad 0 < k < 2. \tag{6.23}$$

On the other hand for all k, $0<k<2$ we have by (6.19), (6.20)

$$\sum_{\nu=1}^{\infty} \alpha_\nu^k q_\nu^{1-k} > \frac{1}{2} \sum_{\nu=1}^{\infty} (\epsilon_\nu^k - \epsilon_{\nu+1}^k) 2^{\nu(1-k/2)} = \infty . \tag{6.24}$$

Now construct the power series

$$G(z) = \sum_{n=1}^{\infty} u_n(z) \equiv \sum_{\nu=1}^{\infty} \gamma_\nu z^\nu . \tag{6.25}$$

Since

$$| u_n(z) | \leqq \alpha_n q_n^{-1/2}, \tag{6.26}$$

formula (6.23) shows that $G(z)$ converges uniformly in $|z| \leqq 1$, hence is continuous in the closed unit circle, so that $g(\theta) = G(e^{i\theta})$ is continuous. On the other hand γ_ν are the Fourier coefficients of $G(\theta)$ and we have

$$\sum_{\nu=1}^{\infty} |\gamma_\nu|^k > \sum_{\nu=1}^{\infty} \alpha_\nu^k q_\nu^{-2k}(1 + 2^k + 3^k + \cdots + (q_\nu - 1)^k)$$

$$> (k+1)^{-1} \sum_{\nu=1}^{\infty} \alpha_\nu^k q_\nu^{-2k}(q_\nu - 1)^{k+1} > 2^{-k-1}(k+1)^{-1} \sum_{\nu=1}^{\infty} \alpha_\nu^k q_\nu^{1-k} = \infty .$$

To complete the proof of Theorem 6.3 it remains to estimate the modulus of continuity $\xi(g; \delta)$ of $g(\theta)$. We have to estimate

$$| g(\theta + t) - g(\theta) | \leqq \sum_{\nu=1}^{\infty} | u_\nu(e^{i(\theta+t)}) - u_\nu(e^{i\theta}) | = \sum_{\nu=1}^{n} + \sum_{\nu=n+1}^{\infty} \equiv S_1^{(0)} + S_2^{(0)} .$$

In view of (6.26), (6.21), and (6.22) we see at once that

$$S_2^{(0)} \leqq 2 \sum_{\nu=n+1}^{\infty} \alpha_\nu q_\nu^{-1/2} = O\left(\sum_{\nu=n+1}^{\infty} \alpha_\nu 2^{-\nu/2} \right),$$

whence

$$S_2^{(0)} = O(\epsilon_{n+1}) = O(\eta(2^{-n})) \quad \text{if} \quad 0 < k \leqq 1,$$

$$S_2^{(0)} = O\left(\int_{2^n}^{\infty} x^{-1} \eta(1/x) dx \right) = O(\eta(2^{-n})), \quad \text{if} \quad 1 < k < 2.$$

To estimate $S_1^{(0)}$ we use Corollary 5.2, (6.26), and (6.20) with the result

$$S_1^{(0)} = tO\left(\sum_{\nu=1}^{n} \alpha_\nu 2^{\nu/2} \right) = tO\left(\int_1^{2^n} \eta(1/x) dx \right), \qquad 0 < k < 2.$$

Now assume $0<t<1$ and choose n so that $2^{n-1}<1/t \leqq 2^n$. Using (6.1) which

is equivalent to assumption (6.9) of Theorem 6.2, and thus must be satisfied also in the case of Theorem 6.3, we have

$$S_1^{(0)} = O(\eta(t)), \qquad S_2^{(0)} = O(\eta(t)) \text{ as } t \to 0,$$

and finally

$$\xi(g; t) = O(\eta(t)).$$

On multiplying $g(\theta)$ by a suitable constant factor we shall obtain a function $f(\theta)$ the existence of which is asserted in Theorem 6.3.

As an illustration of the preceding theory consider the function $\eta(t) = t^\alpha(\log 1/t)^\beta$, $0<\alpha\leqq 1$, β arbitrary real. Since

$$\frac{d}{dt}(\eta(t)t^{-\rho}) = t^{\alpha-\rho-1}(\log 1/t)^{\beta-1}\{(\alpha-\rho)\log(1/t) - \beta\},$$

$\eta(t)t^{-\rho} \downarrow 0$ as $t \downarrow 0$ if $0<\rho<\alpha$, and by remarks 2.1 and 2.2, $\eta(t)$ satisfies conditions (6.1), (6.15), and (6.16). Now

$$\sum_{n=2}^{\infty} n^{-k/2}\eta(1/n)^k = \sum_{n=2}^{\infty} n^{-k/2-\alpha k}(\log n)^{\beta k} < \infty$$

if either $k>2/(2\alpha+1)$ and β arbitrary, or $k=2/(2\alpha+1)$ and $\beta<-1/k = -(2\alpha+1)/2$. In this case, according to Theorem 3.1, condition $\omega_2^{(1)}(f; \delta) = O(\eta(\delta))$ implies $\sum_n |c_n|^k < \infty$.

If however $\beta = -1/k = -(2\alpha+1)/2$, then the power series (6.25) for $z=e^{i\theta}$ represents a continuous function $g(\theta)$ for which

$$\xi(g; \delta) = O(\delta^\alpha \log(1/\delta)^{-\alpha-1/2}) \text{ as } \delta \to 0,$$

but

$$\sum_n |\gamma_n|^{2/(2\alpha+1)} = \infty.$$

Bibliography

1. S. Bernstein. *Sur la convergence absolue des séries trigonométriques*. Comptes Rendus des Séances de l'Académie des Sciences, Paris, vol. 158 (1914), pp. 1661–1663.
2. ———. Same title. Communications de la Société Mathématique de Kharkoff, vol. 14 (1914), pp. 139–144 (in Russian).
3. ———. Same title. Comptes Rendus, vol. 199 (1934), pp. 397–400.
4. G. H. Hardy and J. E. Littlewood. *Some properties of fractional integrals*. I. Mathematische Zeitschrift, vol. 27 (1928), pp. 565–606. II. Ibidem, vol. 34 (1932), pp. 403–439.
5. ———. *A convergence criterion for Fourier series*. Ibidem, vol. 28 (1928), pp. 612–634.
6. ———. *Notes on the theory of series*. IX. *On the absolute convergence of Fourier series*. Journal of the London Mathematical Society, vol. 3 (1928), pp. 250–253.
7. G. H. Hardy, J. E. Littlewood, and G. Pólya. *Inequalities*. Cambridge, 1934.
8. E. Hille and J. D. Tamarkin. *On the summability of Fourier series*. III. Mathematische Annalen, vol. 108 (1933), pp. 525–576.

9. K. NABETANI. *Remarks on some theorems concerning sections of a power series.* Tôhoku Mathematical Journal, vol. 41 (1935–36), pp. 333–336.
10. F. RIESZ. *Über die Randwerte einer analytischen Funktion.* Mathematische Zeitschrift, vol. 18 (1923), pp. 87–95.
11. M. RIESZ. *Eine trigonometrische Interpolationsformel und einige Ungleichungen für Polynome.* Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), pp. 354–368.
12. O. SZÁSZ. *Über den Konvergenzexponenten der Fourierschen Reihen gewisser Funktionenklassen.* Sitzungsberichte der Bayerischen Akademie der Wissenschaften, München, 1922, pp. 135–150.
13. ———. *Korlátos hatványsorokról,* Matematikai és Természettudomanyi Ertesitö, vol. 43 (1926), pp. 504–520.
14. ———. *Über die Fourierschen Reihen gewisser Funktionenklassen.* Mathematische Annalen, vol. 100 (1928), pp. 530–536.
15. E. C. TITCHMARSH. *A note on Fourier transforms.* Journal of the London Mathematical Society, vol. 2 (1927), pp. 148–150.
16. Z. WARASZKIEWICZ. *Sur un théorème de M. Zygmund.* Bulletin International de l'Académie Polonaise, Classe A, Cracovie, 1929, pp. 275–279.
17. A. ZYGMUND. *A remark on conjugate series.* Proceedings of the London Mathematical Society, (2), vol. 34 (1932), pp. 392–400.
18. ———. *Some points in the theory of trigonometric and power series.* These Transactions, vol. 36 (1934), pp. 586–617.
19. ———. *Trigonometrical Series.* Warsaw-Lwów, 1935.

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THE CHARACTERIZATION OF THE CLOSED n-CELL*

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1. **Introduction.** Various characterizations of the closed 1-cell and of the closed 2-cell have been given. But, with the exception of a paper by Alexandroff,† there is no record of previous attempts to give a characterization of the closed n-cell which is uniformly valid for all values of $n>0$.

Characteristic of the present work are (1) the use of the notion of *strong homeomorphism* (§2) by means of which is defined a very useful concept, that of *the descendant of a set*, and (2) the emphasis placed upon an essential property of the closed n-cell as given in Corollary, Theorem P_1 (§3).

In Theorem I (§4) there is presented a characterization of the closed n-cell without reference to the euclidean spaces. The definition of the closed n-cell implied by this theorem, although given by means of recursive statements, is essentially set-theoretic in character. The words and symbols constituting this theorem may be regarded as defining a function of n, $F(n)$, such that if k is a positive integer, $F(k)$ is a closed k-cell. The space $F(n)$ is defined in terms of certain of its subsets as given by $F(n-1)$. By definition, $F(0)$ consists of a single point.

The proof of Theorem I is based upon Theorem I′ (§4). This latter theorem gives a characterization of the closed n-cell in terms of the closed $(n-1)$-cell.

2. **Definitions.** The set $\overline{M}-M$, where $\overline{M}$ designates the closure of the set M, is called the λ-set of M and is denoted by $\lambda[M]$.

A set M_1 is said to be *strongly homeomorphic* with a set M_2 provided there exists a homeomorphism, $H(\overline{M}_1)=\overline{M}_2$, of such nature that $H(M_1)=M_2$.‡

Let P be a non-vacuous subset of a set M such that $M-P=M_1+M_2$, $\overline{M}_1\cdot\overline{M}_2=\overline{P}$ and M_i, $i=1, 2$, is strongly homeomorphic with M. Each of the sets M_i is called a *proper descendant* of M. This relation is expressed symbolically: $M_i=D_i^P(M)$. The set P is said to generate the descendants. The

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† P. Alexandroff, *Zur Begründung der n-dimensionalen mengentheoretischen Topologie*, Mathematische Annalen, vol. 94, p. 296.

‡ The use of the expression "strongly homeomorphic" in this connection was suggested to the author by Professor J. R. Kline.

complex whose elements are the descendants of M generated by P is represented by $\Delta[P]M$.

$$\Delta[P]M = (D_1^P(M), D_2^P(M)).$$

A set P is said to generate *improper descendants* for a set M as follows:

(1) $P \neq 0$, $M \neq 0$, but P does not generate proper descendants for M: $D_i^P(M) = 0$, $i = 1, 2$; $\Delta[P]M = (0, 0)$.

(2) $P = 0$: $D_i^P(M) = M$, $i = 1, 2$; $\Delta[P]M = (M, M)$.

(3) $M = 0$: $D_i^P(M) = 0$, $i = 1, 2$; $\Delta[P]M = (0, 0)$.

We define two additional complexes:

$$\Delta[P](M_1, M_2, \cdots, M_n) = (D_1^P(M_1), D_2^P(M_1), \cdots, D_1^P(M_n), D_2^P(M_n)),$$

$$\overset{n}{\underset{i=1}{\Delta}}[P_i]M = \Delta[P_1]\left(\overset{n}{\underset{i=2}{\Delta}}[P_i]M\right).$$

That the complex $\Delta_{i=1}^n[P_i]M$ contains at least one non-vacuous element is indicated as follows: $\Delta_{i=1}^n[P_i]M \neq 0$. It is to be noted that, if $P_i = 0$ for every value of i, every element of $\Delta_{i=1}^n[P_i]M$ is identical with M.

It will be of advantage to extend the notion of a descendant of a set and speak of every descendant, proper or improper, of a descendant of a set M as a descendant of M. When there is no doubt as to the identity of the set which generates a given descendant, the symbol for the generating set will be omitted. Frequently multiple subscripts will be used in denoting descendants, as in $D_{12}(M)$. In such cases the meaning will be clear from the context.

An n-cell, $n > 0$, is a subset of a space S which is strongly homeomorphic with the set in euclidean n-space which is the interior of the $(n-1)$-sphere whose equation is $\sum_{i=1}^n x_i^2 = 1$. A 0-cell is a set consisting of a single point.

3. **Preliminary Theorems.** We prove first

THEOREM P_1. *Let C^n and K^n be two n-cells, $n > 0$. If there is a homeomorphism, $H_1(\lambda[C^n]) = \lambda[K^n]$, there exists a homeomorphism, $H_2(\overline{C^n}) = \overline{K^n}$, such that $H_2(\lambda[C^n]) = H_1(\lambda[C^n])$.*

In view of the definition of the n-cell (§2), K^n may be taken to be the set in euclidean n-space which is the interior of the $(n-1)$-sphere whose equation is $\sum_{i=1}^n x_i^2 = 1$. Since C^n is strongly homeomorphic with K^n, there is a homeomorphism, $H_a(\overline{C^n}) = \overline{K^n}$, such that $H_a(\lambda[C^n]) = \lambda[K^n]$. Denote by O the point $(0, 0, \cdots, 0)$ in euclidean n-space. Let p be any point of $\lambda[C^n]$, $H_1(p) = q$ and $H_a(p) = q'$. Let Oq and Oq' be the straight line intervals in $\overline{K^n}$ joining O to q and q' respectively. Make the points of Oq and Oq' to correspond in such a manner that a point q_1 of Oq corresponds to a point q_1' of Oq' if, and only if, $d(O, q_1) = d(O, q_1')$.*

* If p and q are two points, the symbol $d(p, q)$ is used to designate the distance from p to q.

Since p is any point of $\lambda[C^n]$, this procedure results in a transformation of $\overline{K^n}$ into itself which is a homeomorphism. If $H_\beta(\overline{K^n})=\overline{K^n}$ is this transformation, $H_\beta(q_1)=q_1'$. Let H_2 denote the transformation $H_{\bar\beta}^{-1}H_\alpha$. Then H_2 is a homeomorphism. $H_2(\overline{C^n})=\overline{K^n}$. Since $H_2(p)=H_{\bar\beta}^{-1}H_\alpha(p)=H_{\bar\beta}^{-1}(q')=q$, $H_2(\lambda[C^n])=H_1(\lambda[C^n])$.

COROLLARY. *If C^n is an n-cell and if there is a homeomorphism, $H_1(\lambda[C^n])=\lambda[C^n]$, there exists a homeomorphism, $H_2(\overline{C^n})=\overline{C^n}$, such that $H_2(\lambda[C^n])=H_1(\lambda[C^n])$.*

THEOREM P_2. *Let S_1^n and S_2 be two topological n-spheres such that $S_i^n=\overline{C_{i1}^n}+\overline{C_{i2}^n}$, $\overline{C_{i1}^n}\cdot\overline{C_{i2}^n}=\lambda[C_{i1}^n]=\lambda[C_{i2}^n]$, $i=1, 2$, and C_{ij}^n is an n-cell. Then there exists a homeomorphism, $H(S_1^n)=S_2^n$, such that $H(\overline{C_{1i}^n})=\overline{C_{2i}^n}$, $i=1, 2$.*

C_{11}^n is strongly homeomorphic with C_{21}^n. There is a homeomorphism $H_1(\overline{C_{11}^n})=\overline{C_{21}^n}$, such that $H_1(\lambda[C_{11}^n])=\lambda[C_{21}^n]$. But $\lambda[C_{i1}^n]=\lambda[C_{i2}^n]$, $i=1, 2$. Then $H_1(\lambda[C_{11}^n])=H_1(\lambda[C_{12}^n])=\lambda[C_{22}^n]$. By Theorem P_1 there is a homeomorphism, $H_2(\overline{C_{12}^n})=\overline{C_{22}^n}$ such that $H_2(\lambda[C_{12}^n])=H_1(\lambda[C_{12}^n])$. The existence of the required transformation is evident.

COROLLARY. *If S^n is an n-sphere, $S^n=\overline{C_{11}^n}+\overline{C_{12}^n}=\overline{C_{21}^n}+\overline{C_{22}^n}$ and $\overline{C_{i1}^n}\cdot\overline{C_{i2}^n}=\lambda[C_{i1}^n]=\lambda[C_{i2}^n]$, $i=1, 2$, there exists a homeomorphism, $H(S^n)=S^n$, such that $H(\overline{C_{1i}^n})=\overline{C_{2i}^n}$, $i=1, 2$.*

THEOREM P_3. *If C^n is an n-cell, $\lambda[C^n]=\overline{C_1^{n-1}}+\overline{C_2^{n-1}}$ and $\overline{C_1^{n-1}}\cdot\overline{C_2^{n-1}}=\lambda[C_i^{n-1}]$, $i=1, 2$, there exists an $(n-1)$-cell C_3^{n-1} such that $\lambda[C_3^{n-1}]=\lambda[C_i^{n-1}]$, $i=1, 2$, and $\Delta[C_3^{n-1}]\,C^n\neq 0$.*

Let K^n be the same subset of euclidean n-space as in the proof of Theorem P_1. The $(n-1)$-dimensional plane $x_1=0$ has in common with K^n the $(n-1)$-cell K_3^{n-1}. The set $\lambda[K^n]$ is the sum of two closed $(n-1)$-cells, $\overline{K_1^{n-1}}$ and $\overline{K_2^{n-1}}$, such that $\overline{K_1^{n-1}}\cdot\overline{K_2^{n-1}}=\lambda[K_i^{n-1}]$, $i=1, 2, 3$. By Theorem P_2, there is a homeomorphism, $H_1(\lambda[C^n])=\lambda[K^n]$, such that $H_1(\overline{C_i^{n-1}})=\overline{K_i^{n-1}}$, $i=1, 2$. By Theorem P_1 there is a homeomorphism, $H_2(\overline{C^n})=\overline{K^n}$, such that $H_2(\lambda[C^n])=H_1(\lambda[C^n])$. The set $H_2^{-1}[K_3^{n-1}]$ is an $(n-1)$-cell C_3^{n-1}. Obviously $\Delta[C_3^{n-1}]C^n\neq 0$. The set C_3^{n-1} separates C^n into two n-cells.

Remark. In the sequel the symbol C^k (or C_δ^k) is always to be understood as designating a k-cell.

4. **Principal theorems.** We state now our main theorems.

THEOREM I. *In order that a space Z^n be a closed n-cell, the following conditions are necessary and sufficient:*

(4.1) *Z^n is a connected and locally compact Hausdorff space* which is de-*

* A Hausdorff space is a space defined by a system of neighborhoods which satisfy the Hausdorff neighborhood axioms. See F. Hausdorff, *Grundzüge der Mengenlehre*, 1914, p. 213.

fined by a countable set of neighborhoods: $\{N_i\}$, $i=1, 2, 3, \cdots$.

(4.2) *Z^n contains a proper subset I^n such that* (i) $\overline{I^n}=Z^n$ *and* (ii) *if there is a homeomorphism, $H_1(\lambda[I^n])=\lambda[I^n]$, there exists a homeomorphism, $H_2(Z^n)=Z^n$, for which $H_2(\lambda[I^n])=H_1(\lambda[I^n])$.*

(4.3) *Let W_1 and W_2 be two sets such that $0\neq W_1\cdot W_2\not\supset W_j$, $j=1, 2$, W_1 is a neighborhood N_ξ and W_2 is an element of $\Delta_{i=1}^{a}[P_i]V$, where V is either I^n or a neighborhood belonging to I^n and $0\subseteq P_i\subset\lambda[N_{\alpha_i}]$, $N_{\alpha_i}\not\subset V$, $\xi\neq\alpha_i$. Then $\overline{W}_2\cdot\lambda[W_1]=\sum_{j=1}^{d}Z_j^{n-1}$, $\Delta_{j=1}^{d}[I_j^{n-1}]W_2\neq 0$, and, if K_{β_j} is a component of $\lambda[W_2]-\lambda[W_2]\cdot Z_j^{n-1}$, there exists $Z_{\beta_j}^{n-1}$ such that $K_{\beta_j}=I_{\beta_j}^{n-1}$.*

(4.4) *$Z^0(\equiv I^0)$ is a set consisting of a single point.*

THEOREM I′. *This theorem differs from Theorem* I *only in the following respects*: (1) *in Condition* (4.3), Z_j^{n-1}, I_j^{n-1}, $Z_{\beta_j}^{n-1}$, *and* $I_{\beta_j}^{n-1}$ *are replaced by* $\overline{C_j^{n-1}}$, C_j^{n-1}, $\overline{C_{\beta_j}^{n-1}}$, *and* $C_{\beta_j}^{n-1}$ *respectively, and* (2) *Condition* (4.4) *is omitted.*

5. **Proof that the conditions in Theorem I′ are sufficient.** We state first

5.1. LEMMA 1. *There exists a set G of compact neighborhoods which is a subset of the set of all neighborhoods in Z^n having the following properties*:

(a) *The set G is equivalent to the set of all neighborhoods.*

(b) *Corresponding to each point p, there exists a subset of G: $G(p)=\{N_{a_i}\}$, $i=1, 2, 3, \cdots$, such that $N_{a_i}\supset p$ and $N_{a_{i+1}}\subset N_{a_i}$.*

(c) *If $\{N_{b_i}\}$, $i=1, 2, 3, \cdots$, is a set of neighborhoods belonging to G for which $N_{b_{i+1}}\subset N_{b_i}$, then the set $\prod_{i=1}^{\infty}\overline{N}_{b_i}$ consists of a single point.**

Hereafter all neighborhoods mentioned will be members of the set G.

5.2. LEMMA 2. *The set $\lambda[I^n]$ is an $(n-1)$-sphere.*

There exists a neighborhood N_a such that $N_a\cdot\lambda[I^n]\neq 0$ and $N_a\not\supset I^n$. By Theorem I′ (4.3), $Z^n\cdot\lambda[N_a]=\sum_{j=1}^{d}\overline{C_j^{n-1}}$ and $\Delta[C_d^{n-1}]I^n\neq 0$.

$$I^n = D_1(I^n) + D_2(I^n) + C_d^{n-1};\ \overline{D_1(I^n)}\cdot\overline{D_2(I^n)}=\overline{C_d^{n-1}}.$$

The set $N_a\cdot\lambda[I^n]$ contains a point p which belongs to one and only one of the sets $\lambda[I^n]\cdot\lambda[D_i(I^n)]$, $i=1, 2$. Suppose that $\lambda[D_1(I^n)]\supset p$. There is a neighborhood N_b such that $N_b\supset p$ and $\overline{N}_b\cdot\overline{D^2(I^n)}=0$.† Then $\lambda[N_b]$ contains a closed $(n-1)$-cell $\overline{C_\alpha^{n-1}}$ such that $\Delta[C_\alpha^{n-1}]\,D_1(I^n)\neq 0$ (Theorem I′ (4.3)).

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* For the proof of this lemma, see I. Gawehn, *Über unberandete 2-dimensionale Mannigfaltigkeiten*, Mathematische Annalen, vol. 98, p. 339. An understanding of the method by which the sets $G(p)$ are obtained is assumed in 5.8 and 5.9.

† The fact that Z^n is a locally compact Hausdorff space assures us of the existence of a neighborhood N_b having the desired properties. In fact, the following proposition, of which we shall make frequent use, holds; *If F is a closed set and p, a point not belonging to F, there exists a neighborhood $N_b\supset p$ such that $\overline{N}_b\cdot F=0$.*

$$D_1(I^n) = D_{11}(I^n) + D_{12}(I^n) + C_\alpha^{n-1};\ \overline{D_{11}(I^n)} \cdot \overline{D_{12}(I^n)} = \overline{C_\alpha^{n-1}}.$$

Every point of $\lambda[D_1(I^n)]$ not belonging to $\overline{C_\alpha^{n-1}}$ belongs to one and only one of the sets $\lambda[D_{1i}(I^n)]$, $i=1, 2$. Since $\overline{C_d^{n-1}}$ is connected and $\overline{C_d^{n-1}} \cdot \overline{C_\alpha^{n-1}} = 0$, $\overline{C_d^{n-1}}$ belongs to one of the sets $\lambda[D_{1i}(I^n)]$. Suppose that $\overline{C_d^{n-1}} \subset \lambda[D_{12}(I^n)]$. Then $\overline{C_d^{n-1}} \cdot \lambda[D_{11}(I^n)] = 0$.

Case 1. $n<3$. Assume that, in this case, $\lambda[I^n] - \lambda[I^n] \cdot \overline{C_d^{n-1}}$ contains an infinite number of components. $\overline{C_d^{n-1}} \subset I^n$. Every component of $\lambda[I^n] - \lambda[I^n] \cdot \overline{C_d^{n-1}}$ is an $(n-1)$-cell (Theorem I′ (4.3)).

If $n=1$, $C_d{}^0$ consists of a single point and $\lambda[I^1] \cdot \overline{C_d{}^0} = 0$. Then $\lambda[I^1]$ is not a connected set and consists of infinitely many points.

Let $n=2$. The set $C_d{}^1$ is a 1-cell and $\lambda[C_d{}^1]$ consists of two points, g_1 and g_2. $\overline{C_d{}^1}$ belongs to a component $C_m{}^1$ of $\lambda[D_1(I^2)] - \lambda[D_1(I^2)] \cdot \overline{C_\alpha{}^1}$. The points g_1 and g_2 separate $C_m{}^1$ into three 1-cells. Of these three 1-cells, one is $C_d{}^1$ and each of the other two 1-cells is a subset of a component of $\lambda[I^2] - \lambda[I^2] \cdot \overline{C_d{}^1}$ which belongs to $\lambda[D_1(I^2)]$. By a similar process it can be proved that each of the points g_i belongs to the λ-set of one and only one component of $\lambda[I^2] - \lambda[I^2] \cdot \overline{C_d{}^1}$ which is contained in $\lambda[D_2(I^2)]$. Every component of $\lambda[I^2] - \lambda[I^2] \cdot \overline{C_d{}^1}$ belongs to one and only one of the sets $\lambda[D_i(I^2)]$. At least one of the sets $\lambda[D_i(I^2)]$ contains an infinite number of these components. Let $\lambda[D_1(I^2)]$ be such a set.

The set g_1+g_2 cannot belong to the λ-set of a single component of $\lambda[I^2] - \lambda[I^2] \cdot \overline{C_d{}^1}$ which belongs to $\lambda[D_1(I^2)]$. For, suppose that $C_e{}^1$ is a component of $\lambda[I^2] - \lambda[I^2] \cdot \overline{C_d{}^1}$ which belongs to $\lambda[D_1(I^2)]$ and whose λ-set is g_1+g_2. There exists N_f such that $N_f \cdot \lambda[D_1(I^2)] \neq 0$ and $\overline{N_f} \cdot (\overline{C_d{}^1} + C_e{}^1) = 0$. Then $\lambda[N_f]$ contains a 1-cell $C_x{}^1$ such that $\Delta[C_x{}^1] D_1(I^2) \neq 0$. $\overline{C_d{}^1} + C_e{}^1$ is a 1-sphere which belongs to a component of $\lambda[D_1(I^2)] - \lambda[D_1(I^2)] \cdot \overline{C_x{}^1}$. But such a component would not be a 1-cell. We can now conclude that $\lambda[D_1(I^2)]$ consists of infinitely many components and that each of these components is a 1-cell. This last statement holds true for $\lambda[I^2]$ which is homeomorphic with $\lambda[D_i(I^2)]$.

Since $\lambda[D_2(I^n)] - \overline{C_d^{n-1}} \neq 0$, there exists a closed 1-cell $\overline{C_\beta^{n-1}}$ belonging to the λ-set of a neighborhood and $\Delta[C_\beta^{n-1}] D_2(I^n) \neq 0$, $n=1, 2$.

$$D_2(I^n) = D_{21}(I^n) + D_{22}(I^n) + C_\beta{}^{n-1};\ \overline{D_{21}(I^n)} \cdot \overline{D_{22}(I^n)} = \overline{C_\beta{}^{n-1}}.$$

Suppose that $\overline{C_d^{n-1}} \subset \lambda[D_{22}(I^n)]$. Then $\overline{C_d^{n-1}} \cdot \lambda[D_{21}(I^n)] = 0$. $\lambda[D_{i1}(I^n)]$, $i=1, 2$, being homeomorphic with $\lambda[I^n]$, consists of infinitely many components, each of which is an $(n-1)$-cell.

There exists a homeomorphism, $H(\lambda[D_{11}(I^n)] = \lambda[D_{21}(I^n)])$. It can be shown that $\lambda[D_{11}(I^n)] = G_1 + G_2$, where $G_i \neq 0$, $\overline{G}_1 \cdot G_2 + G_1 \cdot \overline{G}_2 = 0$,

$G_1 \subset \lambda[D_{11}(I^n)]\cdot\lambda[I^n]$, $H(G_1) \subset \lambda[D_{21}(I^n)]\cdot\lambda[I^n]$ and both of the sets, $\lambda[D_{11}(I^n)]\cdot\lambda[I^n]-G_1$ and $\lambda[D_{21}(I^n)]\cdot\lambda[I^n]-H(G_1)$, are non-vacuous. Let $H(G_1)=G_1'$.

There exists a homeomorphism, $H_1(\lambda[I^n])=\lambda[I^n]$, such that $H_1(G_1)=H(G_1)=G_1'$, $H_1(G_1')=H^{-1}(G_1')=G_1$ and, if x is a point of $\lambda[I^n]-G_1-G_1'$, $H_1(x)=x$. By Theorem I (4.2), there is a homeomorphism, $H_2(Z^n)=Z^n$, for which $H_2(\lambda[I^n])=H_1(\lambda[I^n])$.

Under the transformation H_2, a subset of $\overline{D_{i1}(I^n)}$, $i=1, 2$, is transformed into itself and a subset of $\overline{D_{j1}(I^n)}$ is transformed into a subset of $\overline{D_{k1}(I^n)}$, $j\neq k$; $j, k=1, 2$. The set $\overline{D_{i1}(I^n)}$, being homeomorphic with Z^n, is a connected set. Since H_2 is a homeomorphism, $H_2(\overline{D_{i1}(I^n)})$ is a connected set. $H_2(\overline{D_{i1}(I^n)})$ contains non-vacuous subsets in each of the two sets, $\overline{D_1(I^n)}-\overline{C_d^{n-1}}$ and $\overline{D_2(I^n)}-\overline{C_d^{n-1}}$. Neither of the two last-named sets contains a point or limit point of the other. Therefore, $H_2(\overline{D_{i1}(I^n)})\cdot\overline{C_d^{n-1}}\neq 0$.

Let $n=1$. In this case the point $C_d{}^0$ is the transform of two distinct points belonging respectively to $\overline{D_{11}(I^n)}$ and $\overline{D_{21}(I^n)}$. This is impossible since H_2 is a homeomorphism. Therefore, our assumption that $\lambda[I^1]-\lambda[I^1]\cdot\overline{C_d{}^0}$ has an infinite number of components has led to a contradiction. Hence $\lambda[I^1]$ consists of a finite number of points. Let n_1 be the number of points in $\lambda[I^1]$.

$$\lambda[I^1] = \lambda[D_1(I^1)]\cdot\lambda[I^1] + \lambda[D_2(I^1)]\cdot\lambda[I^1],$$
$$n_1 = 2(n_1 - 1),$$
$$n_1 = 2.$$

Therefore, $\lambda[I^1]$ is a 0-sphere.

Let $n=2$. Since each point of $\lambda[C_d{}^1]$ is a limit point of $\lambda[I^2]-G_1-G_1'$, $H_2(\lambda[C_d{}^1])=\lambda[C_d{}^1]$ and $H_2(\overline{D_{i1}(I^2)})\cdot\overline{C_d{}^1}\subset C_d{}^1$, $i=1, 2$. Then $\overline{C_d{}^1}-\sum_{i=1}^{2}H_2(\overline{D_{i1}(I^2)})\cdot\overline{C_d{}^1}$ is not a connected set. $H_2(\overline{C_d{}^1})\not\subset\overline{C_d{}^1}$. For, if $H_2(\overline{C_d{}^1})\subset\overline{C_d{}^1}$, $H_2(\overline{C_d{}^1})\subset\overline{C_d{}^1}-\sum_{i=1}^{2}H_2(\overline{D_{i1}(I^2)})\cdot\overline{C^1}$. But no subset of $\overline{C_d{}^1}-\sum_{i=1}^{2}H_2(\overline{D_{i1}(I^2)})\cdot\overline{C_d{}^1}$ containing $\lambda[C_d{}^1]$ is a connected set. Therefore, $H_2(C_d{}^1)$ contains a point q not belonging to $C_d{}^1$. This point q belongs either to $D_1(I^2)$ or to $D_2(I^2)$. The discussion is of the same character in either case. Suppose that $D_2(I^2)\supset q$ and $H_2^{-1}(q)=q'$. Since $C_d{}^1$ contains no point which is a limit point of G_1, q' is not a limit point of G_1, $H_2(\lambda[D_1(I^2)]\cdot\lambda[I^2]-G_1)=\lambda[D_1(I^2)]\cdot\lambda[I^2]-G_1$. The point q, which belongs to $D_2(I^2)$, is not a limit point of $H_2(\lambda[D_1(I^2)]\cdot\lambda[I^2]-G_1)$. Therefore, q' is not a limit point of $\lambda[D_1(I^2)]\cdot\lambda[I^2]-G_1$. Then there exists N_r such that $\overline{N_r}\supset q'$ and $\overline{N_r}\cdot(\lambda[D_1(I^2)]-C_d{}^1)=0$. The set $\lambda[N_r]$ contains a closed 1-cell $\overline{C_\kappa^1}$ such that $\Delta[C_\kappa^1]D_1(I^2)\neq 0$. Since the λ-set of each descendant of $D_1(I^2)$ generated by C_κ^1 is homeomorphic with $\lambda[D_1(I^2)]$, each component of the λ-set of such a de-

scendant, under our assumption, is a 1-cell. $\overline{C_\kappa^1}$ belongs to a component of the λ-set of each descendant of $D_1(I^2)$ generated by C_κ^1 and $\overline{C_\kappa^1}\cdot(\lambda[D_1(I^2)]-C_d^1)=0$. Then $\lambda[C_\kappa^1]\subset C_d^1$. The set C_d^1 contains a 1-cell C_μ^1 such that $\lambda[C_\mu^1]=\lambda[C_\kappa^1]$. The λ-set of one of the descendants of $D_1(I^2)$ generated by C_κ^1 contains the 1-sphere $\overline{C_\kappa^1}+C_\mu^1$. This contradicts the fact that every component of the λ-set of a descendant of $D_1(I^2)$ generated by C_κ^1 is a 1-cell. Hence our assumption that $\lambda[I^2]-\lambda[I^2]\cdot\overline{C_d^1}$ contains infinitely many components has led to a contradiction.

The set $\lambda[I^2]-\lambda[I^2]\cdot\overline{C_d^1}$ is non-vacuous and consists of a finite number of components $C_{\alpha_1}^1, C_{\alpha_2}^1, \cdots, C_{\alpha_f}^1$. There exists N_s such that $N_s\cdot C_{\alpha_1}^1\neq 0$ and $\overline{N_s}\cdot\sum_{i=2}^{f}\overline{C_{\alpha_i}^1}=0$. The set $\lambda[N_s]$ contains a closed 1-cell $\overline{C_t^1}$ such that $\Delta[C_t^1]I^2\neq 0$. As in the above, it can be shown that the λ-set of one of the descendants of I^2 generated by C_t^1 contains a 1-sphere. Then $\lambda[I^2]$ contains a 1-sphere S^1. Suppose that $\lambda[I^2]$ contains a point g not belonging to S^1. There exists N_m such that $N_m\supset g$ and $\overline{N_m}\cdot S^1=0$. Then $\lambda[N_m]$ contains a closed 1-cell $\overline{C_\rho^1}$ such that $\Delta[C_\rho^1]I^2\neq 0$. S^1, being a connected set, belongs to a component of $\lambda[I^2]-\lambda[I^2]\cdot\overline{C_\rho^1}$. But such a component, containing a 1-sphere, would not be a 1-cell. Hence $\lambda[I^2]=S^1$.

The set $\lambda[I^2]-\lambda[I^2]\cdot\overline{C_d^1}$ consists of two components belonging to $\lambda[D_1(I^2)]$ and $\lambda[D_2(I^2)]$ respectively and $\lambda[C_d^1]$ is the λ-set of each component.

Case 2. $n\geqq 3$. The set $\overline{C_d^{n-1}}$, being a connected set, belongs to a component C_ξ^{n-1} of $\lambda[D_1(I^n)]-\lambda[D_1(I^n)]\cdot\overline{C_\alpha^{n-1}}$ (see the first part of this proof). By the Jordan-Brouwer Theorem,* $C_\xi^{n-1}-\lambda[C_d^{n-1}]=M_1+M_2$, $\overline{M_1}\cdot\overline{M_2}=\lambda[C_d^{n-1}]$, and M_i, $i=1, 2$, is a connected set. Since C_d^{n-1} is connected, one of the sets M_i contains no point of C_d^{n-1}. Let M_1 be this set. Then M_1 is a subset of a component C_h^{n-1} of $\lambda[I^n]-\lambda[I^n]\cdot\overline{C_d^{n-1}}$ which belongs to $\lambda[D_1(I^n)]$ and which has $\lambda[C_d^{n-1}]$ for its λ-set. Then $\lambda[D_1(I^n)]$ contains the $(n-1)$-sphere $\overline{C_d^{n-1}}+C_h^{n-1}$. Therefore, $\lambda[I^n]$ contains an $(n-1)$-sphere. By an argument used in connection with the proof for the case when $n=2$, it can be shown that $\lambda[I^n]$ is an $(n-1)$-sphere. The set $\lambda[I^n]-\lambda[I^n]\cdot\overline{C_d^{n-1}}$ consists of two components belonging to $\lambda[D_1(I^n)]$ and $\lambda[D_2(I^n)]$ respectively, and $\lambda[C_d^{n-1}]$ is the λ-set of each component.

5.3. LEMMA 3. *If N_α is a neighborhood such that $\overline{N_\alpha}\subset I^n$, the set $\lambda[N_\alpha]$ is an $(n-1)$-sphere.*

Since N_α is an open proper subset of the connected space Z^n, $\lambda[N_\alpha]\neq 0$. $\overline{N_\alpha}$ is compact (Lemma 1).

* L. E. J. Brouwer, *Beweis des Jordanschen Satzes für den n-dimensionalen Raum*, Mathematische Annalen, vol. 71, p. 314; J. W. Alexander, *A proof and extension of the Jordan-Brouwer separation theorem*, these Transactions, vol. 23, p. 333.

Case 1. $n=1$. There exists a set of neighborhoods $\{N_{\phi_i}\}$, $i=1, 2, \cdots, m$, such that $I^n \supset N_{\phi_i} \not\supset N_\alpha$, $N_{\phi_i} \cdot \lambda[N_\alpha] \neq 0$ and $\lambda[N_\alpha] \subset \sum_{i=1}^{m} N_{\phi_i}$. Then $\overline{N}_{\phi_1} \cdot \lambda[N_\alpha] = \sum_{j=1}^{d_i} \overline{C_{ij}^0}$, $i=1, 2, \cdots, m$. Hence $\lambda[N_\alpha]$ consists of a finite number of points. By a method used in Lemma 2, it can be proved that $\lambda[N_\alpha]$ is a 0-sphere.

Case 2. $n>1$. There exists N_ψ such that $N_\psi \cdot \lambda[N_\alpha] \neq 0$, $N_\psi \subset I^n$ and $N_\psi \not\supset N_\alpha$. Then $\overline{N}_\psi \cdot \lambda[N_\alpha] = \sum_{j=1}^{d} \overline{C_j^{n-1}}$ and $\Delta_{j=1}^{d}[C_j^{n-1}] N_\psi \neq 0$. Let p be a point of C_1^{n-1}. The point p is not a limit point of the set $\sum_{j=2}^{d} \overline{C_j^{n-1}}$. There is a neighborhood N_δ such that $N_\delta \supset p$, $\overline{N}_\delta \subset N_\psi$ and $\overline{N}_\delta \cdot \sum_{j=2}^{d} C_j^{n-1} = 0$. Then $\lambda[N_\delta]$ contains a closed $(n-1)$-cell $\overline{C_i^{n-1}}$ such that $\Delta[C_i^{n-1}] N_\alpha \neq 0$. By arguments given in the proof of Lemma 2, it can be proved that C_1^{n-1} contains an $(n-1)$-cell C_a^{n-1} whose λ-set is $\lambda[C_i^{n-1}]$. It is evident that the λ-set of one of the descendants of N_α generated by C_i^{n-1} contains the $(n-1)$-sphere $\overline{C_i^{n-1}} + C_a^{n-1}$ and, furthermore, that the λ-set of this descendant is identical with this $(n-1)$-sphere. Hence $\lambda[N_\alpha]$ is an $(n-1)$-sphere.

5.4. LEMMA 4. *Let W_1, W_2, and V be the sets given in Theorem* I′ (4.3) *with the following restrictions: $\overline{W}_1 \subset I^n$ and, if $V \neq I^n$, $\bar{V} \subset I^n$. Then $W_2 = \sum_{i=1}^{d+1} D_i + \sum_{j=1}^{d} C_j^{n-1}$, D_i is a proper descendant of W_2 and $D_r \cdot D_s = 0$, $r \neq s$. If μ is a fixed value of j, there are two sets, D_α and D_β, such that $\lambda[D_\alpha] \cdot \lambda[D_\beta] = \overline{C_\mu^{n-1}}$ and $C_\mu^{n-1} \cdot \lambda[D_\delta] = 0$, $\delta \neq \alpha, \beta$. The set $\lambda[D_\alpha] - \overline{C_\mu^{n-1}}$ is an $(n-1)$-cell whose λ-set is $\lambda[C_\mu^{n-1}]$.*

Since $\lambda[W_i]$, $i=1, 2$, is an $(n-1)$-sphere, it can be shown that $\lambda[C_j^{n-1}] \subset \lambda[W_2]$. The other desired results can be obtained by referring to the definitions in §2, Theorem P_1 and results previously established.

The set of descendants given in the lemma is called the set of final descendants of W_2 generated by $\lambda[W_1]$.

COROLLARY. *If V is I^n or is strongly homeomorphic with I^n, $\Delta[C_j^{n-1}] W_2 \neq 0$ for all values of j.*

This proposition can be proved by means of the results given in the lemma, Theorem I′ (4.2) and Theorems P_1 and P_2.

5.5. LEMMA 5. *Let C_1^{n-1} and C_2^{n-1} be two $(n-1)$-cells such that $\lambda[I^n] = \overline{C_1^{n-1}} + \overline{C_2^{n-1}}$ and $\overline{C_1^{n-1}} \cdot \overline{C_2^{n-1}} = \lambda[C_i^{n-1}]$, $i=1, 2$. Then there exists an $(n-1)$-cell C_3^{n-1} having the following properties: $\lambda[C_3^{n-1}] = \lambda[C_i^{n-1}]$, $i=1, 2$, and $\Delta[C_3^{n-1}] I^n \neq 0$.*

There exists N_I such that $\lambda[N_I]$ contains a closed $(n-1)$-cell $\overline{C_\alpha^{n-1}}$ and $\Delta[C_\alpha^{n-1}] I^n \neq 0$. From the proof of Lemma 2 it is known that $\lambda[I^n] = \overline{C_{\beta_1}^{n-1}} + \overline{C_{\beta_2}^{n-1}}$, where $\overline{C_{\beta_1}^{n-1}} \cdot \overline{C_{\beta_2}^{n-1}} = \lambda[C_\alpha^{n-1}] = \lambda[C_{\beta_i}^{n-1}]$, $i=1, 2$. By Corollary, Theorem P_2, there exists a homeomorphism, $H_1(\lambda[I^n]) = \lambda[I^n]$, such that $H_1(\overline{C_{\beta_i}^{n-1}}) = \overline{C_i^{n-1}}$, $i=1, 2$. There is a homeomorphism, $H_2(Z^n) = Z^n$, of such

nature that $H_2(\lambda[I^n]) = H_1(\lambda[I^n])$. The set $H_2(C_\alpha^{n-1})$ is an $(n-1)$-cell C_β^{n-1}. Since H_2 is a homeomorphism, $\Delta[C_\beta^{n-1}]I^n \neq 0$.

5.6. LEMMA 6. *I^n is a connected set.*

Case 1. $n=1$. $\lambda[I^1]$ consists of two points, p_1 and p_2. Suppose that I^1 is not connected. Then

$$I^1 = M_1 + M_2;\ \overline{M}_1 \cdot M_2 + M_1 \cdot \overline{M}_2 = 0;\ M_i \neq 0.$$

The two sets, $M_1+p_1+p_2$ and $M_2+p_1+p_2$, cannot both be connected sets. For, assume that each of these sets is connected. There exists N_h such that $N_h \supset p_1$ and $\overline{N}_h \not\supset p_2$. Since $M_1+p_1+p_2$ is a connected set, $M_1+p_1+p_2$ contains a point q belonging to $\lambda[N_h]$ and $\Delta[q]I^1 \neq 0$. $M_1 \supset q$. Then $M_2 \not\supset q$.

$$I^1 = D_1(I^1) + D_2(I^1) + q;\ \overline{D_1(I^1)} \cdot \overline{D_2(I^1)} = q.$$

Each of the points p_i, $i=1, 2$, belongs to one and only one of the sets $\lambda[D_j(I^1)]$, $j=1, 2$. The connected set $M_2+p_1+p_2$ is the sum of two non-vacuous sets belonging respectively to $\overline{D_1(I^1)}-q$ and $\overline{D_2(I^1)}-q$. This is impossible. Therefore, at least one of the sets $M_i+p_1+p_2$ is not connected. Suppose that $M_1+p_1+p_2$ is not connected.

$$M_1 + p_1 + p_2 = P_1 + P_2;\ \overline{P}_1 \cdot P_2 + P_1 \cdot \overline{P}_2 = 0;\ P_i \neq 0.$$

The set p_1+p_2 cannot belong to one of the sets P_i. For, suppose that $P_1 \supset p_1+p_2$. $Z^1=(P_1+M_2)+P_2$. But Z^1 is a connected set and $\overline{P_1+M_2} \cdot P_2 + (P_1+M_2) \cdot \overline{P}_2 = 0$. Then let $P_i \supset p_i$, $i=1, 2$. The set P_1 is connected. For, suppose that

$$P_1 = P_{11} + P_{12};\ \overline{P}_{11} \cdot P_{12} + P_{11} \cdot \overline{P}_{12} = 0;\ P_{1i} \neq 0.$$

The point p_1 belongs to one of the sets P_{1i}. Assume that $P_{11} \supset p_1$. $Z^1=(P_{11}+P_2+M_2)+P_{12}$. Again we have an impossible situation, since $\overline{P_{11}+P_2+M_2} \cdot P_{12} + (P_{11}+P_2+M_2) \cdot \overline{P}_{12} = 0$. Then P_1 is a connected set. Similarly, P_2 is a connected set. Since $M_1 \neq 0$, at least one of the sets P_i contains more than one point. Let P_1 be such a set. Now assume that $M_2+p_1+p_2$ is a connected set. There exists N_g such that $N_g \supset p_1$ and $N_g \not\supset P_1$. Then P_1, being a connected set, contains a point x of $\lambda[N_g]$ and $\Delta[x]I^1 \neq 0$.

$$I^1 = D_{\delta_1}(I^1) + D_{\delta_2}(I^1) + x;\ \overline{D_{\delta_1}(I^1)} \cdot \overline{D_{\delta_2}(I^1)} = x.$$

Since x belongs to M_1, the connected set $M_2+p_1+p_2$ is the sum of two non-vacuous sets belonging respectively to $\overline{D_{\delta_1}(I^1)}-x$ and $\overline{D_{\delta_2}(I^1)}-x$. Then $M_2+p_1+p_2$ is not a connected set.

$$M_2 + p_1 + p_2 = R_1 + R_2;\ \overline{R}_1 \cdot R_2 + R_1 \cdot \overline{R}_2 = 0;\ R_i \neq 0.$$

As in the above, it can be shown that, by a proper choice of subsets, $R_i \supset p_i$, $i=1, 2$. $Z^1=(P_1+R_1)+(P_2+R_2)$ and $\overline{P_1+R_1}\cdot(P_2+R_2)+(P_1+R_1)\cdot\overline{P_2+R_2}=0$. But Z^1 is connected. This final contradiction shows that I^1 is a connected set.

Case 2. $n>1$. There exists N_r such that $\overline{N}_r \subset I^n$. $\lambda[N_r]$ is an $(n-1)$-sphere (Lemma 4). Suppose that $\overline{N}_r$ is not connected. Then

$$\overline{N}_r = B_1 + B_2;\ \overline{B}_1\cdot B_2 + B_1\cdot\overline{B}_2 = 0;\ B_i \neq 0.$$

$\lambda[N_r]$, being a connected set, belongs to one of the sets B_i. Suppose that $\lambda[N_r] \subset B_1$. $Z^n=[B_1+(Z^n-\overline{N}_r)]+B_2$ and $\overline{B_1+(Z^n-\overline{N}_r)}\cdot B_2+[B_1+(Z^n-\overline{N}_r)]\cdot\overline{B}_2=0$. But Z^n is connected. Then $\overline{N}_r$ is connected.

Let T be a component of I^n. $Z^n=T+(\lambda[I^n]+(I^n-T))$. If a point q of T were a limit point of I^n-T, there would be a neighborhood N_b such that $\overline{N}_b \subset I^n$, $N_b \supset q$ and $N_b\cdot(I^n-T)\neq 0$. Then the set $T+\overline{N}_b$ would be a connected set. This contradicts the fact that T is a maximal connected subset of I^n. The set $\lambda[I^n]$ contains a limit point of T.

Let p be a point of $\lambda[T]\cdot\lambda[I^n]$. There exists N_s such that $N_s \supset p$ and $\overline{N}_s \not\supset T$. Then $T\cdot\lambda[N_s]\neq 0$. $Z^n\cdot\lambda[N_s]=\sum_{j=1}^{d}\overline{C_j^{n-1}}$, $\overline{C_j^{n-1}}\cdot I^n=C_j^{n-1}$ and $T\cdot\lambda[N_s] \subset \sum_{j=1}^{d}C_j{}^{-1}$. Let $T\cdot C_1^{n-1}\neq 0$. Then $C_1^{n-1} \subset T$. $\Delta[C_1^{n-1}]I^n\neq 0$ (Corollary, Lemma 4).

$$I^n = D_1(I^n) + D_2(I^n) + C_1^{n-1};\ \overline{D_1(I^n)}\cdot\overline{D_2(I^n)} = \overline{C_1^{n-1}}.$$

There exist two sets, $C_{\alpha_1}^{n-1}$ and $C_{\alpha_2}^{n-1}$ such that $\lambda[I^n]=\overline{C_{\alpha_1}^{n-1}}+\overline{C_{\alpha_2}^{n-1}}$ and $\overline{C_{\alpha_1}^{n-1}}\cdot\overline{C_{\alpha_2}^{n-1}}=\lambda[C_1^{n-1}]=\lambda[C_{\alpha_i}^{n-1}]$, $i=1, 2$. Suppose that $\lambda[D_i(I^n)] \supset C_{\alpha_i}^{n-1}$, $i=1, 2$. Let S^{n-1} be the $(n-1)$-sphere in euclidean n-space E^n whose points have coordinates which satisfy the equation: $\sum_{i=1}^{n}x_i^2=1$. The $(n-1)$-dimensional plane $x_1=0$ separates S^{n-1} into two $(n-1)$-cells, K_1^{n-1} and K_2^{n-1}, such that $S^{n-1}=\overline{K_1^{n-1}}+\overline{K_2^{n-1}}$ and $\overline{K_1^{n-1}}\cdot\overline{K_2^{n-1}}=\lambda[K_i^{n-1}]$, $i=1, 2$. By Theorem P_2 there is a homeomorphism, $H(\lambda[I^n])=S^{n-1}$, such that $H(\overline{C_{\alpha_i}^{n-1}})=\overline{K_i^{n-1}}$, $i=1, 2$.

Let p_1 be a point of $C_{\alpha_1}^{n-1}$ and $H(p_1)=p_1'$. $K_1^{n-1} \supset p_1'$. In E^n there exists an $(n-1)$-dimensional plane E_1^{n-1} such that E_1^{n-1} contains the points p_1' and $(0, 0, \cdots, 0)$. E_1^{n-1} separates S^{n-1} into two $(n-1)$-cells, $K_{\beta_1}^{n-1}$ and $K_{\beta_2}^{n-1}$. $S^{n-1}=\overline{K_{\beta_1}^{n-1}}+\overline{K_{\beta_2}^{n-1}}$, $\overline{K_{\beta_1}^{n-1}}\cdot\overline{K_{\beta_2}^{n-1}}=\lambda[K_{\beta_1}^{n-1}]$, $\lambda[K_{\beta_i}^{n-1}] \supset p_1'$ and $\lambda[K_{\beta_i}^{n-1}]\cdot K_2^{n-1}\neq 0$, $i=1, 2$. Let $H^{-1}(K_{\beta_i}^{n-1})=C_{\beta_i}^{n-1}$. Then $\lambda[C_{\beta_i}^{n-1}] \supset p_1$, $\lambda[C_{\beta_i}^{n-1}]\cdot C_{\alpha_2}^{n-1}\neq 0$ and $\lambda[I^n]=\overline{C_{\beta_1}^{n-1}}+\overline{C_{\beta_2}^{n-1}}$, $\overline{C_{\beta_1}^{n-1}}\cdot\overline{C_{\beta_2}^{n-1}}=\lambda[C_{\beta_i}^{n-1}]$, $i=1, 2$. By Lemma 5, there exists $C_{\beta_3}^{n-1}$ such that $C_{\beta_3}^{n-1} \subset I^n$ and $\lambda[C_{\beta_3}^{n-1}]=\lambda[C_{\beta_i}^{n-1}]$, $i=1, 2$. The connected set $C_{\beta_3}^{n-1}$ contains non-vacuous subsets in each of the two sets, $D_1(I^n)$ and $D_2(I^n)$. Therefore $C_{\beta_3}^{n-1}\cdot C_1^{n-1}\neq 0$. Since $C_1^{n-1} \subset T$, $C_{\beta_3}^{n-1} \subset T$. Then p_1, any point of $C_{\alpha_1}^{n-1}$, is a limit point of T. In the same man-

ner it can be shown that $C_{a_2}^{n-1} \subset \lambda[T]$. Hence $\lambda[T]=\lambda[I^n]$. If I^n contains a component T_1 distinct from T, $\lambda[T_1]=\lambda[I^n]$. From the preceding discussion, it is seen that $T_1 \cdot C_1^{n-1} \neq 0$. Then T_1 cannot be distinct from T.

5.7. LEMMA 7. *There exists a neighborhood N_α such that $\overline{N}_\alpha \subset I^n$ and N_α is strongly homeomorphic with I^n.*

There exists N_t such that $N_t \cdot \lambda[I^n] \neq 0$ and $N_t \not\supset I^n$. Then $Z^n \cdot \lambda[N_t] = \sum_{j=1}^{d} \overline{C_j^{n-1}}$. Let $D_1(I^n)$ and $D_2(I^n)$ be the two members of the set of final descendants of I^n generated by $\lambda[N_t]$ which have C_1^{n-1} on their λ-sets (Lemma 4). Let p be a point of C_1^{n-1}. There is a neighborhood N_α such that $N_\alpha \supset p$ and $\overline{N}_\alpha$ belongs to the set $D_1(I^n)+D_2(I^n)+C_1^{n-1}$. $\lambda[N_t]$ generates a set of final descendants for N_α. $\overline{N}_\alpha \cdot \lambda[N_t] = \sum_{j=1}^{m} \overline{C_{\delta_j}^{n-1}}$. Since $\overline{N}_\alpha \cdot \lambda[N_t] \subset C_1^{n-1}$, $\sum_{j=1}^{m} \overline{C_{\delta_j}^{n-1}} \subset C_1^{n-1}$. Let $D_1(N_\alpha)$ and $D_2(N_\alpha)$ be the two members of the set of final descendants of N_α generated by $\lambda[N_t]$ which have $C_{\delta_1}^{n-1}$ on their λ-sets. $\lambda[N_\alpha]$ generates a set of final descendants for each of the sets $D_i(I^n)$, $i=1, 2$. $C_{\delta_1}^{n-1} \cdot \lambda[N_\alpha]=0$. Then, since $C_{\delta_1}^{n-1}$ is a connected set, $C_{\delta_1}^{n-1}$ belongs to the λ-set of one and only one final descendant of each of the sets $D_i(I^n)$, $i=1, 2$, generated by $\lambda[N_\alpha]$. Let these two final descendants be $D_{11}(I^n)$ and $D_{21}(I^n)$ belonging to $D_1(I^n)$ and $D_2(I^n)$ respectively. There exists N_e such that $N_e \cdot C_{\delta_1}^{n-1} \neq 0$ and N_e belongs to the set $D_{11}(I^n)+D_{21}(I^n)+C_{\delta_1}^{n-1}$. N_e contains a point p_1 of $D_1(N_\alpha)$. The point p_1 belongs to one of the sets $D_{i1}(I^n)$. Suppose that $D_{11}(I^n) \supset p_1$. Since I^n is a connected set, $D_{11}(I^n)$ is connected. But $D_{11}(I^n) \cdot (\lambda[N_\alpha]+\lambda[N_t])=0$. Then $D_{11}(I^n) \subset D_1(N_\alpha)$. Since each of the two sets, $\lambda[I^n]$ and $\lambda[N_\alpha]$, is an $(n-1)$-sphere, each of the sets, $\lambda[D_{11}(I^n)]$ and $\lambda[D_1(N_\alpha)]$, is an $(n-1)$-sphere. $\lambda[D_{11}(I^n)] \subset \lambda[D_1(N_\alpha)]$. Therefore $\lambda[D_{11}(I^n)] \equiv \lambda[D_1(N_\alpha)]$, since no proper subset of an $(n-1)$-sphere is an $(n-1)$-sphere. No point of $D_1(N_\alpha)$ can belong to any final descendant of one of the sets $D_i(I^n)$, $i=1, 2$, generated by $\lambda[N_\alpha]$ other than $D_{11}(I^n)$. For, otherwise, such a descendant would belong to $D_1(N_\alpha)$ and its λ-set would be identical with $\lambda[D_1(N_\alpha)]$ and, therefore, with $\lambda[D_{11}(I^n)]$—an impossible situation. Then $D_1(N_\alpha)=D_{11}(I^n)$. Hence $D_1(N_\alpha)$ is strongly homeomorphic with I^n. N_α is strongly homeomorphic with I^n.

COROLLARY. *Let N_ρ and N_ξ be two neighborhoods such that $\overline{N}_\rho+\overline{N}_\xi \subset I^n$, N_ρ is connected, N_ξ contains a point p of $\lambda[N_\rho]$ and $N_\xi \not\supset N_\rho$. Then N_ξ is strongly homeomorphic with N_ρ and p is a limit point of $Z^n-\overline{N}_\rho$.*

This proposition can be proved by means of the procedure used in the proof of the lemma.

5.8. LEMMA 8. *Let N_α be a neighborhood such that $\overline{N}_\alpha \subset I^n$, N_α is strongly homeomorphic with I^n and $\overline{N}_\alpha = \sum_{k=1}^{c} P_k$, where each set P_k is an open set with*

respect to $\overline{N}_\alpha$. *Then* $\overline{N}_\alpha=\sum_{j=1}^{r}\bar{D}_j$ *such that* $D_\xi\cdot D_\eta=0$, $\xi\neq\eta$, *and* D_j *belongs to at least one set* P_k *and is strongly homeomorphic with* N_α. *If* μ *is a fixed value of* j, $\lambda[D_\mu]=\sum_{i=1}^{b_\mu}\overline{C_{\mu_i}^{n-1}}$, $C_{\mu_s}^{n-1}\cdot C_{\mu_t}^{n-1}=0$, $s\neq t$, *and, if* h *is a fixed value of* i, $\lambda[D_\mu]-\overline{C_{\mu_h}^{n-1}}$ *is an* $(n-1)$*-cell and either* $C_{\mu_h}^{n-1}\subset N_\alpha$ *or* $C_{\mu_h}^{n-1}\subset\lambda[N_\alpha]$. *If* $C_{\mu_h}^{n-1}\subset N_\alpha$, *there is a set* D_κ, $\kappa\neq\mu$, *such that* $\lambda[D_\mu]\cdot\lambda[D_\kappa]=\overline{C_{\mu_h}^{n-1}}$ *and* $C_{\mu_h}^{n-1}\cdot\lambda[D_\delta]=0$, $\delta\neq\mu,\kappa$. *If* $C_{\mu_h}^{n-1}\subset\lambda[N_\alpha]$, $C_{\mu_h}^{n-1}\cdot\lambda[D_\phi]=0$, $\phi\neq\mu$.

The following proof applies when $n>1$. The modifications necessary when $n=1$ are obvious.

The set $\overline{N}_\alpha$ is compact. Assign to each point p of $\overline{N}_\alpha$ the neighborhood N_{α_p} such that N_{α_p} is the neighborhood of $G(p)$ (§5.1) of least subscript having the following properties: $\overline{N}_{\alpha_p}\subset I^n$, $N_{\alpha_p}\not\supset N_\alpha$ and $N_{\alpha_p}\cdot\overline{N}_\alpha$ belongs to at least one of the sets P_k. There exists a finite subset of the set of all such neighborhoods: $T_1=\{N_{\psi_j}\}$, $j=1, 2, \cdots, \rho$, such that $\sum_{j=1}^{\rho}N_{\psi_j}\supset\overline{N}_\alpha$ and $N_{\psi_c}\not\supset N_{\psi_d}$, $c\neq d$. By successive applications of Corollary, Lemma 7, it can be shown that every neighborhood belonging to T_1 is strongly homeomorphic with N_α and is, therefore, a connected set.

Let N_{ψ_m} be any neighborhood of T_1. Every point of the set $\lambda[N_{\psi_m}]\cdot\overline{N}_\alpha$ belongs to at least one of the neighborhoods of T_1. Suppose that the subscripts in the symbols for the neighborhoods constituting T_1 are so chosen that $\{N_{\psi_j}\}$, $j=1, 2, \cdots, m-1$, is the set of neighborhoods such that $\lambda[N_{\psi_m}]\cdot N_{\psi_j}\neq 0$, $j=1, 2, \cdots, m-1$.

Let A_{ψ_1} represent N_{ψ_1}. If $N_{\psi_1}\cdot\lambda[N_{\psi_2}]=0$, let $(A_{\psi_1})_{\psi_2}$ be N_{ψ_1} and, if $N_{\psi_1}\cdot\lambda[N_{\psi_2}]\neq 0$, let $(A_{\psi_1})_{\psi_2}$ be the set of final descendants of A_{ψ_1} generated by $\lambda[N_{\psi_2}]$. If $N_{\psi_2}\cdot\lambda[N_{\psi_1}]=0$, let $B_{\psi_2\psi_1}$ be N_{ψ_2}, and, if $N_{\psi_2}\cdot\lambda[N_{\psi_1}]\neq 0$, let $B_{\psi_2\psi_1}$ be the set consisting of those final descendants of N_{ψ_2} generated by $\lambda[N_{\psi_1}]$ which do not lie in N_{ψ_1}.

Every descendant of N_{ψ_2} is a connected set. No final descendant of N_{ψ_2} generated by $\lambda[N_{\psi_1}]$ contains a point of the set $\lambda[N_{\psi_1}]+\lambda[N_{\psi_2}]$. Then every such final descendant lies wholly in N_{ψ_1} or wholly in $Z^n-\overline{N}_{\psi_1}$.

$$A_{\psi_1\psi_2}=(A_{\psi_1})_{\psi_2}+B_{\psi_2\psi_1},$$
$$\overline{A}_{\psi_1\psi_2}=\overline{N}_{\psi_1}+\overline{N}_{\psi_2}.$$

If $A_{\psi_1\psi_2}\cdot\lambda[N_{\psi_3}]=0$, let $(A_{\psi_1\psi_2})_{\psi_3}$ be $A_{\psi_1\psi_2}$ and, if $A_{\psi_1\psi_2}\cdot\lambda[N_{\psi_3}]\neq 0$, let $(A_{\psi_1\psi_2})_{\psi_3}$ be the set consisting of the final descendants of members of $A_{\psi_1\psi_2}$ generated by $\lambda[N_{\psi_3}]$ and those members of $A_{\psi_1\psi_2}$ which contain no point of $\lambda[N_{\psi_3}]$. If $N_{\psi_3}\cdot\lambda[N_{\psi_2}]=0$, $B_{\psi_3\psi_2}=N_{\psi_3}$ and, if $N_{\psi_3}\cdot\lambda[N_{\psi_2}]\neq 0$, $B_{\psi_3\psi_2}$ is the set of those final descendants of N_{ψ_3} generated by $\lambda[N_{\psi_2}]$ which do not lie in N_{ψ_2}. If $B_{\psi_3\psi_2}\cdot\lambda[N_{\psi_1}]=0$, $B_{\psi_3\psi_2\psi_1}=B_{\psi_3\psi_2}$ and, if $B_{\psi_3\psi_2}\cdot\lambda[N_{\psi_1}]\neq 0$, $B_{\psi_3\psi_2\psi_1}$ is the set consisting of the final descendants of members of $B_{\psi_3\psi_2}$ generated by

$\lambda[N_{\psi_1}]$ which do not lie in N_{ψ_1} and those members of $B_{\psi_3\psi_2}$ which contain no point of $\lambda[N_{\psi_1}]$.

$$A_{\psi_1\psi_2\psi_3} = (A_{\psi_1\psi_2})_{\psi_3} + B_{\psi_3\psi_2\psi_1},$$

$$\overline{A}_{\psi_1\psi_2\psi_3} = \sum_{j=1}^{3} \overline{N}_{\psi_j}.$$

Proceeding in this manner, we obtain $A_{\psi_1\psi_2\cdots\psi_{m-1}}$.

$$A_{\psi_1\psi_2\cdots\psi_{m-1}} = (A_{\psi_1\psi_2\cdots\psi_{m-2}})_{\psi_{m-1}} + B_{\psi_{m-1}\cdots\psi_2\psi_1},$$

$$\overline{A}_{\psi_1\psi_2\cdots\psi_{m-1}} = \sum_{j=1}^{m-1} \overline{N}_{\psi_j}.$$

Considering in the next step the neighborhood N_{ψ_m}, we obtain $A_{\psi_1\psi_2\cdots\psi_{m-1}\psi_m}$. If T_1 contains additional neighborhoods $N_{\psi_{m+j}}$, $j=1, 2, \cdots, s$, $N_{\psi_{m+j}} \cdot (\lambda[N_{\psi_m}] \cdot \overline{N}_\alpha) = 0$. Continuing as in the above, we obtain finally

$$\overline{A}_{\psi_1\psi_2\cdots\psi_m\cdots\psi_\rho} = \sum_{j=1}^{\rho} \overline{N}_{\psi_j}.$$

Let A be the set consisting of those final descendants of members of $A_{\psi_1\psi_2\cdots\psi_m\cdots\psi_\rho}$ generated by $\lambda[N_\alpha]$ which lie in N_α and those members of $A_{\psi_1\psi_2\cdots\psi_m\cdots\psi_\rho}$ which lie in N_α. $\overline{A} = \overline{N}_\alpha \cdot \sum_{j=1}^{\rho} \overline{N}_{\psi_j} = \overline{N}_\alpha$. Since the members of A are connected sets and $A \cdot (\lambda[N_\alpha] + \sum_{j=1}^{\rho} \lambda[N_{\psi_j}]) = 0$, the subdivision of $\overline{N}_\alpha$ obtained by the above process is unique, that is, the subdivision is independent of the order in which the neighborhoods N_{ψ_j} enter into the discussion.

Since $\lambda[N_{\psi_m}]$ generates a set of final descendants for each of the neighborhoods N_{ψ_j}, $j=1, 2, \cdots, m-1$, the set $\lambda[N_{\psi_m}] \cdot \sum_{j=1}^{m-1} \overline{N}_{\psi_j}$ is, topologically, an $(n-1)$-dimensional euclidean set and $\lambda[N_{\psi_m}] \cdot \sum_{j=1}^{m-1} \lambda[N_{\psi_j}]$ is an $(n-2)$-dimensional set. Then $\lambda[N_{\psi_m}] \cdot A_{\psi_1\psi_2\cdots\psi_{m-1}} \neq 0$. Therefore $\lambda[N_{\psi_m}]$ generates sets of final descendants for certain members of $A_{\psi_1\psi_2\cdots\psi_{m-1}}$. Let E be the set of such members of $A_{\psi_1\psi_2\cdots\psi_{m-1}}$. $\overline{E} \cdot \lambda[N_{\psi_m}] = \sum_{k=1}^{t} \overline{C_k^{n-1}}$, $\overline{C_k^{n-1}} \cdot E = C_k^{n-1}$, $C_\xi^{n-1} \cdot C_\eta^{n-1} = 0$, $\xi \neq \eta$, and $\lambda[C_k^{n-1}] \subset \lambda[E]$. Suppose that p is a point of $\lambda[N_{\psi_m}] \cdot \overline{A}_{\psi_1\psi_2\cdots\psi_{m-1}}$ and that $\overline{E} \not\ni p$. Then $\lambda[A_{\psi_1\psi_2\cdots\psi_{m-1}}] \supset p$. But $\lambda[A_{\psi_1\psi_2\cdots\psi_{m-1}}] = \sum_{j=1}^{m-1} \lambda[N_{\psi_j}]$. Hence p belongs to the λ-set of at least one of the neighborhoods N_{ψ_j}, $j=1, 2, \cdots, m-1$. Let $\lambda[N_{\psi_1}] \supset p$. $\overline{N}_{\psi_1} \cdot \lambda[N_{\psi_m}] = \sum_{j=1}^{b} \overline{C_{\delta_j}^{n-1}}$, $N_{\psi_1} \cdot \overline{C_{\delta_j}^{n-1}} = C_{\delta_j}^{n-1}$ and $\sum_{j=1}^{b} \overline{C_{\delta_j}^{n-1}} \supset p$. Suppose that $\overline{C_{\delta_1}^{n-1}} \supset p$. There exists N_σ such that $N_\sigma \supset p$ and $\overline{N}_\sigma \cdot \overline{E} = 0$. Then N_σ contains a point q of $C_{\delta_1}^{n-1}$. There is a neighborhood N_ϵ such that $N_\epsilon \supset q$, $\overline{N}_\epsilon \subset N_{\psi_1}$ and $\overline{N}_\epsilon \cdot \overline{E} = 0$. $\overline{N}_\epsilon \cdot \lambda[N_{\psi_m}] = \sum_{j=1}^{a} \overline{C_{\theta_j}^{n-1}}$. The set $\sum_{j=1}^{a} \overline{C_{\theta_j}^{n-1}}$ is an $(n-1)$-dimensional set. $\sum_{j=1}^{a} \overline{C_{\theta_j}^{n-1}} \subset \overline{A}_{\psi_1\psi_2\cdots\psi_{m-1}} - \overline{E}$. Then $\sum_{j=1}^{a} \overline{C_{\theta_j}^{n-1}} \subset \lambda[N_{\psi_m}] \cdot \lambda[A_{\psi_1\psi_2\cdots\psi_{m-1}}]$. But $\lambda[N_{\psi_m}] \cdot \lambda[A_{\psi_1\psi_2\cdots\psi_{m-1}}]$ is an $(n-2)$-dimensional set. The contradiction

here encountered shows that $\lambda[N_{\psi_m}]\cdot\overline{A}_{\psi_1\psi_2\cdots\psi_{m-1}}=\lambda[N_{\psi_m}]\cdot\overline{E}=\sum_{k=1}^{t}\overline{C_k^{n-1}}$.

Let $C_{k_1}^{n-1}$ be one of the sets C_k^{n-1} and M_1 and M_2 be the two final descendants of a member of $A_{\psi_1\psi_2\cdots\psi_{m-1}}$ generated by $\lambda[N_{\psi_m}]$ such that $\overline{M}_1\cdot\overline{M}_2=\lambda[M_1]\cdot\lambda[M_2]=\overline{C_{k_1}^{n-1}}$. The set $M_1+M_2+C_{k_1}^{n-1}$ is a descendant of one of the neighborhoods N_{ψ_j}, $j=1, 2, \cdots, m-1$. Let M denote the set $M_1+M_2+C_{k_1}^{n-1}$. Considering now the set $\lambda[N_{\psi_{m+1}}]$, two possibilities present themselves:

(1) $\lambda[N_{\psi_{m+1}}]\cdot(M_1+M_2)=0$. In this case M_1 and M_2 are members of $A_{\psi_1\psi_2\cdots\psi_m\psi_{m+1}}$.

(2) $\lambda[N_{\psi_{m+1}}]\cdot(M_1+M_2)\neq 0$. In this situation $\lambda[N_{\psi_{m+1}}]$ generates final descendants for one or both of the sets M_i. If $\lambda[N_{\psi_{m+1}}]\cdot M_i\neq 0$, $\lambda[N_{\psi_{m+1}}]\cdot\lambda[M_i]$ is an $(n-2)$-dimensional euclidean set. Then $\lambda[N_{\psi_{m+1}}]\cdot C_{k_1}^{n-1}$ is at most an $(n-2)$-dimensional set. The set M belongs to one of the members of $A_{\psi_1\psi_2\cdots\psi_{m-1}}$. Let M' be this member. M may be identical with M'. Under our present assumption, $\lambda[N_{\psi_{m+1}}]$ generates a set of final descendants for M'. Let R be the set of these final descendants. $R\subset A_{\psi_1\psi_2\cdots\psi_{m-1}\psi_{m+1}}$, $\overline{C_{k_1}^{n-1}}\subset\overline{R}$ and $C_{k_1}^{n-1}\cdot\lambda[R]=C_{k_1}^{n-1}\cdot\lambda[N_{\psi_{m+1}}]$. Then $C_{k_1}^{n-1}\cdot R\neq 0$. Therefore $\lambda[N_{\psi_m}]$ generates sets of final descendants for certain members of R. Let R_1 be the set of such members of R. $\overline{R}_1\cdot\lambda[N_{\psi_m}]=\sum_{j=1}^{a}\overline{C_{\omega_j}^{n-1}}$ and $\overline{C_{\omega_j}^{n-1}}\cdot R_1=C_{\omega_j}^{n-1}$. Since $C_{k_1}^{n-1}\cdot\lambda[R]$ is at most an $(n-2)$-dimensional set, certain of the sets $\overline{C_{\omega_j}^{n-1}}$ belong to $\overline{C_{k_1}^{n-1}}$. $C_{k_1}^{n-1}$ may be one of the sets $C_{\omega_j}^{n-1}$. Suppose that, by a proper choice of subscripts, $\{\overline{C_{\omega_j}^{n-1}}\}$, $j=1, 2, \cdots, a_1$, is the set of the sets $\overline{C_{\omega_j}^{n-1}}$ which belong to $\overline{C_{k_1}^{n-1}}$. If $C_{k_1}^{n-1}$ contains a point q that does not belong to $\sum_{j=1}^{a_1}\overline{C_{\omega_j}^{n-1}}$, $\lambda[R]\supset q$. It can be shown by a method analogous to one used above that $C_{k_1}^{n-1}$ contains no such point q. Therefore, $\overline{C_{k_1}^{n-1}}=\sum_{j=1}^{a_1}\overline{C_{\omega_j}^{n-1}}$ $A_{\psi_1\psi_2\cdots\psi_{m-1}\psi_{m+1}\psi_m}\equiv A_{\psi_1\psi_2\cdots\psi_{m-1}\psi_m\psi_{m+1}}$. Corresponding to each set $C_{\omega_j}^{n-1}$, $j=1, 2, \cdots, a_1$, $A_{\psi_1\psi_2\cdots\psi_{m-1}\psi_m\psi_{m+1}}$ contains two members, E_{1j} and E_{2j}, such that $\overline{E}_{1j}\cdot\overline{E}_{2j}=\lambda[E_{1j}]\cdot\lambda[E_{2j}]=C_{\omega_j}^{n-1}$.

Let Y be the set consisting of the final descendants of members of $A_{\psi_1\psi_2\cdots\psi_m\cdots\psi_\rho}$ generated by $\lambda[N_\alpha]$ and those members of $A_{\psi_1\psi_2\cdots\psi_m\cdots\psi_\rho}$ which contain no point of $\lambda[N_\alpha]$. It is evident that $\overline{Y}\cdot\sum_{k=1}^{t}\overline{C_k^{n-1}}=\sum_{j=1}^{\beta}C_{\mu_j}^{n-1}$ and, corresponding to each set $C_{\mu_j}^{n-1}$, Y contains two members, B_{1j} and B_{2j}, such that $\overline{B}_{1j}\cdot\overline{B}_{2j}=\lambda[B_{1j}]\cdot\lambda[B_{2j}]=\overline{C_{\mu_j}^{n-1}}$. $\lambda[N_{\psi_m}]\cdot\overline{A}=\lambda[N_{\psi_m}]\cdot\overline{N}_\alpha\subset\sum_{k=1}^{t}\overline{C_k^{n-1}}$. If $N_{\psi_m}\subset N_\alpha$, $\lambda[N_{\psi_m}]\cdot\overline{A}=\sum_{j=1}^{\beta}\overline{C_{\mu_j}^{n-1}}$ and, if $N_{\psi_m}\not\subset N_\alpha$, $\lambda[N_{\psi_m}]\cdot\overline{A}$ is the sum of a certain number of the sets $\overline{C_{\mu_j}^{n-1}}$.

It can be shown that $\lambda[N_\alpha]=\sum_{j=1}^{\nu}\overline{C_{\kappa_j}^{n-1}}$ and, corresponding to each set $C_{\kappa_j}^{n-1}$, Y contains two members, P_{1j} and P_{2j}, such that $\overline{P}_{1j}\cdot\overline{P}_{2j}=\lambda[P_{1j}]\cdot\lambda[P_{2j}]=\overline{C_{\kappa_j}^{n-1}}$. Therefore $\lambda[A]=\lambda[N_\alpha]+\overline{N}_\alpha\cdot\sum_{j=1}^{\rho}\lambda[N_{\psi_j}]=\sum_{i=1}^{w}\overline{C_i^{n-1}}$. Each set C_i^{n-1} belongs wholly to N_α or wholly to $\lambda[N_\alpha]$. $C_{i_1}^{n-1}\cdot C_{i_2}^{n-1}=0$, $i_1\neq i_2$. If $C_{i_1}^{n-1}\subset N_\alpha$, $C_{i_1}^{n-1}$ belongs to the λ-sets of two and only two members of A and no other

member of A contains a point of $C_{i_1}^{n-1}$ on its λ-set. Since every point of $\lambda[N_\alpha]$ is a limit point of $Z^n - \overline{N}_\alpha$, if $C_{i_1}^{n-1} \subset \lambda[N_\alpha]$, $C_{i_1}^{n-1}$ belongs wholly to the λ-set of one and only one member of A and no other member of A contains a point of $C_{i_1}^{n-1}$ on its λ-set. It can be shown without difficulty that, if D is a member of A and $\lambda[D] \supset C_{i_1}^{n-1}$, then $\lambda[D] - \overline{C_{i_1}^{n-1}}$ is an $(n-1)$-cell whose λ-set is $\lambda[C_{i_1}^{n-1}]$.

Let B be any member of A. Evidently $\lambda[B] = \sum_{y=1}^{b} \overline{C_y^{n-1}}$, $C_{y_1}^{n-1} \cdot C_{y_2}^{n-1} = 0$, $y_1 \neq y_2$, and $\lambda[B] - \overline{C_{y_1}^{n-1}}$ is an $(n-1)$-cell whose λ-set is $\lambda[C_{y_1}^{n-1}]$. Since every neighborhood of T_1 is strongly homeomorphic with I^n, it follows that B is strongly homeomorphic with I^n and Lemma 5 holds if B is substituted for I^n. Then there exists C_{11}^{n-1} such that $\overline{C_{11}^{n-1}} \cdot B = C_{11}^{n-1}$, $\lambda[C_{11}^{n-1}] = \lambda[C_1^{n-1}]$ and $\Delta[C_{11}^{n-1}]B \neq 0$.

$$B = D_1(B) + D_2(B) + C_{11}^{n-1};\ \overline{D_1(B)} \cdot \overline{D_2(B)} = \overline{C_{11}^{n-1}}.$$

Suppose that $C_1^{n-1} \subset \lambda[D_1(B)]$. Then $\sum_{y=2}^{b} \overline{C_y^{n-1}} \subset \lambda[D_2(B)]$. It can be shown that the set $\lambda[D_2(B)] - \overline{C_2^{n-1}}$ is homeomorphic with the set $\lambda[B] - \overline{C_2^{n-1}}$. Therefore $\lambda[D_2(B)] - \overline{C_2^{n-1}}$ is an $(n-1)$-cell whose λ-set is $\lambda[C_2^{n-1}]$. There exists C_{21}^{n-1} such that $\lambda[C_{21}^{n-1}] = \lambda[C_2^{n-1}]$ and $\Delta[C_{21}^{n-1}]D_2(B) \neq 0$.

$$D_2(B) = D_{21}(B) + D_{22}(B) + C_{21}^{n-1};\ \overline{D_{21}(B)} \cdot \overline{D_{22}(B)} = \overline{C_{21}^{n-1}}.$$

Suppose that $C_2^{-1} \subset \lambda[D_{21}(B)]$. Then $\sum_{y=3}^{b} \overline{C_y^{n-1}} \subset \lambda[D_{22}(B)]$. Proceeding in this manner, we finally arrive at the following result: $B = \sum_{j=1}^{b+1} B_j + \sum_{j=1}^{b} C_{j1}^{n-1}$, where B_j is a descendant of B, $\lambda[B_j] = \overline{C_j^{n-1}} + \overline{C_{j1}^{n-1}}$, $j = 1, 2, \cdots, b$, and $\lambda[B_{b+1}] = \sum_{j=1}^{b} \overline{C_{j1}^{n-1}}$.

Subject each member of A to the same sort of subdivision. The result is a set of descendants of neighborhoods satisfying the requirements of the lemma.

5.9. Let N_α be the same as in Lemma 8 and the sets P_k, which occur in this lemma, be the sets $\overline{N}_\alpha \cdot N_{\rho_k}$, where N_{ρ_k} is a neighborhood belonging to a finite set of neighborhoods which cover $\overline{N}_\alpha$. Then $\overline{N}_\alpha = \sum_{j=1}^{r} D_j$ (Lemma 8). Each set D_j is the descendant of a neighborhood. As in Lemma 8, denote by T_1 the set of all such neighborhoods. Let $\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots$ be a sequence of positive numbers such that $\epsilon_{n+1} < \epsilon_n$ and $\lim_{n \to \infty} \epsilon_n = 0$.

Designate by K the set in euclidean n-space E^n whose points have coordinates which satisfy the inequality $\sum_{i=1}^{n} x_i^2 < 1$. Then K is an n-cell and $\lambda[K]$ is an $(n-1)$-sphere. It will now be proved that $\overline{K} = \sum_{j=1}^{r} \overline{K}_j$, where the sets K_j are n-cells and that there exists a homeomorphism, $H(\sum_{j=1}^{r} \lambda[D_j]) = \sum_{j=1}^{r} \lambda[K_j]$, such that $H(\lambda[D_j]) = \lambda[K_j]$, $j = 1, 2, \cdots, r$. We shall show that, if the above statement is true for $r = m$, the statement is true for $r = m+1$.

Suppose that we have a subdivision of $\overline{N}_\alpha$ of the sort described in Lemma 8 in which $r=m+1$. Then $\overline{N}_\alpha=\sum_{j=1}^{m+1}\bar{D}_j$. Consider the set D_1. The discussion in Lemma 8 shows that there exists a set D_κ such that $\bar{D}_1\cdot\bar{D}_\kappa=\overline{C_{1\kappa}^{n-1}}$, none of the sets $\lambda[D_j]$, other than $\lambda[D_1]$ and $\lambda[D_\kappa]$, contains a point of $C_{1\kappa}^{n-1}$, $\lambda[D]-\overline{C_{1\kappa}^{n-1}}$ is an $(n-1)$-cell $C_{\mu 1}^{n-1}$, $\lambda[D_\kappa]-\overline{C_{1\kappa}^{n-1}}$ is an $(n-1)$-cell $C_{\mu\kappa}^{n-1}$, and $\overline{C_{\mu 1}^{n-1}}\cdot\overline{C_{\mu\kappa}^{n-1}}=\lambda[C_{1\kappa}^{n-1}]=\lambda[C_{\mu 1}^{n-1}]=\lambda[C_{\mu\kappa}^{n-1}]$. Suppose that $\kappa=2$. Let D represent the set $D_1+D_2+C_{12}^{n-1}$. Since we are assuming that our present sets D_j result from a process similar to that employed in Lemma 8, it is clear that D is strongly homeomorphic with N_α. $\lambda[D]$ is the $(n-1)$-sphere $\overline{C_{\mu 1}^{n-1}}+\overline{C_{\mu 2}^{n-1}}$. $\overline{N}_\alpha=\bar{D}+\sum_{j=3}^{m+1}\bar{D}_j$. The sets D, D_j, $\lambda[D]$, $\lambda[D_j]$, $j=3, 4, \cdots, m+1$, have the properties and the relations among themselves ascribed to the corresponding sets obtained in Lemma 8. Since it is assumed that the statement under consideration is true for $r=m$, $\overline{K}=\sum_{j=1}^{m}\overline{K}_j$ and there exists a homeomorphism, $H_1(\lambda[D]+\sum_{j=3}^{m+1}\lambda[D_j])=\sum_{j=1}^{m}\lambda[K_j]$, such that $H_1(\lambda[D])=\lambda[K_1]$ and $H_1(\lambda[D_j])=\lambda[K_k]$, $j=k+1$, $k=2, 3, \cdots, m$. $H_1(\overline{C_{\mu j}^{n-1}})$ is a closed $(n-1)$-cell $\overline{C_j^{n-1}}$, $j=1, 2$. $\overline{C_1^{n-1}}\cdot\overline{C_2^{n-1}}=\lambda[C_j^{n-1}]$, $j=1, 2$. Then, by Theorem P_3 there exists an $(n-1)$-cell C_3^{n-1} such that $\lambda[C_3^{n-1}]=\lambda[C_j^{n-1}]$, $j=1, 2$, and $\Delta[C_3^{n-1}]K_1\neq 0$. $H_1(\lambda[C_{12}^{n-1}])=\lambda[C_3^{n-1}]$. By Theorem P_1 there is a homeomorphism, $H_2(\overline{C_{12}^{n-1}})=\overline{C_3^{n-1}}$, such that $H_2(\lambda[C_{12}^{n-1}])=H_1(\lambda[C_{12}^{n-1}])$. Since $\Delta[C_3^{n-1}]K_1\neq 0$, $K_1=K_{11}+K_{12}+C_3^{n-1}$. K_{1j}, $j=1, 2$, is an n-cell. There exists a homeomorphism, $H_3(\sum_{j=1}^{m+1}\lambda[D_j])=\lambda[K_{11}]+\lambda[K_{12}]+\sum_{j=2}^{m}\lambda[K_j]$, in which $H_3(\lambda[D_j])=H_3(\overline{C_{\mu j}^{n-1}}+\overline{C_{12}^{n-1}})=H_1(\overline{C_{\mu j}^{n-1}})+H_2(C_{12}^{n-1})=\lambda[K_{1j}]$, $j=1, 2$, and $H_3(\lambda[D_j])=\lambda[K_k]$, $j=k+1$, $k=2, 3, \cdots, m$. The statement under consideration holds for $r=1$. The induction is complete.

We have the desired result $\overline{K}=\sum_{j=1}^{r}\overline{K}_j$, and there is a homeomorphism, $H(\sum_{j=1}^{r}\lambda[D_j])=\sum_{j=1}^{r}\lambda[K_j]$, such that $H(\lambda[D_j])=\lambda[K_j]$, $j=1, 2, \cdots, r$. Suppose that K_ξ is one of the sets K_j and that $d(K_\xi)\geqq\epsilon_1$.* Let R be the set of all points of E^n which have coordinates x_i such that $0<x_i<1$, $i=1, 2, \cdots, n$. There exists a homeomorphism $H_\theta(\overline{R})=\overline{K}_\xi$. The correspondence established by H_θ is uniformly continuous both ways. Therefore, there is a number $\delta_1>0$ such that, if the distance between any two points of $\overline{R}$ is less than δ_1, the distance between the corresponding two points of $\overline{K}_\xi$ as given by H_θ is less than ϵ_1. Let m be the smallest integer greater than $n^{1/2}/\delta_1$ and let $t=1/m$. Then $t<\delta_1/n^{1/2}$.

Consider the $(n-1)$-dimensional planes in E^n whose equations are

$$x_i = yt, \quad i = 1, 2, \cdots, n; \; y = 1, 2, \cdots, m-1.$$

These $(n-1)$-dimensional planes are $n(m-1)$ in number and separate R into m^n n-cells R_j such that $\overline{R}=\sum_{j=1}^{m^n}\overline{R}_j$. Let $H_\theta(\overline{R}_j)=\overline{K}_{\xi j}$, $j=1, 2, \cdots, m^n$. If

* The symbol $d(M)$ is used to denote the diameter of the set M.

two points belong to a set $\overline{R}_j$, the difference between the x_i-coordinates $(i=1, 2, \cdots, n)$ of these two points is at most t. Since $t<\delta_1/n^{1/2}$, the distance between these two points is less than δ_1. Then $d(\overline{K}_{\xi j})<\epsilon_1$, $j=1, 2, \cdots, m^n$. Corresponding to K_ξ is the set D_ξ which belongs to N_α. By successive applications of an argument used above, it can be shown that $\bar{D}_\xi=\sum_{j=1}^{m^n}\bar{D}_{\xi j}$, where $D_{\xi j}$ is a descendant of D_ξ, and there exists a homeomorphism, $H_\xi(\sum_{j=1}^{m^n}\lambda[D_{\xi j}])=\sum_{j=1}^{m^n}\lambda[K_{\xi j}]$, such that $H_\xi(\lambda[D_{\xi j}])=\lambda[K_{\xi j}]$ and $H_\xi(\lambda[D_\xi])=H(\lambda[D_\xi])=\lambda[K_\xi]$.

Suppose that this procedure is followed in the case of every set K_j whose diameter is not less than ϵ_1. The final result is that $\overline{N}_\alpha$ and $\overline{K}$ can be expressed as follows:

$$\overline{N}_\alpha = \sum_{j=1}^{r_1} \overline{D_j^{(1)}},$$

$$\overline{K} = \sum_{j=1}^{r_1} \overline{K_j^{(1)}}.$$

$d(K_j^{(1)})<\epsilon_1$, $D_j^{(1)}$ is strongly homeomorphic with N_α, $K_j^{(1)}$ is an n-cell and $\sum_{j=1}^{r}\lambda[D_j]\subset\sum_{j=1}^{r_1}\lambda[D_j^{(1)}]$. There exists a homeomorphism, $H_{\psi_1}(\sum_{j=1}^{r_1}\lambda[D_j^{(1)}])=\sum_{j=1}^{r_1}\lambda[K_j^{(1)}]$, such that $H_{\psi_1}(\lambda[D_j^{(1)}])=\lambda[K_j^{(1)}]$, $j=1, 2, \cdots, r_1$, and $H_{\psi_1}(\lambda[D_j])=H(\lambda[D_j])$, $j=1, 2, \cdots, r$.

Assign to each point p of $\overline{N}_\alpha$ the neighborhood N_{b_p} such that N_{b_p} is the neighborhood of $G(p)$ (§5.1) of least subscript greater than 2 having the following properties: $\overline{N}_{b_p}$ belongs to every neighborhood of T_1 which contains p; if N_ξ belongs to T_1 and $\overline{N}_\xi \not\ni p$, then $\overline{N}_{b_p}\cdot\overline{N}_\xi=0$; $N_{b_p}\not\ni D_j^{(1)}$ for all values of j. There exists a finite subset of the set of all such neighborhoods: $T_2=\{N_{\phi_j}\}$, $j=1, 2, \cdots, h$, such that $\sum_{j=1}^{h}N_{\phi_j}\supset\overline{N}_\alpha$ and $N_{\phi_s}\not\ni N_{\phi_t}$, $s\neq t$.

Let $D_\mu^{(1)}$ be any one of the sets $D_j^{(1)}$. Suppose that $N_{\phi_m}\cdot D_\mu^{(1)}\neq 0$. Then $N_{\phi_m}\cdot\overline{D_\mu^{(1)}}$ is an open set with respect to $\overline{D_\mu^{(1)}}$. Since $D_\mu^{(1)}$ is strongly homeomorphic with N_α, Lemma 8 is true if $D_\mu^{(1)}$ and the sets $N_{\phi_m}\cdot\overline{D_\mu^{(1)}}$ are substituted for N_α and the sets P_k respectively. Then $D_\mu^{(1)}=\sum_{t=1}^{e}D_{\mu t}^{(1)}$, and the sets $D_{\mu t}^{(1)}$ and $\lambda[D_{\mu t}^{(1)}]$ have the properties and the relations among themselves ascribed to the corresponding sets in Lemma 8. Corresponding to $D_\mu^{(1)}$ is the set $K_\mu^{(1)}$. Then $\overline{K_\mu^{(1)}}=\sum_{t=1}^{e}\overline{K_{\mu t}^{(1)}}$. $K_{\mu t}^{(1)}$ is an n-cell and there exists a homeomorphism, $H_{\rho_1}(\sum_{t=1}^{e}\lambda[D_{\mu t}^{(1)}])=\sum_{t=1}^{e}\lambda[K_{\mu t}^{(1)}]$, such that $H_{\rho_1}(\lambda[D_{\mu t}^{(1)}])=\lambda[K_{\mu t}^{(1)}]$, $i=1, 2, \cdots, e$, and $H_{\rho_1}(\lambda[D_\mu^{(1)}])=\lambda[K_\mu^{(1)}]$. $H_{\psi_1}H_{\rho_1}^{-1}(\lambda[K_\mu^{(1)}])=\lambda[K_\mu^{(1)}]$. Let H_{ρ_2} denote the transformation $H_{\psi_1}H_{\rho_1}^{-1}$. Then by Corollary, Theorem P_1 there is a homeomorphism, $H_{\rho_2}(\overline{K_\mu^{(1)}})=\overline{K_\mu^{(1)}}$, such that $H_{\rho_2}(\lambda[K_\mu^{(1)}])=H_{\rho_2}(\lambda[K_\mu^{(1)}])$. Let $H_{\rho_2}(\overline{K_{\mu t}^{(1)}})=\overline{K_{\mu t}^{(1,1)}}$. $H_{\rho_2}(\lambda[K_{\mu t}^{(1)}])=\lambda[K_{\mu t}^{(1,1)}]$. If H_δ is used to denote the transformation $H_{\rho_2}H_{\rho_1}$,

$$H_\delta\left(\sum_{i=1}^{e}\lambda[D_{\mu i}^{(1)}]\right) = H_{\rho_4}H_{\rho_1}\left(\sum_{i=1}^{e}\lambda[D_{\mu i}^{(1)}]\right) = H_{\rho_2}\left(\sum_{i=1}^{e}\lambda[K_{\mu i}^{(1)}]\right)$$

$$= \sum_{i=1}^{e}\lambda[K_{\mu i}^{(1,1)}],$$

$$H_\delta(\lambda[D_{\mu i}^{(1)}]) = \lambda[K_{\mu i}^{(1,1)}],$$

$$H_\delta(\lambda[D_{\mu}^{(1)}]) = H_{\rho_4}H_{\rho_1}(\lambda[D_{\mu}^{(1)}]) = H_{\rho_3}(\lambda[K_{\mu}^{(1)}]) = H_{\rho_2}(\lambda[K_{\mu}^{(1)}])$$

$$= H_{\psi_1}H_{\rho_1}^{-1}(\lambda[K_{\mu}^{(1)}]) = H_{\psi_1}(\lambda[D_{\mu}^{(1)}]).$$

Similar results can be obtained for any two corresponding sets $D_j^{(1)}$ and $K_j^{(1)}$.

By the process described above, we can obtain the following

$$\overline{N}_\alpha = \sum_{j=1}^{r_2}\overline{D_j^{(2)}}, \qquad \overline{K} = \sum_{j=1}^{r_2}\overline{K_j^{(2)}},$$

$D_j^{(2)}$ is strongly homeomorphic with N_α, $K_j^{(2)}$ is an n-cell, and $d(K_j^{(2)}) < \epsilon_2$, $j=1, 2, \cdots, r_2$. $\sum_{j=1}^{r_1}\lambda[D_j^{(1)}] \subset \sum_{j=1}^{r_2}\lambda[D_j^{(2)}]$ and $\sum_{j=1}^{r_1}\lambda[K_j^{(1)}] \subset \sum_{j=1}^{r_2}\lambda[K_j^{(2)}]$. There exists a homeomorphism, $H_{\psi_2}(\sum_{j=1}^{r_2}\lambda[D_j^{(2)}]) = \sum_{j=1}^{r_2}\lambda[K_j^{(2)}]$, such that $H_{\psi_2}(\lambda[D_j^{(2)}]) = \lambda[K_j^{(2)}]$ and $H_{\psi_2}(\sum_{j=1}^{r_1}\lambda[D_j^{(1)}]) = H_{\psi_1}(\sum_{j=1}^{r_1}\lambda[D_j^{(1)}])$.

Continuing the process, we can obtain the sets $F_1 = \sum_{k=1}^{\infty}\sum_{j=1}^{r_k}\lambda[D_j^{(k)}]$ and $F_2 = \sum_{k=1}^{\infty}\sum_{j=1}^{r_k}\lambda[K_j^{(k)}]$, such that

$$\overline{N}_\alpha = \sum_{j=1}^{r_k}\overline{D_j^{(k)}}, \qquad \overline{K} = \sum_{j=1}^{r_k}\overline{K_j^{(k)}},$$

$D_j^{(k)}$ is strongly homeomorphic with N_α, $K_j^{(k)}$ is an n-cell, $d(K_j^{(k)}) < \epsilon_k$, $\sum_{j=1}^{r_{k-1}}\lambda[D_j^{(k-1)}] \subset \sum_{j=1}^{r_k}\lambda[D_j^{(k)}]$, $\sum_{j=1}^{r_{k-1}}\lambda[K_j^{(k-1)}] \subset \sum_{j=1}^{r_k}\lambda[K_j^{(k)}]$ and, for every value of k, there is a homeomorphism, $H_{\psi_k}(\sum_{j=1}^{r_k}\lambda[D_j^{(k)}]) = \sum_{j=1}^{r_k}\lambda[K_j^{(k)}]$, such that $H_{\psi_k}(\lambda[D_j^{(k)}]) = \lambda[K_j^{(k)}]$ and $H_{\psi_k}(\sum_{j=1}^{r_{k-1}}\lambda[D_j^{(k-1)}]) = H_{\psi_{k-1}}(\sum_{j=1}^{r_{k-1}}\lambda[D_j^{(k-1)}])$.

Corresponding to T_2, the set of neighborhoods used in obtaining the sets $D_j^{(2)}$, there is a set of neighborhoods T_k employed in a similar manner to obtain the sets $D_j^{(k)}$. The neighborhoods of T_k are determined as follows: Assign to each point p of $\overline{N}_\alpha$ the neighborhood N_{c_p} such that N_{c_p} is the neighborhood of $G(p)$ of least subscript greater than k having the following properties: $\overline{N}_{c_p}$ belongs to every neighborhood of T_j, $j=1, 2, \cdots, k-1$, which contains p; if N_η belongs to T_j, $j=1, 2, \cdots, k-1$, and $N_\eta \not\ni p$, then $\overline{N}_{c_p} \cdot \overline{N}_\eta = 0$; $N_{c_p} \not\ni D_j^{(k-1)}$ for all values of j. There exists a finite subset of the set of all such neighborhoods: $T_k = \{N_{\kappa_j}\}$, $j=1, 2, \cdots, g$, such that $\sum_{j=1}^{g}N_{\kappa_j} \supset \overline{N}_\alpha$ and $N_{\kappa_a} \not\ni N_{\kappa_b}$, $a \neq b$.

We can now conclude without difficulty that, if p is a point of $\overline{N}_\alpha$, there exists a sequence of neighborhoods $\{N_{\delta_j}\}$, $j=1, 2, \cdots$, such that $N_{\delta_j} \supset p$,

$N_{\delta_a} \neq N_{\delta_b}$, $a \neq b$, N_{δ_j} belongs to T_{k_j}, $k_j < k_{j+1}$, and $N_{\delta_{j+1}} \subset N_{\delta_j}$. By Lemma 1, $\prod_{j=1}^{\infty} \overline{N}_{\delta_j} = p$.

Every neighborhood of T_k which contains a given point p of $\overline{N}_\alpha$ contains all the sets $D_j^{(k)}$ for which $\overline{D_j^{(k)}} \supset p$. If the sets $D_{\mu_k}^{(k)}$, $k=1, 2, \cdots$, are such that $D_{\mu_k}^{(k)} \subset D_{\mu_{k-1}}^{k-1}$, $k>1$, then $\prod_{k=1}^{\infty} \overline{D_{\mu_k}^{(k)}}$ contains a point p, since $\overline{D_{\mu_k}^{(k)}}$ is a compact set. Then, by means of the result stated in the preceding paragraph, we can infer that $\prod_{k=1}^{\infty} \overline{D_{\mu_k}^{(k)}} = p$.

Let p be a point of $\overline{N}_\alpha - F_1$ and $D_{\mu_k}^{(k)}$, the set $D_j^{(k)}$ which contains p. It is evident that $p = \prod_{k=1}^{\infty} D_{\mu_k}^{(k)} = \prod_{k=1}^{\infty} \overline{D_{\mu_k}^{(k)}}$. Let $K_{\mu_k}^{(k)}$ be the set which corresponds to $D_{\mu_k}^{(k)}$. $\prod_{k=1}^{\infty} K_{\mu_k}^{(k)}$ consists of one point p'. The point p' cannot belong to F_2. For, if $F_2 \supset p'$, the point p' appears in F_2 for the first time in the set $\lambda[K_{\mu_m}^{(m)}]$, $m \geq 1$. Then $p' = \prod_{k=m}^{\infty} \lambda[K_{\mu_k}^{(k)}]$. Under the circumstances, however, $\prod_{k=m}^{\infty} \lambda[D_{\mu_k}^{(k)}]$ would be non-vacuous. This is impossible, since $\prod_{k=m}^{\infty} \lambda[D_{\mu_k}^{(k)}] = 0$. Then $\overline{K} - F_2 \supset p'$.

Make every point p of $\overline{N}_\alpha - F_1$ to correspond to the point p' of $\overline{K} - F_2$ obtained in the manner just described. The process used in obtaining the sets F_1 and F_2 establishes a (1, 1) correspondence between the points of F_1 and the points of F_2. We now have a (1, 1) correspondence between the points of $\overline{N}_\alpha$ and the points of $\overline{K}$. This correspondence is continuous both ways. Then $\overline{N}_\alpha$ is homeomorphic with $\overline{K}$. Therefore Z^n, which is homeomorphic with $\overline{N}_\alpha$ is a closed n-cell and I^n is an n-cell.

6. **Proof that the conditions in Theorem I′ are necessary.** The proof which follows is applicable when $n>1$. The slight modifications necessary when $n=1$ are obvious.

In euclidean n-space let K be the set of all points whose coordinates x_j satisfy the condition $0 < x_j < 1$, $j=1, 2, \cdots, n$. K is an n-cell and $\lambda[K]$ is an $(n-1)$-sphere. Let $N_k^{(9)}$ be the irrational number $(.99 \cdots 9)^{1/2}$ in which the symbol 9 occurs k times and $N_k^{(8)}$, the irrational number similarly defined in which the symbol 8 takes the place of 9. The neighborhoods N_i to be defined belong to certain sets L_i, $i=1, 2, 3, \cdots$.

The neighborhoods belonging to the set L_1 are obtained as follows: Denote by e_1 the number $1 - N_1^{(9)}$. Let h_1 be the smallest integer greater than $1/e_1$ and m_1, the number $1/h_1$. Then $m_1 < e_1$. The $(n-1)$-dimensional planes $x_j = tm_1$ ($j=1, 2, \cdots, n$; $t=1, 2, \cdots, h_1-1$) separate K into h_1^n sets K_{y_1} such that $\overline{K} = \sum_{y_1=1}^{h_1^n} \overline{K}_{y_1}$. Each set K_{y_1} is the interior of an $(n-1)$-dimensional cube whose edges are equal in length to m_1. Denote by r_{y_1} the smallest positive integer such that $N_{y_1}^{(8)}/r_{y_1} < m_1/2$, and by g_{y_1} the number $N_{y_1}^{(8)}/r_{y_1}$. Let the point $(x_1^{y_1}, x_2^{y_1}, \cdots, x_n^{y_1})$ be the point of $\overline{K}_{y_1}$ which is equidistant from the vertices of $\overline{K}_{y_1}$. Denote by M_{y_1} the set whose points have coordinates

x_j which satisfy the condition $x_j^{y_1}+g_{y_1}-e_1<x_j<x_j^{y_1}+g_{y_1}+e_1$. Then $\overline{K}_{y_1}\subset M_{y_1}$. The neighborhoods of L_1 are the sets $\overline{K}\cdot M_{y_1}$, $y_1=1, 2, \cdots, h_1^n$.

The neighborhoods belonging to the set L_2 are obtained as follows: Let E_{21}^{n-1} be an $(n-1)$-dimensional plane containing an $(n-1)$-dimensional face of $\overline{K}$ or of a set $\overline{M}_{y_1}$ and E_{22}^{n-1}, an $(n-1)$-dimensional plane containing an $(n-1)$-dimensional face of a set $\overline{M}_{y_1}$ such that $E_{21}^{n-1}\cdot E_{22}^{n-1}=0$. Designate by δ_2 the lower bound of the set of numbers which give the distances between all such pairs of $(n-1)$-dimensional planes. Denote by e_2 the number $1-N_{q_2}^{(9)}$, where q_2 is the smallest integer such that $6e_2<\delta_2$. Define the numbers h_2, m_2, r_{y_2} and g_{y_2} by the method used in obtaining the corresponding numbers in the preceding paragraph and obtain the sets K_{y_2} and M_{y_2}, $y_2=1, 2, \cdots, h_2^n$, which correspond to the sets K_{y_1} and M_{y_1}. The neighborhoods which belong to L_2 are the sets $\overline{K}\cdot M_{y_2}$, $y_2=1, 2, \cdots, h_2^n$.

In general, the neighborhoods belonging to the set L_i are obtained as follows:

Let E_{i1}^{n-1} be an $(n-1)$-dimensional plane containing an $(n-1)$-dimensional face of $\overline{K}$ or of a set $\overline{M}_{yj}$, $j=1, 2, \cdots, i-1$, and E_{i2}^{n-1}, an $(n-1)$-dimensional plane containing an $(n-1)$-dimensional face of a set $\overline{M}_{yk}$, $k=1, 2, \cdots, i-1$, such that $E_{i1}^{n-1}\cdot E_{i2}^{n-1}=0$. Denote by δ_i the lower bound of the set of numbers which give the distances between all such pairs of $(n-1)$-dimensional planes. Let e_i be the number $1-N_{q_i}^{(9)}$, where q_i is the smallest integer such that $6e_i<\delta_i$. Define the numbers h_i, m_i, r_{yi}, and g_{yi} by the method used in obtaining the corresponding numbers in the preceding paragraph and obtain the sets K_{yi} and M_{yi}. The neighborhoods belonging to L_i are the sets $\overline{K}\cdot M_{yi}$, $y_i=1, 2, \cdots, h_i^n$.

If a is a given value of i, no $(n-1)$-dimensional face of one set $\overline{M}_{ya}$ is in the same $(n-1)$-dimensional plane with an $(n-1)$-dimensional face of a second set $\overline{M}_{ya}$. No $(n-1)$-dimensional face of a set $\overline{M}_{yb}$ is in the same $(n-1)$-dimensional plane with an $(n-1)$-dimensional face of a set $\overline{M}_{yc}$, $b\neq c$.

When $\overline{K}$ and K are substituted for Z^n and I^n respectively in Theorem I′, all of the conditions of the theorem are satisfied.

7. **Proof of Theorem I.** We can conclude from the result obtained in (5.9) that, *if Z^{n-1} is a space such that I^{n-1} is an $(n-1)$-cell, Z^n is a closed n-cell and I^n is an n-cell.* By Theorem I (4.4), Z^0 is a closed 0-cell and I^0 is a 0-cell. The proof by induction that the conditions in Theorem I are sufficient follows immediately. The proof that the conditions are necessary is found in §6.

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PARTIALLY ORDERED SETS*

BY

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1. **Introduction.** A relation, $\subset$, *orders* a set, K, of elements, $a, b, c, \cdots$, if Postulates 1.1 and 1.2 hold for any elements of K.

1.1 POSTULATE. *$a \subset b$ and $b \subset c$ imply $a \subset c$.*

1.2 POSTULATE. *$a \subset a$.*

Such a set need not possess simple or linear order and for this reason is commonly called a partially ordered set.‡ A large part of the theory of simply ordered sets applies, with little or no change, to partially ordered sets. Although the principal objectives of the paper are certain properties of partial order which have either a trivial or no counterpart in the theory of simple order, considerable use is made of the parallelism between the two theories without a systematic development of it. This parallelism makes it natural to call any set which satisfies Postulates 1.1 and 1.2 an *ordered set*, dropping the word "partially." This convention leads to no confusion in this paper as but scant reference is made to simple order. $a \subset b$ is read *a is contained in b.*

The relations of *equality* and *equivalence* order any set in which they occur. *Being a subset of* is a relation ordering the subsets of a given set. The relation *less than or equal to* orders the set of real numbers. A set of propositions is ordered by the relation of *implication*. An ordering relation can be assigned to any system, such as a Boolean algebra, which has an associative operation with respect to which every element is idempotent. The relations of *being homomorphic to* and *being a subsystem of* order any aggregate of classes with a common set of operations. These examples do not begin to exhaust even the

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† This paper contains the results of a doctoral dissertation, *Extensions of partially ordered sets*, accepted by Harvard University in May, 1935. It also contains further results obtained while the author was a Sterling Fellow at Yale University, 1935–36.

‡ Hausdorff (10), p. 139, Ore (18) p. 408, von Neumann (17), p. 94, and Alexandroff (1) p. 1650. Alexandroff (1) shows that discrete spaces in which $\overline{A} = \overline{B}$ implies $A = B$ are abstractly equivalent to partially ordered sets where $A \subset B$ if, and only if, A is an element of $\overline{B}$. Bennett (2) speaks of such sets as having semi-serial order. Tucker (26) calls such sets cell spaces and makes the relation $<$ (§2), instead of $\subset$, fundamental.

The numbers in parentheses refer to the bibliography at the end of the paper.

common instances of ordered sets in mathematics.*

This paper presupposes the calculus of classes without restriction as to cardinal number. The classes employed are defined by such properties that the admission or rejection of a candidate does not depend upon the disposition of candidates under consideration or not yet considered. Hence, the paradoxes connected with inconsistent aggregates do not arise and no recourse to the theory of types is necessary. In fact, no classes are employed where the admission or rejection of a candidate depends upon the disposition of elements already considered. The definition of such classes depends upon a particular well ordering of the candidates. For the classes here considered, the possibility of well ordering the candidates is not assumed.

The concept of a class or set, K, can be specialized by the imposition of correspondences between subsets of its elements. A correspondence can be considered as a subclass of the class of ordered pairs of subsets of the given set. Hence, the only new undefined concept needed for the introduction of correspondences is that of an ordered pair.† If, under a correspondence, a subset A corresponds to a subset B, $A \rightarrow B$, then B must be uniquely determined by A and the definition of the correspondence. A is called the *original* of B and B is called the *image* of A. The *domain* of a correspondence in K is the class of subsets which have an image in K. The *range*‡ is the class of subsets which have at least one original in K. The definition of a correspondence may restrict the subsets admissible to the domain both as to the particular elements and the number of elements they may contain. The correspondence may also impose such restrictions on the subsets admissible to the range. A set K is *closed* under a correspondence if the image of every admissible original exists in K. The ordering relation may be defined as a correspondence if the domain is restricted to single elements of K and the image of any element a is the set B of all elements b_j such that $a \subset b_j$. The closure of K under this correspondence is assured by Postulate 1.2. The relations, operations, and homomorphisms, the systematic exploitation of which is the object of this paper, are further instances of correspondences.

In terms of the ordering relation, new relations are defined in §2 and operations in §3. An ordered set is not necessarily closed under the opera-

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* For example, the order preserving mappings, $a \rightarrow D(a)$, of an ordered set K upon the interval $[0, 1]$ of real numbers, such that $D(a)+D(b)=D(a+b)+D(ab)$, where addition and multiplication in K are defined in §3 and ordinary addition is used for the real numbers, have as special cases the theories of probability, measure, and dimensionality in projective geometries. See von Neumann (17) and Freudenthal (9).

† Quine (20), p. 10, for classes, p. 18 et seq.

‡ Stone (22), p. 33.

tions of §3, though closure under certain operations implies closure under others. The relations between closure under various subsets of the operations of §3 are worked out in §4 and §7. This forms a basis for the classification of ordered sets with respect to the operations under which they are closed, as equivalent types can now be recognized. Multiplicative systems, lattices or structures, and Boolean algebras are among the distinct types of ordered sets included in this classification.

The object of this classification is to establish content for the extensions which form the principal part of this paper. The purpose of an extension is to adjoin elements to a given set so as to increase the number of operations under which it is closed. This is usually accomplished by the construction of a new set with the desired closure properties and with a subset isomorphic with the given set. These constructions depend upon the theory of homomorphisms developed in §5. Methods of constructing and methods of combining extensions are discussed in §9. In the following sections, the adjunction of units, unrestricted sums and products, unrestricted distributive sums, and complements is attained. By successive application of these extensions, any ordered set can be imbedded in a complete Boolean algebra. §9 contains a more detailed statement of the program of extensions carried out in this paper.

2. **Definition of new relations.** In an ordered set, if $a \subset b$ does not hold, we write $a \not\subset b$ and read *a is not contained in b*. The proofs of theorems, in cases where they are practically automatic, have been omitted. The propositions which follow refer to a set, K, of elements, $a, b, c, \cdots$, ordered by the relation $\subset$.

2.1 THEOREM. *Every subset of an ordered set is an ordered set.*

2.2 DEFINITION. $a \supset b$ if $b \subset a$.

2.3 THEOREM. *The relation* $\supset$ *orders the set* K.

By virtue of Definition 2.2, an exact duality exists between the relations $\subset$ and $\supset$. Hence, any general proposition holding for the relation $\subset$ implies a dual proposition for the relation $\supset$. The relations $\subset$ and $\supset$ are not mutually exclusive. For example, $a \subset a$ and $a \supset a$. To obtain, mutually exclusive relations we give the following four definitions:

2.4 DEFINITION. $a = b$ if $a \subset b$ and $b \subset a$.

2.5 DEFINITION. $a < b$ if $a \subset b$ and $b \not\subset a$.

2.6 DEFINITION. $a > b$ if $a \not\subset b$ and $b \subset a$.

2.7 DEFINITION. $a \| b$ if $a \not\subset b$ and $b \not\subset a$.

2.8 THEOREM. *Exactly one of the relations* $=, <, >,$ *and* $\|$ *holds for any ordered pair of* K *elements.*

These four mutually exclusive relations are usually used in defining ordered sets. An ordered set in which the relation $\|$ does not occur is a simply ordered set or a set with linear order. It frequently happens that the relations $<$ and $>$ do not occur in an ordered set. There is also the trivial case where all elements are equal. With these exceptions, all four relations occur in any ordered set. The equality defined by Definition 2.4 satisfies the usual postulates for equality and is an ordering relation.

2.9 THEOREM. *Postulates* 1.1 *and* 1.2 *are consistent and completely independent.*

Proof: The necessary examples can be constructed with three elements.

2.10 THEOREM. *If* $a=b$ *and* $c=d$, *then* $a \subset c$ *implies* $b \subset d$ *and* $a \not\subset c$ *implies* $b \not\subset d$. *More generally, equals may be substituted for equals in an ordered set without effecting the ordering relation.*

Let A be an ordered set of elements, a_i, to each distinct element of which an ordered set, $A_i \longleftrightarrow a_i$, of elements, a_{ij}, has been assigned. The set B of elements $b_{ij} \longleftrightarrow a_{ij}$ is defined as any set of elements in a one-to-one correspondence with the totality of elements in the sets A_i. A relation, $\subset$, for B is defined as follows: $b_{ij} \subset b_{k1}$ if either $a_i \subset a_k$ and $a_i \not\supset a_k$, or $a_i = a_k$ and $a_{ij} \subset a_{k1}$. This is the product (in the sense of Hausdorff (11), p. 46), ordered lexicographically, of two ordered sets. The sum (in the sense of Hausdorff (11), p. 44) is a special case of the product. This is an example of a property commonly restricted to simply ordered sets which generalizes at once to partially ordered sets.

2.11 THEOREM. *The set* B, *as defined above, is ordered by the relation* $\subset$, *as defined above, if, and only if, the sets* A *and* A_i *are ordered.*

3. **Definition of operations.** Given an ordered set, K, it is possible in various ways to define correspondences, $A \rightarrow B$, between subsets of K. For example, let $A \rightarrow B$ if b_j is in B if, and only if, $b_j \subset a_i$ for every a_i in A. In such situations, we can regard the correspondence as defining B as a function of A or as defining an operation upon A. We are particularly interested in certain correspondences where the elements of the set B are all necessarily equal. By Theorem 2.10, equals may be substituted for equals in ordered sets, so we can represent such sets by a single element. In general, correspondences between subsets and single elements will be referred to as operations. As a rule, these operations are relative to the set K for which they are defined and are not necessarily the same when defined for subsets of K. Thus, to avoid ambiguity, it is necessary to specify the set relative to which the operations are defined. We shall denote subsets of K by capital letters with or without subscripts and elements of these subsets by the same small letter with one

more subscript. For example, a_i is an element of A, b_{jk} of B_j, and so on. Because of the duality between the relations $\subset$ and $\supset$ each definition yields two operations, one for each relation. The operations have been defined for the relation $\subset$ and the changes necessary to obtain the dual have been indicated in parentheses. The following definitions of operations are made with no assumption as to the closure of the arbitrary set, K, under them. The question of closure is treated in later sections. In the theorems of this section it is tacitly assumed that the subsets involved lie in the domain of the operations involved.

3.1 DEFINITION. a is a *product*, $\prod a_i$ (*sum*, $\sum a_i$) of A if $a \subset a_i$ ($a \supset a_i$) for every a_i in A and $x \subset a_i$ ($x \supset a_i$) for every a_i in A implies $x \subset a$ ($x \supset a$).†

3.2 THEOREM. *If a and b are both products (sums) of A, then $a=b$.*

3.3 THEOREM. *If $a=\prod a_i$ ($\sum a_i$) and $b=\prod b_j$ ($\sum b_j$) and, for each a_i in A, b_j exists in B such that $b_j \subset a_i$ ($b_j \supset a_i$), then $b \subset a$ ($b \supset a$).*

3.4 COROLLARY. *If $a=b$, then $ac=bc$ ($a+c=b+c$). More generally, equals may be substituted for equals in products (sums) without changing the value of the product (sum).*

3.5 COROLLARY. $a[bc]=[ab]c=abc$ $(a+[b+c]=[a+b]+c=a+b+c)$. *More generally, if each b_j is the product (sum) of a subset of A and, for each a_i in A, b_j exists in B such that $b_j \subset a_i$ ($b_j \supset a_i$), then $\prod a_i=\prod b_j$ ($\sum a_i=\sum b_j$).*

3.6 COROLLARY. $ab=ba$ $(a+b=b+a)$. *More generally, the value of a product (sum) is independent of any ordering or arrangement of the factors (summands).*

3.7 COROLLARY. $aa=a$ $(a+a=a)$. *More generally,* $\prod a=a$ $(\sum a=a)$.

3.8 DEFINITION. $\prod b_j$ ($\sum b_j$) *distributes* b with respect to $\prod a_i$ ($\sum a_i$) if $b=\prod b_j$ ($b=\sum b_j$) and, for each b_j, a_i exists such that $a_i \subset b_j$ ($a_i \supset b_j$).

3.9 DEFINITION. ab $(a+b)$ is *modular* with respect to a if every element, x, such that $ab \subset x \subset a$ $(a+b \supset x \supset a)$ can be distributed with respect to ab $(a+b)$.

3.10 DEFINITION. $\prod a_i$ ($\sum a_i$) is *distributive* if every element, x, such that $\prod a_i \subset x$ ($\sum a_i \supset x$) can be distributed with respect to $\prod a_i$ ($\sum a_i$).

3.11 THEOREM. *If $\prod a_i$ ($\sum a_i$) is distributive and $\prod a_i \subset b$ ($\sum a_i \supset b$), then $\prod b_j$ ($\sum b_j$) exists such that $\prod b_j$ ($\sum b_j$) is distributive and distributes b with respect to $\prod a_i$ ($\sum a_i$).*

† For the case of two factors this goes back to Peirce (19), pp. 32–33. See also Schröder (21) p. 196, Huntington (12), and Bennett (2).

3.12 THEOREM. *If* $a=\prod a_i$ $(a=\sum a_i)$ *and each* $a_i=\prod a_{ij}$ $(a_i=\sum a_{ij})$ *and if all these products (sums) are distributive, then* $a=\prod a_{ij}$ $(a=\sum a_{ij})$ *is distributive.*

3.13 THEOREM. $a=\prod a_i$ $(a=\sum a_i)$ *and* $\prod a_i$ $(\sum a_i)$ *is distributive, if* A *is the set of all elements* $a_i \supset a$ $(a_i \subset a)$.

3.14 DEFINITION. 0 (I) is a *zero* (*one*) of K if $0 \subset a$ $(\mathrm{I} \supset a)$ for every a in K.

3.15 THEOREM. *If* 0 (I) *and* b *are both zeros* (*ones*) *of* K, *then* $0=b$ $(\mathrm{I}=b)$.

Collectively, the elements 0 and I are called the *units* of K.

3.16 THEOREM. $a0=0$ $(a+\mathrm{I}=\mathrm{I})$ *and* $a+0=a$ $(a\mathrm{I}=a)$. *Furthermore,* $a0$ $(a+\mathrm{I})$ *and* $a+0$ $(a\mathrm{I})$ *are distributive.*

3.17 DEFINITION. a' (a^*) is a *product* (*sum*) *complement*† of a if $aa'=0$ $(a+a^*=\mathrm{I})$ and $ax=0$ $(a+x=\mathrm{I})$ implies $x \subset a'$ $(x \supset a^*)$.

3.18 THEOREM. *If* a' (a^*) *and* b *are both product* (*sum*) *complements of* a, *then* $a'=b$ $(a^*=b)$.

3.19 THEOREM. $a \subset [a']'$ $(a \supset [a^*]^*)$.

3.20 THEOREM. $a \subset b$ *implies* $b' \subset a'$ *and* $b^* \subset a^*$.

3.21 DEFINITION. b is a *complement* of a with respect to c and d if $a+b=c$ and $ab=d$.

Complements, unlike sum and product complements, are not, in general, unique. Sum and product complements may be defined with respect to elements other than 0 and I. This further distinction, though not needed in this paper, would prove useful in developing Stone's generalized Boolean algebras along the lines of this paper.

3.22 THEOREM. *If* $a'=a^*$, *then* a' *is the unique complement of* a *with respect to* I *and* 0.

4. **Classification of ordered sets by closure under operations.** We proceed to the classification of ordered sets with respect to the operations under which they are closed. The discussion of the different types of complements is postponed until after the treatment of lattices. We further limit the number of types to be considered by distinguishing only between operations on finite and on infinite sets. The classification might be continued indefinitely by discriminating between different orders of infinity in restricting the subsets admissible to the domain of an operation. Under these restrictions, there remain to be considered six possibilities of closure with respect to multiplication, together with their duals for addition.

† Collectively called pseudo-complements by Birkhoff (3), p. 459.

4.1 POSTULATE. *If a and b are in K, then ab $(a+b)$ is in K.*

4.2 POSTULATE. *If ab $(a+b)$ is in K, it is modular with respect to a.*

4.3 POSTULATE. *If ab $(a+b)$ is in K, it is distributive.*

4.4 POSTULATE. *K contains a unit* 0 (I), *that is, the product (sum) of all its elements.*

4.5 POSTULATE. *If A is a subset of K, then $\prod a_i$ $(\sum a_i)$ is in K.*

4.6 POSTULATE. *If $\prod a_i$ $(\sum a_i)$ is in K, it is distributive.*

Clearly the dual postulates are not equivalent to the originals, so we shall denote them by asterisks. For example, Postulate 4.1 relates to products while Postulate 4.1* is the corresponding postulate relating to sums.

These postulates, together with the postulates for ordered sets give us a system of 14 postulates. We wish to investigate the independence and consistency of all subsets of this system. Not all these postulates are independent. In each case we must either establish a relation between some combination of these postulates or show their independence by examples. The consistency of the entire set is demonstrated by an example. Fortunately, the large number of examples required can all be constructed by combining a few simple examples in various ways. The following examples are the ones required for these combinations.

4.7 **Example.** K consists of a single element, a, such that $a \subset a$.

4.8 **Example.** K consists of two elements, a and b, such that $a \subset a$ and $b \subset b$ and no further relations hold.

4.9 **Example.** K consists of three elements, a, b, and c, such that $a \subset a$, $b \subset b$, and $c \subset c$ and no further relations hold.

4.10 **Example.** 4.9 with the additional relation $a \subset b$.

4.11 **Example.** K consists of the positive integers in their natural order.

4.12 **Example.** K consists of the positive integers where $m \subset 2n-1$, if $m \leqq 2n-1$ in the natural order; $2m \subset 2n$, if $m \leqq n$ in the natural order; and no further relations hold.

4.13 **Example.** K consists of three elements, a, b, and c, such that $a \subset a$, $a \subset b$, $b \subset b$, $b \subset c$, and $c \subset c$ and no additional relations hold.

4.14 **Example.** K consists of the positive integers where $m \subset n$ if $m < n$ in the natural order.

Examples 4.7, 4.8, 4.9, 4.10, and 4.13 are self dualistic. We denote the duals of the remaining examples with an asterisk. All the examples except 4.13 and 4.14 are ordered sets. These examples may be combined in sequences, such that the assigned relation holds between any two elements of the same example and such that all the elements of any example are contained in every

element of every succeeding example. We shall enclose such sequences in brackets, for instance, [4.11, 4.7] consists of the integers in their natural order and a single element containing all of them. It follows from Theorem 2.11 that if the components of such a sequence are ordered sets, then the entire sequence is an ordered set.

The definitions of operations do not require that the relation $\subset$ be an ordering relation. Hence these definitions can be applied equally well to the examples which are not ordered sets and the postulates for closure under operations can be considered independently from those of ordering.

We shall study each postulate separately, determining all the combinations of the remaining postulates which imply it and constructing examples which show its independence from the remaining postulates by which it is not implied. By suitably combining these examples, an example can be constructed in which any given subset of the postulates, and those postulates shown to be implied by this subset, hold, while the remaining postulates do not hold.

4.15 THEOREM. *Postulate* 4.6 (4.6*) *is implied by no combination of the remaining postulates.*

Proof: Consider the example [4.7, 4.12*, 4.7]. Clearly, the product of the even integers is the zero of the set and every odd integer contains it, but no odd integer can be distributed with respect to it. The remaining postulates are satisfied.

4.16 LEMMA. *Postulates* 4.4 *and* 4.5* (4.4* *and* 4.5) *imply Postulate* 4.5 (4.5*).

Proof: If A is any subset of K and B is the set of all K elements such that $b_j \subset a_i$ for every element of A, then B is not void for it contains the element 0, which exists by 4.4, and $\sum b_j$ exists by 4.5*. But, by 3.1 and 3.1*, $\sum b_j = \prod a_i$. Therefore, 4.5 holds.

4.17 THEOREM. *Postulate* 4.5 (4.5*) *is implied by no combination of the remaining postulates except by Postulates* 4.4 *and* 4.5* (4.4* *and* 4.5) *jointly.*

Proof: Consider Example 4.11*. All the postulates except 4.4 and 4.5 are satisfied. Consider Example [4.11, 4.11*]. All the postulates except 4.5 and 4.5* are satisfied.

4.18 LEMMA. *Postulate* 4.5 (4.5*) *implies Postulate* 4.4 (4.4*).

Proof: The product of all the elements of K, which exists by 4.5, is the 0 of K.

4.19 THEOREM. *Postulate* 4.4 (4.4*) *is implied by no combination of the remaining postulates except by Postulate* 4.5 (4.5*).

Proof: Consider Example 4.11*. All the postulates except 4.4 and 4.5 are satisfied.

4.20 LEMMA. *Postulate* 4.5 (4.5*) *implies Postulate* 4.1 (4.1*).

4.21 THEOREM. *Postulate* 4.1 (4.1*) *is implied by no combination of the remaining postulates except by Postulate* 4.5 (4.5*) *and by Postulates* 4.4 *and* 4.5* (4.4* *and* 4.5) *which jointly imply Postulate* 4.5.

Proof: Consider Example [4.8, 4.7]. All the postulates except 4.1, 4.5, and 4.4 are satisfied. Consider Example [4.11, 4.8, 4.7]. All the postulates except 4.1, 4.5, and 4.5* are satisfied.

4.22 LEMMA. *Postulates* 4.1* *and* 4.2* (4.1 *and* 4.2) *imply Postulate* 4.2 (4.2*).

Proof: We assume that ab exists and that $ab \subset c \subset a$. We must distribute c with respect to ab. By 4.1*, $b+c$ exists. Let X be the set of all elements, x, such that $x \subset a$ and $x \subset b+c$. X is not void, for c is in X. We next show that x in X implies $x \subset c$. We have $c \subset c+x \subset b+c$, and the sums exist by 4.1*. By 4.2*, $c+x=c+y$, where $y \subset b$. Also, since $c \subset a$ and $x \subset a$, $c+x=c+y \subset a$ and $y \subset a$. But, $y \subset a$, $y \subset b$, and $ab \subset c$ imply $y \subset ab \subset c$. Hence, $c+x=c+y=c$ and $x \subset c$. By 3.1, $c=a(b+c)$. $a(b+c)$ exists, since c does, and distributes c with respect to ab.

4.23 LEMMA. *Postulates* 4.1* *and* 4.3* (4.1 *and* 4.3) *imply Postulate* 4.3 (4.3*).

Proof: We assume that ab exists and that $ab \subset c$. We must distribute c with respect to ab. By 4.1*, $a+c$ and $b+c$ exist. Let X be the set of all elements, x, such that $x \subset a+c$ and $x \subset b+c$. X is not void for c is in X. We next show that x in X implies $x \subset c$. We have, by 4.3*, since $x \subset a+c$, $x=u+v$ where $u \subset a$ and $v \subset c$, and, since $u \subset x \subset b+c$, $u=w+y$ where $w \subset b$ and $y \subset c$. But $w \subset b$, $w \subset u \subset a$, and $ab \subset c$ imply $w \subset ab \subset c$. Hence, $x=w+y+v \subset c$. By 3.1, $c=(a+c)(b+c)$. The product exists, since c does, and distributes c with respect to ab.

4.24 LEMMA. *Postulate* 4.3 (4.3*) *implies Postulate* 4.2 (4.2*).

4.25 LEMMA. *Postulate* 4.6 (4.6*) *implies Postulate* 4.3 (4.3*).

4.26 THEOREM. *Postulate* 4.3 (4.3*) *is implied by no combination of the remaining postulates except by Postulate* 4.6 (4.6*), *by Postulates* 4.1* *and* 4.3* (4.1 *and* 4.3) *jointly, and by combinations implying one of these.*

Proof: Consider Example [4.7, 4.9]. All the postulates except 4.3, 4.6, 4.1*, 4.4*, and 4.5* are satisfied. Consider Example [4.7, 4.9, 4.7]. All the postulates except 4.3, 4.6, 4.3*, and 4.6* are satisfied. Consider Example [4.7, 4.9, 4.12*]. All the postulates except 4.3, 4.5, 4.6, 4.1*, and 4.5* are satisfied.

4.27 THEOREM. *Postulate* 4.2 (4.2*) *is implied by no combination of the remaining postulates except by Postulate* 4.3 (4.3*), *by Postulates* 4.1* *and* 4.2* (4.1 *and* 4.2) *jointly, and by combinations implying one of these.*

Proof: Substitute 4.10 and 4.9 in the examples given for 4.26. In addition to the postulates listed in 4.26 as failing to hold in each case, 4.2 and, in the second case, 4.2* are no longer satisfied.

4.28 THEOREM. *Postulate* 1.1 *is not implied by any combination of the remaining postulates.*

Proof: Consider Example [4.7, 4.13, 4.7]. All the postulates except 1.1 are satisfied.

4.29 THEOREM. *Postulate* 1.2 *is not implied by any combination of the remaining postulates.*

Proof: Consider Example [4.7, 4.14*, 4.14, 4.7]. All the postulates except 1.2 are satisfied.

4.30 THEOREM. *Postulates* 1.1, 1.2, 4.1–4.6, *and* 4.1*–4.6* *are consistent.*

Proof: Consider Example [4.7, 4.7, 4.7]. All the postulates are satisfied.

Given any subset, Q (admitting the void set as a subset), of the postulates so far considered, the closure of Q is defined as the subset of the postulates which contains Q and all postulates implied by combinations of the postulates in Q. A subset, Q, of the postulates is said to be closed if it is identical with its closure. Clearly, the subset consisting of all the postulates which hold in any given ordered set, must be a closed subset. It remains to exhibit, for each closed subset, Q, of the postulates, an ordered set in which the postulates, Q, and no others, hold.

To do this we select examples from those used in proving the theorems of this section (specifically, 4.15, 4.17, 4.19, 4.21, 4.26, 4.27, and their duals) and arrange these examples in a sequence, ordering the result lexicographically as in 2.11. It is necessary that the examples selected have units, with the possible exceptions that the first example in the sequence may lack a zero and the last example may lack a one.

First, consider the case where Postulates 4.4 and 4.4* are in Q. Then for each postulate not in Q select an example in which it fails to hold and all the postulates in Q hold. Since Q is closed, this is always possible, as no postulate

not in Q is implied by any combination of postulates in Q. Suppose there are n postulates not in Q, then we have n examples, A_i each with units 0_i, I_i, which we shall arrange in a sequence $A = [A_1, A_2, \cdots, A_i, \cdots, A_n]$. For this example, the postulates Q and no others hold. In the first place, by 2.11, A is ordered if, and only if, every A_i is ordered. Postulates 4.4 and 4.4* hold in A, since they are in Q and hence hold for A_1 and A_n. Next, consider any subset, B, of A. The elements of B come from the sets A_i. Suppose that j is the lowest index for which A_j contributes at least one element to B. Then $\prod b_k$, if it exists, is in A_j as 0_j is contained in this product and 0_{j+1} is not. Furthermore, every element of A appearing in some A_i, $i>j$, can be distributed with respect to $\prod b_k$. For this element contains I_j which contains some b_k by the selection of j. Hence the existence and distributive properties of $\prod b_j$ depends solely on the corresponding properties of the subset of B in A_j. Hence any of the postulates regarding the existence and distributive properties of products which holds for every A_i holds for A. A similar statement is true for sums. Furthermore, as the sum or product with respect to A of any subset of any A_i, being bounded between 0_i and I_i, cannot be in A_j for any $j \neq i$, hence the failure of a sum or product to exist or have certain distributive properties in some A_i guarantees the same failure in A. But every postulate of Q holds in every A_i, hence in A, and every postulate not in Q fails in some A_i and hence in A.

We now consider the case where Postulate 4.4 is not in Q but Postulate 4.4* is. If Postulate 4.5* is in Q, then the closure $\bar{Q}$ of the set formed by adjoining 4.4 to Q must contain 4.1 and 4.5. In no case need other postulates be adjoined to Q to obtain $\bar{Q}$. Since $\bar{Q}$ contains 4.4 and 4.4* an example, A, can be found for it under the case first considered. If 4.1 is not in Q, then $[4.8, A]$ is an example of Q. If 4.1 is in Q, then $[4.11^*, A]$ is an example of Q. Postulate 4.5 cannot be in Q as it would imply 4.4 in Q. By similar reasoning $[A, 4.8]$ and $[A, 4.11]$ are examples of the case where Q contains Postulate 4.4 and not Postulate 4.4* and A is an example of the closure of Q and Postulate 4.4*. If neither Postulate 4.4 nor Postulate 4.4* are in Q, and A is an example for the closure of Q, 4.4 and 4.4*, then $[4.8, A, 4.8]$, $[4.11^*, A, 4.8]$, $[4.8, A, 4.11]$ and $[4.11^*, A, 4.11]$ are the examples required.

5. **Homomorphisms, isomorphisms, automorphisms, and subsystems.** In the theory of ordered sets, so many operations are involved that it is necessary to formulate the concepts of homomorphism and subsystem with generality and precision to provide, without ambiguity, for the different cases which arise. A homomorphism is a correspondence between the elements, the ordering relations, and the operations of two ordered sets. Isomorphisms and automorphisms are particular types of homomorphism. We consider two

ordered sets, K and $\overline{K}$, and a set, Θ, of univocal operations defined by an ordering relation.

5.1 DEFINITION. A *homomorphism* relative to Θ from K to $\overline{K}$ is a correspondence, $\rightarrow$, between the elements, ordering relation, and operations Θ of K and similar entities of $\overline{K}$ such that:

a. If a is in K, then a uniquely defined element, $\bar{a}$, exists in $\overline{K}$ such that $a \rightarrow \bar{a}$;

b. If $\bar{a}$ is in $\overline{K}$, then at least one element, a, exists in K such that $a \rightarrow \bar{a}$;

c. An ordering relation, $\overline{\subset}$, is defined for K such that $a \subset b$ in K implies $\bar{a} \overline{\subset} \bar{b}$ in $\overline{K}$;

d. If θ is in Θ and $\theta(a_i)$ exists in K, then $\bar{\theta}(\bar{a}_i)$ exists in $\overline{K}$ and $\theta(a_i) \rightarrow \bar{\theta}(\bar{a}_i)$.

$\overline{K}$ is homomorphic to K relative to Θ if there is a homomorphism relative to Θ from K to $\overline{K}$. Since the operations Θ are defined in terms of the relation $\subset$ and $\overline{\Theta}$ in terms of $\overline{\subset}$, the correspondence $\theta \rightarrow \bar{\theta}$ is determined by the correspondence $\subset \rightarrow \overline{\subset}$. Thus a homomorphism is determined by the correspondence between elements and between ordering relations. It can be classified by the operations Θ which it is said to preserve.

We have already remarked that the definition of operations is relative to the entire set to which they are applied and not just to the elements immediately involved. Thus, although any subset of an ordered set is an ordered set, there is no assurance that the operations defined for a subset will be identical with operations defined for the entire set when operating on elements of the subset. This fact motivates our definition of subsystems.

5.2 DEFINITION. $\overline{K}$ is a subsystem of K relative to Θ if $\overline{K}$ is a subset of K and if, whenever $\theta(\bar{a}_i)$ exists in K, $\bar{\theta}(\bar{a}_i)$ exists in $\overline{K}$ and $\bar{\theta}(\bar{a}_i) = \theta(\bar{a}_i)$.

Nothing in the definitions of homomorphisms and subsystems requires that the sets involved be closed under any operations. In many cases the sets involved are not closed under the operations Θ. Every homomorphism preserves units. A homomorphism does not necessarily preserve sums, products, complements, or sum and product complements. If a subsystem preserves units, it preserves complements and sum and product complements. A subsystem does not necessarily preserve units, sums, and products.

Taking the complement is not always univocal† and, hence, is not always admissible to the set Θ. The other operations considered are always univocal. We next consider some of the properties of homomorphisms and subsystems.

5.3 THEOREM. *A given aggregate of ordered sets, K_i, is ordered by the relation $\subset$, if $K_i \subset K_j$ whenever K_i is homomorphic to K_j relative to Θ.*

† See Theorems 7.28 and 7.29.

5.4 THEOREM. *A given aggregate of subsystems, K_i, relative to Θ of a given set is ordered by the relation $\subset$ if $K_i \subset K_j$ whenever K_i is a subsystem relative to Θ of K_j.*

5.5 THEOREM. *A given aggregate of ordered sets, K_i, is ordered by the relation $\subset$ if $K_i \subset K_j$ whenever K_i is homomorphic relative to Θ to a subsystem relative to Θ of K_j.*

5.6 DEFINITION. K and $\overline{K}$, two ordered sets, are isomorphic if a 1–1 correspondence exists between the elements of K and the elements of $\overline{K}$ such that $a \subset b$ implies $\bar{a} \bar{\subset} \bar{b}$ and $\bar{a} \bar{\subset} \bar{b}$ implies $a \subset b$.

We remark that an isomorphism is a symmetric relation. The expressions, an isomorphism exists between K and $\overline{K}$ and K is isomorphic to $\overline{K}$, are equivalent to K and $\overline{K}$ are isomorphic. All operations are preserved by an isomorphism, so it is not necessary to specify them.

5.7 THEOREM. *If K and $\overline{K}$ are isomorphic, then each is homomorphic to the other relative to any set of operations.*

For homomorphisms and isomorphisms it is not necessary that the elements of K and $\overline{K}$ be distinct. Furthermore, if K and $\overline{K}$ have elements in common, it is not necessary that their ordering relations be the same in their common part. We call an isomorphism between K and $\overline{K}$ an automorphism if the elements of K and $\overline{K}$ are the same, though the ordering relations need not be the same. Among automorphisms, we are particularly interested in dual automorphisms.

5.8 DEFINITION. An isomorphism between K and $\overline{K}$ is a dual automorphism if the elements of K and $\overline{K}$ are the same and the ordering relation of $\overline{K}$ is the dual of the ordering relation of K.

5.9 THEOREM. *If K is an ordered set and $\overline{K}$ is any subset of K such that a in $\overline{K}$ and $b \subset a$ ($b \supset a$) imply b in $\overline{K}$, then $\overline{K}$ is a subsystem of K with respect to multiplication (addition).*

5.10 COROLLARY. *If A is a subset of K and x is in $\overline{K}$ if, and only if, $x \subset a_i$ ($x \supset a_i$) for every a_i in A, then $\overline{K}$ is a subsystem of K with respect to multiplication (addition).*

5.11 COROLLARY. *If a is in K and x is in $\overline{K}$ if, and only if, $x \subset a$ ($x \supset a$), then $\overline{K}$ is a subsystem of K with respect to multiplication (addition).*

5.12 THEOREM. *If A is any subset of an ordered set K and $\overline{K}$ is the subset of K such that x is in $\overline{K}$ if, and only if, $a_i \subset x$ ($a_i \supset x$) for every a_i in A, then $\overline{K}$ is a subsystem of K with respect to multiplication (addition).*

5.13 COROLLARY. *If a is in K and x is in $\overline{K}$ if, and only if, $a \subset x$ ($a \supset x$), then $\overline{K}$ is a subsystem of K with respect to multiplication (addition).*

5.14 THEOREM. *Given a set of subsystems, $\overline{K}_i$, of K with respect to multiplication (addition) and $\overline{K}$ the section, or common part, of the subsystems $\overline{K}_i$, then $\overline{K}$ is a subsystem of K with respect to multiplication (addition).*

5.15 COROLLARY. *If A and B are subsets of K such that $a_i \subset b_j$ for every a_i in A and every b_j in B and x is in $\overline{K}$ if, and only if, $a_i \subset x \subset b_j$ for every a_i in A and every b_j in B, then $\overline{K}$ is a subsystem of K with respect to addition and multiplication.*

5.16 COROLLARY. *If a and b are in K, $a \subset b$, and x in $\overline{K}$ if, and only if, $a \subset x \subset b$, then $\overline{K}$ is a subsystem of K with respect to addition and multiplication.*

5.17 THEOREM. *If $\overline{K}$ is homomorphic to K with respect to certain types, Θ, of addition and multiplication and A is the set of elements, a_i, in K which have the same image, $\bar{a}$, in $\overline{K}$, then A is a subsystem of K with respect to Θ.*

5.18 COROLLARY. *If K is closed under a subset Φ of Θ then A is closed under Φ.*

6. **Multiplicative systems and lattices.** An ordered set which satisfies Postulate 4.1 for the existence of finite products is called a *multiplicative system*. If Postulate 4.5 for the existence of unrestricted products is satisfied, the ordered set is said to be a *complete multiplicative system*. In this section, the dual theorems apply to systems in which the dual postulates are satisfied.

6.1 THEOREM. *If K is a multiplicative system and a, b, and c are elements in K, then*

(a) $a = a$;
(b) $a = b$ *implies* $b = a$;
(c) $a = b$ *and* $b = c$ *implies* $a = c$;
(d) ab ($a+b$) *exists in* K;
(e) $a = b$ *implies* $ac = bc$ ($a+c = b+c$);
(f) $ab = ba$ ($a+b = b+a$);
(g) $a[bc] = [ab]c$ ($a+[b+c] = [a+b]+c$);
(h) $aa = a$ ($a+a = a$);
(i) $ab = a$ ($a+b = a$) *implies* $a \subset b$ ($a \supset b$), *and conversely*;
(j) $ab = a$ ($a+b = a$) *implies that* $a+b$ (ab) *exists and that* $a+b = b$ ($ab = b$).

6.2 THEOREM. *If* 6.1 (b–h) *are taken as postulates and* $\subset$ ($\supset$) *is defined by* 6.1 (i), *then a set, K, satisfying* 6.1 (b–h) *is ordered by the relation* $\subset$ ($\supset$), *is a multiplicative system, and the equality and multiplication (addition) defined by* $\subset$ *can be identified with the equality and multiplication (addition) postulated by* 6.1 (b–h).

Theorems 6.1 and 6.2 establish the equivalence between the approach to multiplicative systems in which equality and multiplication are made fundamental and that in which the ordering relation is made fundamental. Theorem 6.1 serves to show the greater simplicity of the latter method of approach and gives the more important elementary identities of such systems.

6.3 THEOREM. *If, in a multiplicative system,* $\sum a_i$ $(\prod a_i)$ *exists and is distributive, then* $\sum a_i b = [\sum a_i]b$ $(\prod [a_i+b] = \prod a_i+b)$.

Proof: $a_j \subset \sum a_i$ for every a_j in $\sum a_i$, hence, $a_j b \subset [\sum a_i]b$ and $\sum a_i b \subset [\sum a_i]b$. On the other hand, since $\sum a_i$ is distributive and $[\sum a_i]b \subset \sum a_i$, $[\sum a_i]b = \sum c_k$ where, for each c_j in $\sum c_k$, a_j exists such that $c_j \subset a_j$. Also, $c_j \subset [\sum a_i]b$, hence, $c_j \subset a_j[\sum a_i]b = a_j b$. It follows that $[\sum a_i]b = \sum c_k \subset \sum a_i b$. Therefore, $\sum a_i b = [\sum a_i]b$.

6.4 COROLLARY. *If, in a multiplicative system,* $a+b$ *exists and is distributive, then* $ac+bc = [a+b]c$ $([a+c][b+c] = ab+c)$.

6.5 THEOREM. *If, in a multiplicative system,* $\sum a_i$ $(\prod a_i)$ *exists and for every element,* b, *in the system* $\sum a_i b = [\sum a_i]b$ $(\prod [a_i+b] = \prod a_i+b)$, *then* $\sum a_i$ $(\prod a_i)$ *is distributive.*

6.6 THEOREM. *If, in a multiplicative system,* $\sum a_i$ $(\prod a_i)$ *exists and is distributive, then* $\sum a_i b$ $(\prod [a_i+b])$ *exists and is distributive.*

Proof: $\sum a_i b = [\sum a_i]b$ which exists. If $x \subset \sum a_i b$, then, since $\sum a_i b = [\sum a_i]b$, $x \subset \sum a_i$ and $x \subset b$. Since $\sum a_i$ is distributive, $x = [\sum a_i]x = \sum a_i x$. Since $x \subset b$, $x = bx$, hence, $x = \sum a_i bx$ and $\sum a_i b$ is distributive.

6.7 THEOREM. *If, in a multiplicative system,* $a+b$ (bc) *exists, is modular with respect to* a (c), *and* $a \subset c$, *then* $a+bc = [a+b]c$.

Theorem 6.7 is stated so as to exhibit the intrinsic self duality of the modular property.

6.8 THEOREM. *If, in a multiplicative system,* $a+b$ (bc) *exists and for every element* c (a) *such that* $a \subset c$, $a+bc = [a+b]c$, *then* $a+b$ (bc) *is modular with respect to* a (c).

An ordered set which is a multiplicative system with respect to both the relations $\subset$ and $\supset$ is called a *lattice.*† If the multiplicative systems defined by the relations $\subset$ and $\supset$ are complete, then the lattice is *complete.* If but one of these systems is complete, we speak of a lattice with unrestricted products or sums. However, these latter categories are not very general as Lemma

† Birkhoff (3), p. 442. Ore (18) calls such sets structures; Klein (14 and 15), Verbände; Dedekind (6), Dualgruppe. Grell (9), Menger (16), and von Neumann (17) deal with special cases of such sets.

4.16* implies that any complete multiplicative system with a unit I is a complete lattice.

If Postulate 4.2 holds in a lattice, the lattice is *modular*. Lemma 4.22 implies that in a lattice Postulates 4.2 and 4.2* are equivalent. Theorem 6.9 is self dualistic.

6.9 THEOREM. *A lattice is modular if, and only if,* $a \subset c$ *implies* $a+bc = [a+b]c$.

If Postulate 4.3 holds in a lattice, the lattice is *distributive*. In a lattice, Postulates 4.3 and 4.3* are equivalent, by Lemma 4.23.

6.10 THEOREM. *A lattice is distributive if, and only if,* $ac+bc = [a+b]c$ $([a+c][b+c] = ab+c)$.

Actually, $ac+bc \subset [a+b]c$ is sufficient to imply that a lattice be distributive, for $ac+bc \subset [a+b]c$ in any lattice.

A complete lattice in which Postulate 4.6 (4.6*) holds is called a lattice with *completely distributive products* (*sums*). The situation in the complete case is not analogous to that in the finite case, for Postulate 4.6 does not imply Postulate 4.6* in a complete lattice. If both Postulates 4.6 and 4.6* hold, the lattice is said to be *completely distributive*.

6.11 THEOREM. *A lattice has completely distributive products* (*sums*) *if, and only if,* $\prod [a_i+b] = \prod a_i + b$ $([\sum a_i]b = \sum a_i b)$.

Some properties of subsystems and homomorphisms in multiplicative systems and lattices are now considered.†

6.12 THEOREM. *If K is a multiplicative system and* $\overline{K}$ *is a subset of K belonging to one of the types considered in* 5.10–5.16, *then* $\overline{K}$ *is a subsystem of K with respect to multiplication, addition, modular addition, and distributive addition.*

Proof: By 5.10–5.16, 6.5, and 6.8.

6.13 COROLLARY. $\overline{K}$ *is a multiplicative system and if K is complete, so is* $\overline{K}$.

6.14 COROLLARY. *If K is a lattice, modular lattice, distributive lattice, complete lattice, or lattice with completely distributive sums* (*products*), *then so is* $\overline{K}$.

A subsystem with respect to multiplication of a multiplicative system will be called a *multiplicative subsystem*. Similarly, a subsystem with respect to addition and multiplication of a lattice will be called a *sublattice*.

6.15 THEOREM. *If K is a multiplicative system and b is an element of K, then the correspondence* $x \to bx$, *carrying K into the multiplicative subsystem* $\overline{K}$

† Compare Ore (18), p. 416 et seq.

of elements $y \subset b$, is a homomorphism with respect to multiplication and distributive addition. The sum $\sum a_i$ is preserved if, and only if, $[\sum a_i]b = \sum a_i b$.

Proof: $a \subset c$ implies $ab \subset cb$. $ac \to acb = abcb$. $\sum a_i \to [\sum a_i]b = \sum a_i b$ if, and only if, $[\sum a_i]b = \sum a_i b$. If $\sum a_i$ is distributive, then $[\sum a_i]b = \sum a_i b$ and $\sum a_i b$ is distributive, by 6.3 and 6.6.

6.16 COROLLARY. *If K is a modular lattice, then all sums $a+c$, where $c \subset b$, are preserved as modular sums.*

6.17 THEOREM. *If K is a modular lattice, X the sublattice of all elements x_i such that $a \subset x_i \subset a+b$, and Y the sublattice of all elements y_j such that $ab \subset y_j \subset b$, then the correspondence $x_i \to bx_i$ between X and Y is the inverse of the correspondence $y_j \to a+y_j$ between Y and X. Together, these correspondences define an isomorphism between X and Y.*

Proof: $x_i \to bx_i \to a+bx_i = x_i$ and $x_i \subset x_j$ implies $bx_i \subset bx_j$. Similarly, $y_j \to a+y_j \to b[a+y_j] = y_j$ and $y_i \subset y_j$ implies $a+y_i \subset a+y_j$.

6.18 COROLLARY.† *The correspondence $x_i \longleftrightarrow bx_i$, or $y_j \longleftrightarrow a+y_j$, between X and Y preserves all types of sums and products defined relative to K.*

Proof: An isomorphism preserves all operations. Hence, relative to X and Y, all types of sums and products are preserved. However, X and Y are subsystems of K with respect to these operations by 5.16 and 6.14.

7. **Complements.** We are now ready to continue the classification of ordered sets with respect to the existence of complements. We consider the following postulates.

7.1 POSTULATE. *If a is in K, then a' (a^*) is in K.*

7.2 POSTULATE. *A dual automorphism exists in K.*

7.3 POSTULATE. *The correspondence $a \longleftrightarrow a'$ ($a \longleftrightarrow a^*$) is a dual automorphism.*

7.4 LEMMA. *Postulate* 7.1 (7.1*) *implies Postulates* 4.4 *and* 4.4*.

Proof: The existence of a product complement demands a 0. The product complement of 0 is I.

7.5 THEOREM. *If K belongs to any of the classifications satisfying Postulates* 4.4 *and* 4.4* *considered in* §4, *then an example of the same classification in* §4 *can be found in which Postulate* 7.1 (7.1*) *holds.*

Proof: Consider Example [4.7, K].

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† This is a generalization of Ore (18), p. 418, Theorem 2, as it shows that infinite, as well as finite, sums and products are preserved.

7.6 LEMMA. *Postulates* 4.4, 4.5*, *and* 4.6* (4.4*, 4.5, *and* 4.6) *imply Postulate* 7.1 (7.1*).

Proof: 0 exists by 4.4, hence, the set B of all elements b_j such that $ab_j=0$, for any fixed a in K, is not void. By 4.5*, $\sum b_j$ exists and, by 4.6*, $a(\sum b_j)=\sum ab_j=0$. Therefore, $\sum b_j$ is the product complement of a.

7.7 **Example.** K consists of the finite, or void, subsets of the positive integers with the relation $a \subset b$ if every integer in a is an integer in b.

7.8 THEOREM. *Postulate* 7.1 (7.1*) *is implied by no combination of the postulates of* §1 *and* §4 *except by Postulates* 4.4, 4.5*, *and* 4.6* (4.4*, 4.5, *and* 4.6) *jointly and combinations implying these postulates.*

Proof: Consider Example 4.11*. Postulates 4.4, 4.5, and 7.1 are not satisfied. Consider Example 7.7. Postulates 4.4*, 4.5*, 7.1, and 7.1* are not satisfied. Consider Example [7.7, 4.11*]. Postulates 4.5, 4.5*, and 7.1 are not satisfied. Consider Example [7.7, 4.7]. Postulates 4.6* and 7.1 are not satisfied. In each case the remaining postulates of §1 and §4 are satisfied.

7.9 LEMMA. *If the corresponding* $a \rightarrow \bar{a}$ *is a dual automorphism and* $\sum a_i$ $(\prod a_i)$ *exists, then* $\prod \bar{a}_i$ $(\sum \bar{a}_i)$ *exists and* $[\overline{\sum a_i}]=\prod \bar{a}_i$ $([\overline{\prod a_i}]=\sum \bar{a}_i)$.

Proof: $a_i \subset \sum a_i$, hence $[\overline{\sum a_i}] \subset \bar{a}_i$. If $y \subset \bar{a}_i$ for every a_i in $\sum a_i$, then x is uniquely determined so that $x \longleftrightarrow y$, or $y=\bar{x}$, and $a_i \subset x$. Therefore, $\sum a_i \subset x$ and $y=\bar{x} \subset [\overline{\sum a_i}]$.

7.10 THEOREM. *If Postulate* 7.2 *holds, then the dual of any other postulate which holds is also satisfied.*

Proof: Apply 7.9.

7.11 THEOREM. *Except for duals, no new postulates are implied by combining Postulate* 7.2 *with a given set of the remaining postulates.*

Proof: Consider any subset of the postulates, excluding 7.2 and 7.3, and adjoin the duals. Next adjoin all postulates implied by these. The dual of each such postulate will be such a one and thus adjoined at the same time. By the methods of §4 and this section an example, K, can be constructed in which just these postulates hold. The Example $[K, K^*]$ also has these properties and in addition satisfies 7.2.

7.12 THEOREM. *Postulate* 7.2 *is implied by no combination of the remaining postulates except Postulate* 7.3.

Proof: Consider Example [4.7, 4.7, 4.8, 4.7]. All the postulates except 7.2 and 7.3 are satisfied.

7.13 THEOREM. *Postulate* 7.3 (7.3*) *implies* $[a']'=a$ $([a^*]^*=a)$.

Proof: Since $a'a=0$, $a \subset [a']'$. Similarly, $a' \subset [(a')']'$, and by the dual automorphism, $[a']' \subset a$. Therefore, $[a']'=a$.

7.14 THEOREM. *Postulates* 7.3 (7.3*) *and* 1.1 *imply Postulate* 1.2.

Proof: $a \subset [a']'$ and $[a']' \subset a$ imply $a \subset a$.

7.15 LEMMA. *Postulate* 7.3 (7.3*) *implies Postulates* 7.1, 7.1*, 4.4, 4.4*, 7.2, *and* 7.3*.

7.16 THEOREM. *If Postulates* 7.1 (7.1*), 1.1 *and* 1.2 *are satisfied and* $\sum a_i(\prod a_i)$ *exists, then* $\prod a_i'$ $(\sum a_i^*)$ *exists and* $\prod a_i' = [\sum a_i]'(\sum a_i^* = [\prod a_i]^*)$.

Proof: Since $[\sum a_i]'$ exists by hypothesis, the existence of $\prod a_i'$ follows from the equality. For every a_i, $a_i[\sum a_i]' \subset \sum a_i[\sum a_i]'=0$ and $[\sum a_i]' \subset a_i'$. If, for every a_i, $x \subset a_i'$, then $xa_i=0$, $a_i \subset x'$, $\sum a_i \subset x'$, and $x \subset [\sum a_i]'$.

When $a \longleftrightarrow a'$ $(a \longleftrightarrow a^*)$ is a dual automorphism 7.16 is a special case of 7.9.

7.17 THEOREM. *If Postulates* 7.3 (7.3*) *and* 1.1 *are satisfied, then* $a'=a^*$.

Proof: If $a \subset x$ and $a' \subset x$, then $x' \subset a'$, $x' \subset [a']'$, $x' \subset a'[a']'=0$, and $x=\mathrm{I}$. Therefore, $a+a'=\mathrm{I}$. On the other hand, if $a+y=\mathrm{I}$, then, by 7.16, $a'y'=0$ and $y' \subset [a']'$. Therefore, $a' \subset y$.

7.18 COROLLARY. *If Postulates* 7.3 (7.3*) *and* 1.1 *are satisfied, then* a' *is the unique complement of* a *with respect to* I *and* 0.

7.19 THEOREM. *Postulates* 7.3 (7.3*) *and* 1.1 *imply Postulate* 4.6* (4.6).

Proof: If $\sum a_i$ exists and b is any element such that $b \subset \sum a_i$, then let B consist of all elements b_k such that $b_k \subset b$ and $b_k \subset a_i$ for some a_i, and let X consist of all elements x_i such that $b_k \subset x_i$ for every b_k in B. If we show that $b \subset x_i$ for every x_i in X, then $b=\sum b_k$ and the theorem is proved. It remains to prove that $b \subset x_i$ for every x_i in X. Let x be any element of X and y any element such that $y \subset b$ and $y \subset x'$. Then $ya_i=0$ for every a_i, since $z \subset y$ and $z \subset a_i$ imply z in B, $z \subset x$, $z \subset x'$, and $z=0$. Hence $y \subset a_i'$ and $y \subset \prod a_i' = [\sum a_i]'$. But $y \subset b \subset \sum a_i$. Therefore, $y=0$, $bx'=0$, and $b \subset x$. Since this holds for any x in X, there can be no x for which it fails, and it must be true for every x in X.

7.20 COROLLARY. *Postulates* 7.3 (7.3*) *and* 1.1 *imply Postulates* 4.2, 4.2*, 4.3, 4.3*, *and* 4.6 (4.6*).

7.21 LEMMA. *If Postulates* 7.3 (7.3*) *and* 1.1 *are satisfied, then either Postulate* 4.5 *or* 4.5* *implies the other and* 4.1 *and* 4.1*, *and either* 4.1 or 4.1* *implies the other.*

7.22 THEOREM. *None of the Postulates* 4.1, 4.1*, 4.5, *and* 4.5* *are implied by Postulates* 7.3 (7.3*) *and* 1.1.

Proof: Consider the following set, K, consisting of units, 0 and I and four elements a_i, $i=1, 2, 3, 4$, and their product complements a_i' where $a_i' \subset a_j$ for $i \neq j$. $a_i a_j$ and $a_i' + a_j'$, for $i \neq j$, do not exist, therefore none of the Postulates 4.1, 4.1*, 4.5, and 4.5* hold. However, Postulates 7.3 and 1.1 are satisfied.

7.23 THEOREM. *Postulates* 4.5 *and* 4.5* *are not implied by Postulates* 7.3 (7.3*), 1.1, 4.1, *and* 4.1*.

Proof: Consider the set, K, of finite collections of intervals with rational end points in the closed interval $[0, 1]$, where $a \subset b$ if every point of a is a point of b. $[0, r_i]$ and $[r_i, 1]$, where R consists of all rational numbers less than $(1/2)^{1/2}$, do not exist. Hence Postulates 4.5 and 4.5* are not satisfied. Postulates 7.3, 1.1, 4.1, and 4.1* are fulfilled.

7.24 THEOREM. *No subset of the postulates of* §1, §4, *and the remaining postulates of* §7 *imply Postulate* 7.3 (7.3*).

Proof: Consider Example [4.7, 4.7, 4.7].

7.25 THEOREM. *All the postulates of* §1, §4, *and* §7 *are consistent.*

Proof: Consider Example [4.7, 4.8, 4.7].

7.26 THEOREM. *If Postulates* 7.1 (7.1*) *and* 1.1 *are satisfied and* $[a']' \subset a$ $(a \subset [a^*]^*)$, *then Postulate* 7.3 (7.3*) *is satisfied.*

Proof: Apply 3.18 and 3.19.

7.27 THEOREM. *If Postulates* 7.3 (7.3*) *and* 1.1 *are satisfied, then* $a \subset b$ *if, and only if,* $ab' = 0$ $(a^* + b = \mathrm{I})$.

Proof: If $a \subset b$, then $a = ab$ and $ab' = abb' = a0 = 0$. If $ab' = 0$, then $a \subset (b')' = b$.

Sum complements and product complements are unique. We now consider the necessary and sufficient conditions that complements be unique.

7.28 THEOREM. *If* K *is a distributive lattice and* b *and* $\bar{b}$ *are complements of* a *with respect to* c *and* d, *then* $b = \bar{b}$.

Proof: By hypothesis, $a+b=c=a+\bar{b}$ and $ab=d=a\bar{b}$. Hence, $b=b(a+b)=b(a+\bar{b})=ba+b\bar{b}=a\bar{b}+b\bar{b}=(a+b)\bar{b}=(a+\bar{b})\bar{b}=\bar{b}$.

7.29 THEOREM. *If* K *is a lattice, but not distributive, then* K *contains a set of elements* a, b, $\bar{b}$, c, *and* d, *such that* b *and* $\bar{b}$ *are complements of* a *with respect to* c *and* d, *but* $b \neq \bar{b}$.

Proof: If K is modular, it contains a subset of the form [4.7, 4.9, 4.7].† If K is not modular, it contains a subset of the form [4.7, 4.10, 4.7].†

† Birkhoff (4), p. 617, Theorem 4 and Corollary.

7.30 THEOREM. *In any distributive lattice, with units, if a has a complement b with respect to* I *and* 0, *then b is the product (sum) complement of a.*

Proof: $ab=0$. If b is not the product complement of a, then c exists in K such that $ac=0$ but $c \not\subset b$. This implies that $c+b \neq b$ and that $c+b$ is also a complement of a, contrary to Theorem 7.28.

7.31 THEOREM. *If, for every element a_i of a subset, A, of an ordered set, K, the product (sum) complement a_i' (a_i^*) exists, $\sum a_i$ and $\prod a_i'$ ($\prod a_i$ and $\sum a_i^*$) exist, and $\sum a_i$ ($\prod a_i$) is distributive, then $\prod a_i'$ ($\sum a_i^*$) is the product (sum) complement of $\sum a_i$ ($\prod a_i$).*

Proof: $[\sum a_i]\prod a_i' = \sum [a_i \prod a_i'] = 0$. If $[\sum a_i]x = 0$, then $a_i x = 0$, $x \subset a_i'$ for every a_i', and $x \subset \prod a_i'$.

7.32 THEOREM. *If K is an ordered set and A is a subset of K such that each element a_i in A has a complement b_i with respect to c and d and $\prod a_i$ and $\sum b_i$ ($\sum a_i$ and $\prod b_i$) exist and are distributive, then $\prod a_i$ and $\sum b_i$ ($\sum a_i$ and $\prod b_i$) are complementary with respect to c and d.*

Proof: By hypothesis, $a_i+b_i=c$ and $a_ib_i=d$. Hence, $a_i \subset c$, $b_i \subset c$, and $\prod a_i + \sum b_i \subset c$. On the other hand, $\prod a_i + \sum b_i = \prod (a_i + \sum b_i) \supset \prod (a_i+b_i) = c$. Therefore, $\prod a_i + \sum b_i = c$. Similarly, $\prod a_i \, (\sum b_i) = d$.

8. **Boolean algebras.** Systems satisfying Postulates 1.1, 4.1, and 7.3 are considered next. These Postulates imply all the postulates considered in §1, §4, and §7, except Postulates 4.5 and 4.5*. Hence, such a system is a distributive structure with units and, by Theorem 7.18, complements. The postulates for a Boolean algebra of Huntington,† Stone, and Tarski can be readily verified in such a system. Moreover, Postulates 1.1, 4.1, and 7.3 are well known properties of a Boolean algebra. Therefore Theorem 8.1 follows.

8.1 THEOREM. *For a set, K, to be a Boolean algebra, it is necessary and sufficient that Postulates* 1.1, 4.1, *and* 7.3 *be satisfied.*

It follows from Theorem 7.23 that Postulate 4.5 does not necessarily hold in a Boolean algebra. If Postulate 4.5 does hold, the Boolean algebra is said to be *complete*. Since Postulate 4.5 implies Postulate 4.1, it is no longer necessary to affirm Postulate 4.1 and Theorem 8.2 follows.

8.2 THEOREM. *For a set, K, to be a complete Boolean algebra, it is necessary and sufficient that Postulates* 1.1, 4.5, *and* 7.3 *be satisfied.*

It is interesting to note that Postulates 4.6 and 4.6* are always satisfied in a Boolean algebra, hence, it is not possible to classify Boolean algebras according to the extent to which the distributive property holds, as was done

† Huntington (12 and 13), Stone (23), and Tarski (25).

in the case of lattices. Stone† considers generalized Boolean algebras which are ordered sets with a unit 0 and such that for every b in the set, the subsystem of elements, x, such that $x \subset b$ is a Boolean algebra.

9. **Extensions of ordered sets.** The object of an extension is to complete or close a given ordered set with respect to specified operations. Whenever this can be done in some uniquely defined manner it becomes unnecessary to postulate closure under these operations as closure may be obtained by an extension. To secure the uniqueness of an extension, we require that it be minimal in a sense made precise by Definition 9.4. Furthermore, it is important that a minimal extension be determined by the given ordered set and the operations under which it is to be closed and not depend upon the particular method of extension employed. By achieving this latter goal, we are able to make an extension by any sequence of steps we desire without affecting the uniqueness of the result. Naturally, an extension must preserve in the given set the operations under which it is to be closed. With these points in mind, we are lead to the following definitions and theorems.

9.1 DEFINITION. If K and L_i are ordered sets and Φ is a set of univocal operations defined by an ordering relation, then L_i is an extension to Φ of K provided that:

a. L_i is closed under Φ;

b. L_i contains a subsystem, K_i, relative to Φ such that K and K_i are isomorphic.

We now seek an ordering relation for the set of extensions, L_i, to Φ of a given set, K. It is with respect to this ordering relation that we require an extension to be minimal. Let L_i and L_j be extensions to Φ of K.

9.2 DEFINITION. $L_i \subset L_j$ if a correspondence exists such that:

a. L_i is isomorphic to a subset of L_j;

b. K_i is isomorphic to K_j.

We remark that the image of L_i in L_j need not be a subsystem relative to Φ of L_j for the relation of Definition 9.2 to hold. In fact, Definition 9.2 and also Theorem 9.3 will hold equally well if L_i and L_j are not required to be extensions to Φ of K, but merely closed under Φ and containing a subset isomorphic to K.

9.3 THEOREM. *The relation* $\subset$ *of Definition* 9.2 *orders the set of extensions to* Φ *of* K.

9.4 DEFINITION. L_0 is a *minimal extension* to Φ of K if L_0 is a zero of the ordered set of extensions to Φ of K.

† Stone (23 and 24).

If L_0 and $\bar{L}_0$ are both zeros of the set of extensions to Φ of K, then they are equal. But the equality in this case means only that each is isomorphic to a subset of the other under certain restrictions. We desire a stricter equality, namely, that L_0 and $\bar{L}_0$ be isomorphic. This result is established by considering required elements.

9.5 DEFINITION. An element of L_i, an extension to Φ of K, is *required* by Φ and K if it can be represented by a finite succession of operations of Φ on subsets of K_i.

From the definition it is clear that the set of elements required by Φ and K depends only upon Φ and K and not upon the particular extension, L_i, under consideration. Furthermore, each L_i contains the set of elements required by Φ and K. The representation of required elements in terms of Φ and K and the ordering relation of K impose certain restrictions on their ordering. Such restrictions we call required relations and define precisely in Definition 9.6.

9.6 DEFINITION. The *relation* of a to b, elements of an extension to Φ of K and required by Φ and K, is *required* if either $a \subset b$ or $a \not\subset b$ is implied by their representation in terms of Φ and K.

If the relation of a to b is not required we say that it is optional. It is a characteristic property of minimal extensions that whenever the relation of a to b is optional then $a \subset b$. At the other extreme, we have free extensions which are characterized by the property that if the relation of a to b is optional then $a \not\subset b$. For examples of required elements and relations let addition and multiplication be in Φ and let a, b, and c be unrelated elements of K, then $a+b$ and ab are required elements and $ab \subset a+b$ and $a+b \not\subset ab$ are required relations. The relation of ab to ac is optional.

In each case which we consider we are able to find an extension in which all the elements and relations are required. In fact, we are able to find a unique representation for each element of each extension such that all the relations are required by the unique representations and the given set. We call such an extension a canonical extension.

9.7 DEFINITION. An extension, L, to Φ of K is *canonical* if it consists entirely of required elements and a unique representation can be assigned to each of its formally different elements such that all the relations of L are required by this representation and K.

It should be noted that the representation need be unique only for elements which differ in form. It may well be that some of the required relations will introduce additional equalities.

9.8 THEOREM. *If L is a canonical extension to Φ of K, then L is a minimal extension to Φ of K.*

Proof: Let L_i be any extension to Φ of K. Then L_i contains all elements required by Φ and K. In particular, L_i contains elements with the representations assigned to elements of L. To each element of L let correspond the element of L_i with the same representation. Since all the relations of L are required, this 1–1 correspondence between L and a subset of L_i is an isomorphism. Furthermore, under this correspondence, the subsets of L and L_i which correspond to K are isomorphic. Hence, L is a minimal extension to Φ of K.

The proof of Theorem 9.8 goes through equally well if we do not demand that L_i be an extension to Φ of K, but merely that L_i be closed under Φ and contain a subset isomorphic to K. Since the ordering relation for extensions applies to this type of set too, we can assert that L is minimal with respect to all sets closed under Φ and containing a subset isomorphic with K.

9.9 LEMMA. *If L is a canonical extension to Φ of K, then any isomorphism which carries L into L', a subset of itself, so that the image of K corresponds to itself, element for element, carries each element of L into itself and L' is the entire set L.*

Proof: Since the image of K is preserved, element for element, and since all the relations of L are required by the unique representations and K, each element of L must correspond to itself.

9.10 THEOREM. *If L is a canonical extension to Φ of K and L_0 is a minimal extension to Φ of K, then L and L_0 are isomorphic.*

Proof: Since L_0 is a minimal extension to Φ of K, it is isomorphic to a subset of L, and the images of K in L and L_0 are also isomorphic under this correspondence. Using this isomorphism between the images of K, an isomorphism can be constructed between L and a subset of L_0 by letting each element of L correspond to the element of L_0 with the same representation. Applying this isomorphism from L to a subset of L_0 and carrying this subset back into L by the isomorphism from L_0 to a subset of L, we have an isomorphism between L and a subset of L which preserves the image of K, element for element. By Lemma 9.9, this subset of L must be the set L, itself. Thus every element of L is the image of some element of L_0 under the isomorphism from L_0 to a subset of L and the subset of L in question is the entire set L. Hence, L and L_0 are isomorphic.

The independence of a canonical extension from the particular steps by which it is constructed is established by repeated applications of the following theorem.

9.11 THEOREM. *If L_i is a canonical extension to Φ_i of K; L_{ij}, a canonical extension to Φ_j of L_i where Φ_i is a subset of Φ_j; and K is a subsystem relative to Φ_j of L_i: then L_{ij} is a canonical extension to Φ_j of K.*

9.12 THEOREM. *If K is closed under Φ, then K is a canonical extension to Φ of itself.*

In particular, it follows from either Theorem 9.11 or 9.12 that the iterated application of a canonical extension yields nothing new. Thus a canonical extension has all the properties which we require.

It is not in general true that a canonical extension exists for arbitrary Φ and K. For instance, a lattice, K, which is not distributive cannot be extended to a distributive lattice, L, preserving both addition and multiplication, for some $a(b+c) \neq ab+ac$ exists in K, while the preservation, in L, of this inequality denies the distributive law. On the other hand, if only the distributive sums and distributive products of K are preserved in L, then the extension is not uniquely defined. We find, however, that the problem has a unique solution if all products, but only distributive sums, are preserved. Furthermore, the form of an extension may require that the given set be closed under certain operations and it is natural to require that the extension be closed under these operations, too. Hence, for an extension, we are given two suitably chosen sets of operations, Θ and Φ, of which Θ is a subset of Φ, and an arbitrary ordered set, K, closed under Θ. We are to construct a set, L, which is a canonical extension to Φ of K.

Theorem 9.11 gives us the conditions under which canonical extensions may be combined. We now seek the conditions under which canonical extensions may be split up.

9.13 THEOREM. *If L is a canonical extension to Φ of K, where the only restrictions on K are that it be an ordered set closed under Θ, a specified subset of Φ, and Φ', a subset of Φ, contains Θ, then the set L' of elements of L required by Φ' and K is a canonical extension to Φ' of K, and K is a subsystem relative to Φ of L'.*

Theorems 9.11 and 9.13 enable us to derive a large number of extensions from the few which we actually give. It is our aim to give extensions which may be used as a basis for all extensions possible under the operations which we consider, in that all such extensions may be derived from the given ones. We fall short of this goal in a few cases.

An ordered set is said to be *given* if its elements and ordering relation are defined. The operations, which we consider, are determined by this information. We assume that this determination can actually be carried out. In per-

forming a sequence of extensions it is necessary to work out the internal properties of each before proceeding to the next. An ordered set is said to be *constructed* when its elements and ordering relation have been defined, that is, it can be used as a given set for successive constructions.

The elements of an extension, L, to Φ of a given set, K, closed under Θ are usually suitably restricted subsets of K, such that the ordering relation of L can be defined in terms of the known properties of the elements of L immediately involved. We shall characterize three types of extension by the methods allowed in defining these subsets. The first type, an *absolute* extension, is algebraic in nature and is characterized by the fact that the elements required by Φ and a subsystem of K relative to Φ and closed under Θ can be defined as subsets within the subsystem. Such an extension has the property that the extension of a subsystem of K relative to Φ and closed under Θ is isomorphic with the corresponding subsystem of the extension of the entire set. An absolute extension does not require the calculus of classes. The second type, a *relative* extension, depends upon the calculus of classes, including the theory of infinite sections, but does not involve the well ordering hypothesis. That is, elements of L are classes of K where the classes of K are so defined that the admission, or rejection, of an element to a class does not depend upon the disposition of elements already considered. On the other hand, the subsets of K by which the elements of L are represented involve the entire set K in their definition and the property of absolute extensions with respect to the invariance of subsystems need not hold. The third type of extension employs the well ordering hypothesis† in the definition of subclasses of K to represent elements of L.

Each successive type of extension permits greater liberty in the definition of classes or, to put it in another way, requires stronger assumptions as to the existence of classes. Each type of extension includes its predecessors. In order to get by with as few assumptions as possible, we give preference to an absolute extension. If an absolute extension is not possible, we have recourse to a relative extension. In no case do we require the well ordering hypothesis.

In a canonical extension it frequently happens that equal elements of L can be represented by different subsets of K. If a uniquely defined member is selected to represent each class of equal elements in L, then the extension has been reduced to normal form. In other words, if the elements of an extension, L, have been reduced to normal form, then elements are equal only if the subsets of K by which they are represented are identical. A normal form is found for each extension which follows. In each case the normal form is neces-

† Fraenkel (7) and Zermelo (27).

sarily relative to the entire set K, even though the extension may be absolute. Here again we have no recourse to the axiom of choice.

In the light of this discussion we now list the steps which are required in the construction of a canonical extension. In each case we consider the construction as de facto evidence that such an extension is possible. We then take up the cases where an extension cannot be made and point out the cases which have not, as yet, been solved. In presenting the extensions which follow, we shall pass from step to step without comment, assuming that the motives behind the various steps have been made sufficiently clear in this section.

9.14 CONSTRUCTION. A canonical extension to Φ of K, where K is closed under a subset, Θ, of Φ has been constructed when the following steps have been completed:

a. The elements of L have been defined;

b. The ordering relation of L has been defined;

c. The closure of L under Φ has been established;

d. An isomorphism has been established between K and a subset of L such that the image of K is a subsystem of L relative to Φ;

e. All operations not in Φ relative to which the image of K is a subsystem of L have been found; (Strictly speaking, this information is not needed for the construction of a canonical extension, but it is needed in applying 9.11.)

f. A unique representation, required by Φ and K has been assigned to each element of L;

g. All the relations of L have been shown to be required by K and the unique representations of the elements of L;

h. The possibility of adjoining new operations to both Θ and Φ has been investigated; (Strictly speaking, this is not necessary except to show the full power of the method of extension under consideration. Theorem 9.13 shows merely the possibility of adjoining operations of Φ to Θ.)

i. If the extension is relative, the impossibility of an absolute extension has been shown;

j. A normal form has been found for the elements of L.

10. **The adjunction of units.** It has been shown in §4 that Postulates 1.1 and 1.2 for an ordered set do not imply Postulates 4.4 and 4.4* for units. The adjunction of units may be done directly and presents no difficulty. However, we make a precise statement of the construction in order to avoid ambiguity in the sections which follow. We are given an ordered set, K, closed under specified operations, but not satisfying Postulate 4.4. An extension, L, satisfying Postulate 4.4 is constructed as follows.

10.1 DEFINITION. The elements of L are the elements of K and an element 0.

10.2 DEFINITION. $a \subset b$ in L if a and b are in K and $a \subset b$ in K and $0 \subset a$ in L for every element, a, in L.

10.3 THEOREM. *L is an ordered set, closed under the operations for which the closure of K is specified, and satisfies Postulate* 4.4.

10.4 THEOREM. *If every element of K corresponds to itself in L, then all the operations of* §3 *which can be performed in K are preserved in L.*

10.5 THEOREM. *Every set, closed under the operations for which the closure of K is specified, which satisfies Postulate* 4.4 *and has a subset isomorphic with K has a subset of isomorphic with L.*

It is clear that if K is a multiplicative system, a lattice, or a modular or distributive lattice, then so is L. If K does not satisfy Postulate 4.4*, then a unit I can be adjoined in an entirely analogous manner.

11. **Imbedding an ordered set in a complete lattice.** Given an arbitrary ordered set, K, with units, we construct a canonical extension, L, of K to a complete lattice. The assumption that K have units involves no loss of generality, for units can be adjoined to any ordered set by the method of §10. The extension to a complete lattice is accomplished by means of a generalization of the cuts used by Dedekind† to define irrational numbers. This method has the advantage of preserving in form the essential dualism between sums and products. A cut can be interpreted either as a sum or as a product. This extension is necessarily relative.

11.1 DEFINITION. A *cut* $[A, B]$, of K consists of two sets, A and B, of elements, a_i and b_j, of K which satisfy the following conditions:

a. $a_i \subset b_j$ for every a_i in A and every b_j in B,

b. $a_i \subset x$ for every a_i in A implies x in B,

c. $y \subset b_j$ for every b_j in B implies y in A.

The unit 0 of K is always in the initial set, A, of a cut and the unit I is always in the final set, B, of a cut. By assuming that K has units we have eliminated the possibility that either of the sets of a cut be void. Furthermore, if the two sets of a cut have elements in common, the common elements are equal.

11.2 DEFINITION. The *section*, $|A_i|$, of an aggregate of sets, A_i, is the set of elements common to every A_i.

11.3 LEMMA. *If $[A_i, B_i]$ is any set of cuts in K, A is $|A_i|$ (B is $|B_i|$), and*

† Dedekind (5).

B (A) is the set of all elements b_k (a_j) such that $a_j \subset b_k$ for every a_j in A (b_k in B), then $[A, B]$ is a cut.

Proof: $[A, B]$ satisfies 11.1a and 11.1b by definition of B. Since $[A_i, B_i]$ is a cut and A is a subset of A_i, $a_j \subset b_{ik}$ for every a_j in A and every b_{ik} in B_i. Therefore, B_i is a subset of B. Hence, $y \subset b_k$ for every b_k in B implies $y \subset b_{ik}$ for every b_{ik} in B_i and, since $[A_i, B_i]$ is a cut, y is in A_i. This holds for every A_i. Therefore, y is in A and $[A, B]$ satisfies 11.1c.

We are now ready to construct the complete lattice extension, L, of the given ordered set, K, with units.

11.4 DEFINITION. The elements of L are the cuts in K.

11.5 DEFINITION. $[A, B] \subset [C, D]$ if every element of A is an element of C.

11.6 THEOREM. *$[A, B] \subset [C, D]$ if, and only if, every element of D is an element of B.*

Proof: Since D is a subset of B and $[A, B]$ is a cut, $a_i \subset d_j$ for every a_i in A and every d_j in D. Hence, since $[C, D]$ is a cut, A is a subset of C. On the other hand, if D is not a subset of B, then $a_i \not\subset d_j$ for some a_i and d_j, and A is not a subset of C.

11.7 THEOREM. *L is a complete lattice.*

Proof: $[A, B] \subset [C, D]$ and $[C, D] \subset [E, F]$ imply $[A, B] \subset [E, F]$, for if A is a subset of C and C, of E, then A is a subset of E. $[A, B] \subset [A, B]$, for every element of A is an element of A. Given any set of cuts $[A_i, B_i]$ and, if $[A, B]$ is the cut defined in 11.3, then $[A, B] = \prod [A_i, B_i]$. For, since every element of A is an element of A_i for every A_i, $[A, B] \subset [A_i, B_i]$ for every $[A_i, B_i]$ and, since any subset of every A_i is a subset of A, $[X, Y] \subset [A_i, B_i]$ for every $[A_i, B_i]$ implies $[X, Y] \subset [A, B]$. Similarly, if $[A, B]$ is the cut defined in 11.3*, then $[A, B] = \sum [A_i, B_i]$.

11.8 LEMMA. *If a is any element of K, A consists of all elements $a_i \subset a$, and $\overline{A}$ consists of all elements $a \subset \bar{a}_j$, then $[A, \overline{A}]$ is a cut.*

11.9 THEOREM. *The correspondence $a \longleftrightarrow [A, \overline{A}]$, where $[A, \overline{A}]$ is defined as in* 11.8, *establishes an isomorphism between K and a subset of L which preserves the relation $\subset$, units, sums, and products.*

Proof: $[A, \overline{A}] \subset [B, \overline{B}]$ if, and only if, $a \subset b$. For $a_i \subset a \subset b$ implies $a_i \subset b$ and a_i in B, for every a_i in A. Conversely, if every a_i in A is in B; then, in particular, a is in A; hence, in B; and $a \subset b$. $0 \longleftrightarrow [0, K]$ and $I \longleftrightarrow [K, I]$, preserving units. $[B, \overline{B}] = \prod [A_i, \overline{A}_i]$ if, and only if, $b = \prod a_i$. For $b = \prod a_i$ implies $b \subset a_i$ and $[B, \overline{B}] \subset [A_i, \overline{A}_i]$. If $[X, Y] \subset [A_i, \overline{A}_i]$, then $x_j \subset a_i$, $x_j \subset b$, and x_j

is in B. Hence, $[X, Y] \subset [B, \overline{B}]$. Conversely, if $[B, \overline{B}] = \prod [A_i, \overline{A}_i]$, then $b \subset a_i$ and $x \subset a_i$ implies $[X, \overline{X}] \subset [A_i, \overline{A}_i]$, $[X, \overline{X}] \subset [B, \overline{B}]$, and $x \subset b$. An analogous proof holds for sums.

11.10 LEMMA. $[A, B] = \prod [B_i, \overline{B}_i]$ ($[A, B] = \sum [A, \overline{A}_i]$) *and* $\prod [B_i, \overline{B}_i]$ ($\sum [A_i, \overline{A}_i]$) *is distributive.*

11.11 THEOREM. *The correspondence* $a \longleftrightarrow [A, \overline{A}]$ *preserves modular and distributive sums and products, complements, and sum and product complements.*

Proof: $[B, \overline{B}] = \prod [A_i, \overline{A}_i]$ is distributive if, and only if, $b = \prod a_i$ is distributive. For, if $[B, \overline{B}] \subset [X, Y]$, then $[B, \overline{B}] \subset [Y_j, \overline{Y}_j]$ and $b \subset y_j$. Therefore, since $\prod a_i$ is distributive, $y_j = \prod z_{jk}$ where, for each z_{jk}, a_i exists such that $a_i \subset z_{jk}$. But, since $[X, Y] = \prod [Y_j, \overline{Y}_j]$, $[X, Y] = \prod\prod [Z_{jk}, \overline{Z}_{jk}]$ and, for each $[Z_{jk}, \overline{Z}_{jk}]$, $[A_i, \overline{A}_i]$ exists such that $[A_i, \overline{A}_i] \subset [Z_{jk}, \overline{Z}_{jk}]$. Therefore, $\prod [A_i, \overline{A}_i]$ is distributive. Conversely, if $[B, \overline{B}] = \prod [A_i, \overline{A}_i]$ is distributive and $b \subset x$, then $[B, \overline{B}] \subset [X, \overline{X}]$ and x can be distributed since $[X, \overline{X}]$ can. An analogous proof holds for distributive sums. By a similar proof it can be shown that $[A, \overline{A}][B, \overline{B}]$ is modular with respect to $[A, \overline{A}]$ if, and only if, ab is modular with respect to a. In this case we have $[A, \overline{A}][B, \overline{B}] \subset [X, Y] \subset [A, \overline{A}]$. Hence, by deleting from $\prod [Y_j, \overline{Y}_j]$ all factors such that $[A, \overline{A}] \subset [Y_j, \overline{Y}_j]$, the proof goes through as before. An analogous proof holds for modular sums. Complements are preserved, since sums, products, and units are. Let $a \longleftrightarrow [A, \overline{A}]$ and $a' \longleftrightarrow [A', \overline{A}']$. Then $[A', \overline{A}']$ is the product complement of $[A, \overline{A}]$ if, and only if, a' is the product complement of a. If a' is the product complement of a, then $aa' = 0$ and $[A, \overline{A}][A', \overline{A}'] = [0, K]$. Furthermore, if $[A, \overline{A}][X, Y] = [0, K]$ then $[A, \overline{A}][X_i, \overline{X}_i] = [0, K]$, $ax_i = 0$, $x_i \subset a'$, and $[X_i, \overline{X}_i] \subset [A', \overline{A}']$. Since $[X, Y] = \sum [X_i, \overline{X}_i]$ by 11.8*, $[X, Y] \subset [A', \overline{A}']$ follows. Hence, $[A', \overline{A}']$ is the product complement of $[A, \overline{A}]$. Conversely, if $[A', \overline{A}']$ is the product complement of $[A, \overline{A}]$ and $ax = 0$, then $[A, \overline{A}][X, \overline{X}] = [0, K]$, $[X, \overline{X}] \subset [A', \overline{A}']$, $x \subset a'$, and a' is the product complement of a.

From Lemma 11.10 we see that any cut $[A, B]$ can be represented in terms of Φ and K either as $\prod b_i$ or $\sum a_i$. We choose $\sum a_i$ as the unique representation of $[A, B]$.

11.12 THEOREM. *All the relations of* L *are required by* K *and the representation* $\sum a_i$ *of* $[A, B]$.

Proof: If $[A, B] \subset [C, D]$, then A is a subset of C and $\sum a_i \subset \sum c_j$. If $\sum a_i \subset \sum c_j$, then d_k in D implies $c_j \subset d_k$, $\sum c_j \subset d_k$, $\sum a_i \subset d_k$, and $a_i \subset d_k$. Therefore, d_k is in B, D is a subset of B, and $[A, B] \subset [C, D]$.

Since K is an arbitrary ordered set with units, modular and distributive

lattices are eligible for extension by cuts. The question as to whether the extension by cuts of a modular or distributive lattice is itself a modular or distributive lattice has not yet been answered. The corresponding question for Boolean algebras is answered by the next theorem.

11.13 THEOREM. *If K is a Boolean algebra, then the extension, L, by cuts of K is a complete Boolean algebra.*

Proof: Postulates 1.1 and 4.5 hold by Theorem 11.7. If $[A, B]$ is a cut and A' and B' consist of the complements of the elements of A and B, then $[B', A']$ is a cut, the product complement of $[A, B]$ and $[A, B] \longleftrightarrow [B', A']$ is a dual automorphism.

From Theorem 9.13 we find that the canonical extension of K to a multiplicative system is the subset of L consisting of the image of K and products of finite numbers of these elements. The canonical extension of K to a complete multiplicative system is the subset of L consisting of the image of K and products of any number of these elements. The subset of L consisting of the image of K and all sums and products of finite numbers of these elements is the canonical extension of K to a lattice. However, since a complete multiplicative system with a unit I is a complete lattice, the canonical extension of K to a complete multiplicative system is the entire set L.

11.14 THEOREM. *The extension to a complete lattice of an ordered set with units is necessarily relative.*

Proof: Consider the set of elements a, b, c, and d satisfying 1.2 and no additional relations except $d \subset a$ and $d \subset b$. In the entire set, $ab = d \neq abc$. In the extension of the subset a, b, and c, $ab = abc$.

The elements of this extension are already in normal form.

12. **Extension of a multiplicative system to a distributive lattice.** The next problem is the construction of a canonical extension, L, to a lattice with completely distributive sums of an arbitrary multiplicative system with units, K. To be precise, the operations Θ under which K is closed are units and finite products. The operations Φ under which L is to be closed are units, finite products, and unrestricted distributive sums. As usual, subsets of K are denoted by capital letters.

12.1 DEFINITION. The elements of L are the subsets of K.

12.2 DEFINITION. $A \subset B$ if, for every a_i in A, $a_i = \sum_j a_i b_j$, where the sum is taken over every b_j in B, and $\sum_j a_i b_j$ is distributive.

12.3 THEOREM. *The relation $\subset$ of Definition 12.2 orders the elements of L.*

Proof: If $A \subset B$ and $B \subset C$, then $a_i = \sum_j a_i b_j$ for every a_i in A and

$b_j = \sum_k b_j c_k$ for every b_j in B. Substitute for each b_j in the first set of equations its value as a sum from the second set of equations, then use the first set of equations multiplied through by an element of C and distributed to eliminate the elements of B. That is, $a_i = \sum_j a_i b_j = \sum_j \sum_k a_i b_j c_k = \sum_k a_i c_k$. Since $\sum_j a_i b_j$ and $\sum_k b_j c_k$ are distributive, $\sum_k a_i b_j c_k$, $\sum_j \sum_k a_i b_j c_k$, and $\sum_k a_i c_k$ are distributive. Therefore, $A \subset C$. Clearly, $A \subset A$. Hence, L is ordered by the relation $\subset$.

12.4 THEOREM. *L contains units, 0 and I, namely, the elements of L consisting solely of the 0 or I of K.*

12.5 THEOREM. *$B = \prod A_i$ for any finite set of elements A_i of L, if B is the set of all products with one component from each A_i as factors.*

Proof: For each b_k in B there exists some a_{ik} in each A_i such that $b_k a_{ij} = b_k$, for each b_k has a factor from each A_i. Therefore, $b_k = \sum_j b_k a_{ij}$ and $\sum_j b_k a_{ij}$ is distributive, since b_k appears as a summand. This implies $B \subset A_i$ for every A_i. If $X \subset A_i$ for every A_i, then $x_m = \sum_j x_m a_{ij}$ and $\sum_j x_m a_{ij}$ is distributive. Hence, $x_m = \prod_i \sum_j x_m a_{ij} = \sum x_m \prod_i a_{ij} = \sum_k x_m b_k$ and $\sum_k x_m b_k$ is distributive. This implies $X \subset B$. Therefore, $B = \prod A_i$.

12.6 THEOREM. *$B = \sum A_i$ for any set of elements A_i of L and $\sum A_i$ is distributive, if B is the set of all elements which are components of some A_i.*

Proof: Since every component a_{ij} of every A_i is an element b_k of B, $a_{ij} = \sum_k a_{ij} b_k$ and $\sum_k a_{ij} b_k$ is distributive, for a_{ij} appears as a b_k. This implies $A_i \subset B$ for every A_i. If $A_i \subset X$ for every A_i, then $a_{ij} = \sum_m a_{ij} x_m$ and $\sum_m a_{ij} x_m$ is distributive for every a_{ij} in every A_i. But every b_k is an a_{ij}, hence, $b_k = \sum_m b_k x_m$ and $\sum_m b_k x_m$ is distributive for every b_k in B. This implies $B \subset X$. Therefore, $B = \sum A_i$. $X \subset \sum A_i$ implies $x_m = \sum_i \sum_j x_m a_{ij}$ and that $\sum_i \sum_j x_m a_{ij}$ is distributive. It follows that $x_m = \sum_i \sum_j \sum_k x_m x_k a_{ij}$ and that $\sum_i \sum_j \sum_k x_m x_k a_{ij}$ is distributive. Hence $X \subset \sum_i XA_i$, but $\sum_i XA_i \subset X$ in any ordered set. Therefore, $X = \sum_i XA_i$ and $\sum A_i$ is distributive.

Since L is closed under unrestricted sums and has a unit 0, it is a complete lattice. That is, L is closed under unrestricted products. If K is a complete multiplicative system and the multiplicative axiom is assumed, then Theorem 12.5 holds for unrestricted products and we have a representation of all products in terms of the components of their factors. However, we make no use of this representation. Furthermore, by Lemma 7.6, every element of L has a product complement.

12.7 THEOREM. *The correspondence $a \longleftrightarrow A$, where A is the subset of K consisting solely of the element a, is an isomorphism between K and a subset of L such that units, finite products, and unrestricted distributive sums are preserved.*

Proof: $a \subset b$ implies $a = ab$ which is trivially distributive. Hence, $A \subset B$, where $a \longleftrightarrow A$ and $b \longleftrightarrow B$. Conversely, $A \subset B$ implies $a = ab$ and $a \subset b$. Units are preserved, as shown in 12.4, and finite products are preserved, as shown in 12.5. Distributive sums are preserved, for, if $a = \sum a_i$ is distributive and $a \longleftrightarrow A$ and $a_i \longleftrightarrow A_i$, then $a_i \subset a$ and $A_i \subset A$. If $A_i \subset X$ for every A_i, then $a_i = \sum_m a_i x_m$ and $\sum_m a_i x_m$ is distributive. Adding and applying the distributive property of $\sum a_i$, $a = \sum_m a x_m$ and $\sum_m a x_m$ is distributive. Therefore $A \subset X$ and $A = \sum A_i$.

12.8 THEOREM. *The isomorphism of Theorem* 12.7 *preserves unrestricted products and product complements.*

Proof: If $a = \prod a_i$, then $a \subset a_i$ and $A \subset A_i$. If $X \subset A_i$ for every A_i, then $x_m = x_m a_i$ for every x_m and a_i. Multiplying, $x_m = x_m a$ for every x_m and $X \subset A$. Therefore, $A = \prod A_i$. If a' is the product complement of a and $a \longleftrightarrow A$ and $a' \longleftrightarrow A'$, then $AA' = 0$. By Theorem 12.5, $AX = 0$ implies $ax_m = 0$ for every x_m in X. Hence, $x_m \subset a'$, $x_m = x_m a'$, and $X \subset A'$. Therefore, A' is the product complement of A.

12.9 THEOREM. *The isomorphism of Theorem* 12.7 *does not preserve, in general, sums which are not distributive or sum complements.*

Proof: Consider Example [4.7, 4.10, 4.7]. This set is a multiplicative system. When extended to a distributive lattice, neither $a+c$ nor c^*, of 4.10, are preserved.

Complements are not necessarily preserved, since sums are not preserved, in general. From Theorem 12.6, it follows that each element of L can be represented as the distributive sum of its components. We choose this representation. Thus an arbitrary element, X, of L is represented by $\sum x_m$, where $\sum x_m$ is distributive.

12.10 THEOREM. *All the relations of L are required by K and the representation* $\sum x_m$, *where* $\sum x_m$ *is distributive, of* X.

Proof: If $\sum x_m \subset \sum y_n$, then, since $x_i \subset \sum x_m$ for every x_i in X, $x_i \subset \sum y_n$ and $x_i = x_i \sum y_n = \sum_n x_i y_n$, by Theorem 6.3, and $\sum_n x_i y_n$ is distributive, by Theorem 6.6. Hence, $X \subset Y$. If $X \subset Y$, then $x_i = \sum_n x_i y_n = x_i \sum y_n$ and $x_i \subset \sum y_n$. Therefore, $\sum x_m \subset \sum y_n$.

Since L is closed for unrestricted products and product complements and the isomorphism of Theorem 12.7 preserves these operations, they may perfectly well be added to the set Φ of operations to which K has been extended, without any further changes. Since the isomorphism of Theorem 12.7 does not, in general, preserve non-distributive sums and sum complements, these operations may not be added to the set Φ.

If the finite subsets of K are taken as the elements of L, the canonical extension of K to a distributive lattice is obtained.

This is an absolute extension.

We now seek a normal form for this extension. Although it is desirable that a normal form preserve the finiteness of an element which has a finite representation, in this case it is easy to construct examples to show that this is not possible without the introduction of a finite descending chain condition or some other type of well ordering. For example, let K be the set of proper subsets of the positive integers where $a \subset b$ if a is a subset of b. K does not have a unit I. The extension, L, has a unit I which can be represented in infinitely many ways as the sum of two or any finite number of elements of K, but, lacking any form of the axiom of choice, we are unable to choose one from among them. The normal form is obtained by selecting a subsystem of L in which the converse of the following lemma is true.

12.11 LEMMA. *If A and B are in L and A is a subset of B, then $A \subset B$.*

12.12 DEFINITION. An element A of L is in normal form if $x = \sum a_i x$ and $\sum a_i x$ distributive imply x in A.

$\bar{L}$ denotes the subset of L containing all the elements of L which are in normal form. If an element is in $\bar{L}$, it will be denoted with a bar as $\bar{A}$. $\bar{L}$, being a subset of L, is ordered by the ordering relation of L.

12.13 LEMMA. *If $\bar{A} \subset \bar{B}$, then $\bar{A}$ is a subset of $\bar{B}$.*

12.14 THEOREM. *$\bar{A} \subset \bar{B}$ if, and only if, $\bar{A}$ is a subset of $\bar{B}$.*

12.15 COROLLARY. *$\bar{A} = \bar{B}$ if, and only if, $\bar{A}$ and $\bar{B}$ are identical.*

12.16 THEOREM. *If A is in L and C consists of all elements c_k such that $c_k = \sum a_i c_k$ where $\sum a_i c_k$ is distributive, then C is in normal form and $A = C$.*

Proof: If $x = \sum c_k x$ and $\sum c_k x$ is distributive, then $x = \sum_k \sum_i a_i c_k x = \sum_i \sum_k a_i c_k x = \sum_i a_i x$ and $\sum a_i x$ is distributive. Hence, C is in normal form. $C \subset A$ by definition of C and 12.2 and, since $a_k = \sum_i a_i a_k$ where $\sum_i a_i a_k$ is distributive, A is a subset of C and $A \subset C$ by 12.11. Therefore, $A = C$.

The elements of $\bar{L}$ can be characterized as follows.

12.17 DEFINITION. A subset, A, in K is an *ideal* in K, where K is any partially ordered set, if a_i in A and $x \subset a_i$ imply x in A and $x = \sum_j a_{ij}$, where $\sum_j a_{ij}$ is the distributive sum of a subset A_i of A, implies x in A.

12.18 THEOREM. *If K is a multiplicative system with unit 0, then $\bar{L}$ is the set of ideals in K.*

Proof: If A is an ideal, then $x = \sum a_i x$ and $\sum a_i x$ distributive imply x in A,

since, for each a_i, $a_i x \subset a_i$. Hence, A is in $\bar{L}$. If A is in $\bar{L}$, then $x \subset a_i$ implies $x = \sum a_i x$, where $\sum a_i x$ is distributive, and $x = \sum_j a_{ij}$, where $\sum_j a_{ij}$ is distributive and A_i is a subset of A, implies $x = \sum a_i x$ and $\sum a_i x$ distributive. Hence, A is an ideal.

The term ideal has been used because, if A is an ideal in K, then the product of any element in A with any element in K, if it exists in K, is in A and the sum, interpreted as distributive sum, of any subset of A, if it exists in K, is in A. The extension of any partially ordered set K, with unit 0, to a lattice with completely distributive sums can be effected at once by letting L be the set of ideals in K and defining $A \subset B$ if A is a subset of B. The initial set of a cut is an ideal. In fact, by Theorem 5.17, the sum, not necessarily distributive, of any subset of the initial set of a cut, if it exists in K, is in the initial set of the cut. However, cuts cannot be so characterized.

Extension by ideals is necessarily relative.

13. **Extension of a distributive lattice to a Boolean algebra.** We now construct the canonical extension to a Boolean algebra of a distributive lattice with units. The first step is to imbed the given distributive lattice, K, in a multiplicative system, K', where the product complements of the image of K exist and are isomorphic to K with respect to the dual relation. The second step is the application to K' of the previously established extension to a distributive lattice of a multiplicative system. The resulting distributive lattice is shown to be the sought for Boolean algebra.

13.1 DEFINITION. The elements of K' are the ordered pairs, $(a, \bar{a})$, where a and $\bar{a}$ are arbitrary elements of K.

13.2 DEFINITION. $(a, \bar{a}) \subset (b, \bar{b})$ if $a \subset \bar{a}+b$ and $a\bar{b} \subset \bar{a}$.

13.3 THEOREM. *The relation* $\subset$ *of Definition* 13.2 *orders the elements of* K'.

Proof: $(a, \bar{a}) \subset (b, \bar{b})$ and $(b, \bar{b}) \subset (c, \bar{c})$ imply $a \subset \bar{a}+b$, $a\bar{b} \subset \bar{a}$, $b \subset \bar{b}+c$, and $b\bar{c} \subset \bar{b}$. Hence, $a \subset \bar{a}+b \subset \bar{a}+\bar{b}+c$, $a = a(\bar{a}+\bar{b}+c) = a\bar{a}+a\bar{b}+ac \subset \bar{a}+a\bar{b}+c = \bar{a}+c$, and $a\bar{c} = a\bar{c}(\bar{a}+b) = a\bar{a}\bar{c}+ab\bar{c} \subset a\bar{a}\bar{c}+a\bar{b} \subset a\bar{a}\bar{c}+\bar{a} = \bar{a}$. Therefore, $(a, \bar{a}) \subset (c, \bar{c})$. $(a, \bar{a}) \subset (a, \bar{a})$, since $a \subset \bar{a}+a$ and $a\bar{a} \subset \bar{a}$.

13.4 THEOREM. $\prod(a_i, \bar{a}_i) = (\prod a_i, \sum \bar{a}_i)$, *if* $\prod a_i$ *and* $\sum \bar{a}_i$ *exist in* K *and are distributive.*

Proof: $\prod a_i \subset \sum \bar{a}_i + a_n$ and $\bar{a}_n \prod a_i \subset \bar{a}_n \subset \sum \bar{a}_i$ for every a_n and $\bar{a}_n$. If $(x, \bar{x}) \subset (a_n, \bar{a}_n)$ for every element in $\prod(a_i, \bar{a}_i)$, then $x \subset \bar{x}+a_n$ and $x\bar{a}_n \subset \bar{x}$. Multiplying or adding and distributing we get $x \subset \prod(\bar{x}+a_i) = \bar{x}+\prod a_i$ and $x\sum \bar{a}_i = \sum x\bar{a}_i \subset \bar{x}$.

13.5 COROLLARY. K' *is a multiplicative system.*

In general, sums do not exist in K', so a theorem for sums corresponding to Theorem 13.4 for products must necessarily be much more restricted. Theorem 13.6 is not the most general theorem possible, but it covers the cases which we require.

13.6 THEOREM. *$\sum(a_i, \bar{a}_i) = (\sum a_i, \prod \bar{a}_i)$ and $\sum(a_i, \bar{a}_i)$ is distributive if $\sum a_i$ and $\prod \bar{a}_i$ exist in K and are distributive and $a_j\bar{a}_j \subset a_k + \bar{a}_k$ whenever $(a_j, \bar{a}_j)$ and $(a_k, \bar{a}_k)$ are summands of $\sum(a_i, \bar{a}_i)$.*

Proof: Since $a_j \subset \sum a_i + \bar{a}_j$ and $a_j \prod \bar{a}_i \subset \bar{a}_j$ whenever $(a_j, \bar{a}_j)$ is a summand of $\sum(a_i, a_i)$, $(a_j, \bar{a}_j) \subset (\sum a_i, \prod \bar{a}_i)$. If $(a_j, \bar{a}_j) \subset (x, \bar{x})$ for every summand of $\sum(a_i, a_i)$, then $a_j \subset \bar{a}_j + x$ and $a_i\bar{x} \subset \bar{a}_j$. Hence, $a_j = a_j\bar{a}_j + a_jx \subset a_j\bar{a}_j + x \subset a_k + \bar{a}_k + x \subset \bar{a}_k + x$, $\sum a_i \subset \bar{a}_k + x$, and $\sum a_i \subset \prod[\bar{a}_i + x] = \prod \bar{a}_i + x$. Similarly, $a_j\bar{x} \subset a_j\bar{a}_j\bar{x} \subset [a_k + \bar{a}_k]\bar{x} \subset a_k\bar{x} + \bar{a}_k\bar{x} \subset \bar{a}_k$, $[\sum a_i]\bar{x} = \sum a_i\bar{x} \subset \bar{a}_k$, and $[\sum a_i]\bar{x} \subset \prod \bar{a}_i$. Therefore, $(\sum a_i, \prod \bar{a}_i) \subset (x, \bar{x})$ and $(\sum a_i, \prod \bar{a}_i) = \sum(a_i, \bar{a}_i)$. Furthermore, by Theorems 6.6 and 6.6*, $\sum a_i b$ and $\prod[\bar{a}_i + b]$ exist and are distributive and $a_j\bar{a}_j \subset a_k + \bar{a}_k$ implies $ba_j[b + \bar{a}_j] \subset ba_k + b + \bar{a}_k$. Therefore, $(b\sum a_i, b + \prod \bar{a}_i) = (\sum ba_i, \prod[b + \bar{a}_i]) = \sum(ba_i, b + \bar{a}_i)$ and $\sum(a_i, \bar{a}_i)$ is distributive.

13.7 COROLLARY. *$\sum(a_i, \bar{a}_i) = (\sum a_i, \prod \bar{a}_i)$ if $\sum a_i$ and $\prod \bar{a}_i$ exist and are distributive and if, for every two different summands $(a_j, \bar{a}_j)$ and $(a_k, \bar{a}_k)$ of $\sum(a_i, \bar{a}_i)$, at least one of the relations $a_j \subset a_k$, $a_j \subset \bar{a}_k$, $\bar{a}_j \subset a_k$, or $\bar{a}_j \subset \bar{a}_k$ holds.*

13.8 COROLLARY. *$\sum(a_i, \bar{a}) = (a, \bar{a})$ and $\sum(a_i, \bar{a})$ is distributive if $a = \sum a_i$ and $\sum a_i$ is distributive.*

13.9 COROLLARY. *$\sum(a, \bar{a}_i) = (a, \bar{a})$ and $\sum(a, \bar{a}_i)$ is distributive if $\bar{a} = \prod \bar{a}_i$ and $\prod \bar{a}_i$ is distributive.*

13.10 COROLLARY. $(a + \bar{a}, \bar{a}) = (a, \bar{a}) = (a, a\bar{a})$.

13.11 COROLLARY. *$(a, \bar{a}) + (\bar{a}, \bar{a}) = (a + \bar{a}, \bar{a}\bar{a})$ and $(a, \bar{a}) + (\bar{a}, \bar{a})$ is distributive. In particular, if $\bar{a} \subset \bar{a} \subset a$, then $(a + \bar{a}, \bar{a}\bar{a}) = (a, \bar{a})$.*

13.12 THEOREM. *$(0, 0)$ is a zero and $(\mathrm{I}, 0)$ is a one of K'.*

Proof: $0 \subset 0 + a$, $0\bar{a} \subset 0$, $a \subset \bar{a} + \mathrm{I}$, and $a0 \subset \bar{a}$, where $(a, \bar{a})$ is any element of K'.

It follows from Corollary 13.10 that we could have restricted the elements, $(a, \bar{a})$, of K' to those ordered pairs of K elements which satisfy the condition $\bar{a} \subset a$. We make use of this fact later in finding a normal representation of this extension.

13.13 THEOREM. *The correspondence $a \longleftrightarrow (a, 0)$ is an isomorphism between K and a subset of K' which preserves units, distributive sums, and complements.*

Proof: $a \subset b$ implies $(a, 0) \subset (b, 0)$ and conversely. $0 \longleftrightarrow (0, 0)$ and $\mathrm{I} \longleftrightarrow (\mathrm{I}, 0)$.

Distributive sums are preserved, by 13.8. If a and b are complementary in K with respect to I and 0, then $ab=0$ and $a+b=\mathrm{I}$. Since K is a distributive lattice, ab, $a+b$, and $0+0$ are distributive. Hence, by 13.4, $(a, 0)(b, 0)=(ab, 0)=(0, 0)$, the zero of K' and, by 13.8, $(a, 0)+(b, 0)=(a+b, 0)=(\mathrm{I}, 0)$ the one of K'.

13.14 THEOREM. *The correspondence $a\longleftrightarrow(a, 0)$ preserves distributive products in K as products in K'.*

Proof: If $a=\prod a_i$ and $\prod a_i$ is distributive, then, by 13.4, since $\sum 0=0$ is distributive, $\prod(a_i, 0)=(a, 0)$.

The images in K' of distributive products in K are not, in general, distributive. Sum and product complements in K are not, in general, preserved in K'.

13.15 THEOREM. *(I, a) and $(a, 0)$ are mutually complementary with respect to* I *and* 0, *and each is the product complement of the other.*

Proof: $(\mathrm{I}, a)(a, 0)=(a, a)=(0, 0)$. By 13.11, $(\mathrm{I}, a)+(a, 0)=(\mathrm{I}, 0)$. If $(a, 0)(y, \bar{y})=(0, 0)$, then $ay\subset\bar{y}$ implying $(y, \bar{y})\subset(\mathrm{I}, a)$, and if $(\mathrm{I}, a)(z, \bar{z})=(0, 0)$, then $z\subset a+\bar{z}$ implying $(z, \bar{z})\subset(a, 0)$.

13.16 THEOREM. *The correspondence $(a, 0)\longleftrightarrow(\mathrm{I}, a)$ is an isomorphism between the image of K and its product complements ordered by the dual relation.*

Proof: $a\subset b$ is necessary and sufficient for both $(a, 0)\subset(b, 0)$ and $(\mathrm{I}, b)\subset(\mathrm{I}, a)$.

Since $(a, \bar{a})=(a, 0)(\mathrm{I}, \bar{a})$, any element, $(a, \bar{a})$, in K' can be represented in terms of K, products, and product complements as $a\bar{a}'$, where $\bar{a}'$ is the product complement of $\bar{a}$. We choose this as the unique representation of the elements of K' in terms of K and the operations of a Boolean algebra. We recollect that in a Boolean algebra complements are identified with product complements and any element is equal to the product complement of its product complement.

13.17 THEOREM. *All the relations of K' are required by K and the representation $a\bar{a}'$ of $(a, \bar{a})$.*

Proof: If $a\bar{a}'\subset b\bar{b}'$, then, multiplying through by b' and by $\bar{b}$, $a\bar{a}'b'=0$ and $a\bar{a}'\bar{b}=0$. These conditions imply $a\subset\bar{a}+b$ and $a\bar{b}\subset\bar{a}$, by 7.13, 7.16, and 7.30. Hence, $(a, \bar{a})\subset(b, \bar{b})$. If $(a, \bar{a})\subset(b, \bar{b})$, then $a\subset\bar{a}+b$, $a\bar{b}\subset\bar{a}$, $a\bar{a}'b'=0$, $a\bar{a}'\bar{b}=0$, $a\bar{a}'\subset b$, $a\bar{a}'\subset\bar{b}'$, and $a\bar{a}'\subset b\bar{b}'$.

We have shown that every Boolean algebra with a subset isomorphic with K has a subset isomorphic with K'. Since K' is not necessarily closed with respect to product complements, it is not, in general, a Boolean algebra. Let

L be the canonical extension of K' to a distributive lattice. Then the elements of L can be represented as distributive sums of finite subsets of K', and L satisfies Postulates 1.1 and 4.1 of the postulates for a Boolean algebra. Furthermore, since a Boolean algebra is a distributive lattice, every Boolean algebra with a subset isomorphic with K has a subset isomorphic with L. In particular, if L is a Boolean algebra, then it is the sought for extension of K.

13.18 THEOREM. *L, the canonical extension to a distributive lattice of K', is a Boolean algebra.*

Proof: Postulates 1.1 and 4.1 hold in L. Since the extension from K' to L preserves distributive sums and products, 13.11 and 13.15 imply that $(a, 0)$ and (I, a) are mutually complementary with respect to I and 0 in L even though the extension from K' to L need not, in general, preserve complements. Since $(a, \bar{a}) = (a, 0)(\mathrm{I}, \bar{a})$, $(a, \bar{a})$ has a complement in L, by 7.32 remembering that all finite products in L are distributive. Again applying 7.32, all finite sums, $\sum(a_i, \bar{a}_i)$, have complements in L. But these sums include all of L. By 7.30, each of two complementary elements in L is the product complement of the other. Hence, each element of L has a product complement and, applying 3.20, 7.3 holds. Therefore, L is a Boolean algebra.

The extension from K to L is absolute.

Applying Theorem 11.13, L can be extended by cuts to a complete Boolean algebra. This can also be done directly by extending K' to a lattice with completely distributive sums. Both these extensions are canonical and are, therefore, isomorphic. The first of these methods is a relative extension, the second, absolute. We can now state the following general theorem.

13.19 THEOREM. *Any ordered set, K, can be extended by a canonical extension to a complete Boolean algebra, L, so as to preserve units, finite products and unrestricted distributive sums.*

Theorem 13.19 is not a complete statement of the results of the extensions which we have considered, for further restrictions on K or fewer restrictions on L may make it possible to preserve in L more of the properties of K. To to ascertain this, it is necessary to refer to the operations preserved by each extension, separately.

We now seek a normal form for the extension from a distributive lattice, K, to a Boolean algebra, L. That is, the elements $\sum(a_i, \bar{a}_i)$ of L can be divided into mutually exclusive and jointly exhaustive classes of equal elements, and we propose to select a unique representative from each of these

classes. Since the elements of L are finite sums of elements of K', we may use the integers from 0 to n as subscripts for an element of $n+1$ terms. We first show that the subset, $\bar{L}$, of elements $\sum_{i=0}^{n}(a_i, \bar{a}_i)$ of L which satisfy the conditions $a_j \subset \bar{a}_k \subset a_k$ for $k<j$ is an extension of K to a Boolean algebra and must, therefore, contain elements from each class of equal elements in L. Elements of $\bar{L}$ will be denoted with a minus sign, $\sum_{i=0}^{n}(a_i-\bar{a}_i)$. This is in keeping with the fact that elements of $\bar{L}$ may be considered as the symmetric difference of the elements of K involved. Abstractly, the elements of $\bar{L}$† are the set of all finite descending sequences of K elements with an even number of terms.

13.20 THEOREM. *If* $\sum_{i=0}^{n}(a_i-\bar{a}_i)$ *is in* $\bar{L}$, *then* $\sum_{i=0}^{n+1}(\bar{a}_{i-1}-a_i)$, *where* $\bar{a}_{-1}=\mathrm{I}$ *and* $a_{n+1}=0$, *is in* $\bar{L}$ *and* $\sum_{i=0}^{n}(a_i-\bar{a}_i)$ *and* $\sum_{i=0}^{n+1}(\bar{a}_{i-1}-a_i)$ *are mutually complementary in* L.

Proof: $\sum_{i=0}^{n}(a_i-\bar{a}_i)+\sum_{i=0}^{n+1}(\bar{a}_{i-1}-a_i)=(\mathrm{I}-0)$, by repeated application of 13.11.

$$\left[\sum_{i=0}^{n}(a_i-\bar{a}_i)\right]\left[\sum_{i=0}^{n+1}(\bar{a}_{i-1}-a_i)\right]=\sum_{i=0}^{n}\sum_{j=0}^{n+1}(a_i\bar{a}_{j-1},\ \bar{a}_i+a_j)=(0-0),$$

since $a_i\bar{a}_{j-1} \subset \bar{a}_i+a_j$ for all values of i and j.

13.21 THEOREM. *If* $\sum_{i=0}^{n}(a_i-\bar{a}_i)$ *and* $\sum_{j=0}^{m}(b_j-\bar{b}_j)$ *are in* $\bar{L}$, *then* $\sum_{k=0}^{m+n}(c_k-\bar{c}_k)$, *where* $c_k=\sum_{i=0}^{k}a_ib_{k-i}$ *and* $\bar{c}_k=\sum_{i=0}^{k}(\bar{a}_ib_{k-i}+a_i\bar{b}_{k-i})$, *is in* $\bar{L}$ *and* $[\sum_{i=0}^{n}(a_i-\bar{a}_i)][\sum_{j=0}^{m}(b_j-\bar{b}_j)]=\sum_{k=0}^{m+n}(c_k-\bar{c}_k)$.

Proof: $\bar{c}_k \subset c_k$, for $\bar{a}_ib_{k-i} \subset a_ib_{k-i}$ and $a_i\bar{b}_{k-i} \subset a_ib_{k-i}$. $c_j \subset \bar{c}_k$ if $k<j$, for $a_ib_{j-i} \subset a_i\bar{b}_{k-i} \subset \bar{a}_ib_{k-i}+a_i\bar{b}_{k-i}$. Hence $\sum(c_k-\bar{c}_k)$ is in $\bar{L}$. Furthermore,

$$\sum_{i=0}^{k}(a_ib_{k-i},\ \bar{a}_i+\bar{b}_{k-i})=\left(\sum_{i=0}^{k}a_ib_{k-i},\ \prod_{i=0}^{k}[\bar{a}_i+\bar{b}_{k-i}]\right)=(c_k-\bar{c}_k),\quad \text{by 13.7},$$

since $a_ib_{k-i} \subset \bar{a}_j+\bar{b}_{k-j}$ for $i=j$, and by 13.10, since $[\sum_{i=0}^{k}a_ib_{k-i}]\prod_{i=0}^{k}[\bar{a}_i+\bar{b}_{k-i}]=\bar{c}_k$. Hence,

$$\left[\sum_{i=0}^{n}(a_i-\bar{a}_i)\right]\left[\sum_{j=0}^{m}(b_j-\bar{b}_j)\right]=\sum_{i=0}^{n}\sum_{j=0}^{m}(a_ib_j,\ \bar{a}_i+\bar{b}_j)=\sum_{k=0}^{m+n}(c_k-\bar{c}_k).$$

We remark that if $\sum_{i=0}^{n}(a_i-\bar{a}_i)$ is represented symbolically as the ordered pair of polynomials $(A-\overline{A})$, where $A=a_0x^n+a_1x^{n-1}+\cdots+a^n$ and $\overline{A}=\bar{a}_0x^n+\bar{a}_1x^{n-1}+\cdots+\bar{a}_n$, then, using the usual laws for the addition and multiplication of polynomials, $(A-\overline{A})(B-\overline{B})=(AB-[\overline{A}B+A\overline{B}])$. This

† Hausdorff (11), p. 78 defines $\bar{L}$ in this way for the case where the elements of K are subsets of a set of points.

symbolic formula is given merely as an aid to carrying out multiplications in $\bar{L}$.

By Theorem 7.30, we know that complements and product complements are identified in L. Hence, since $\bar{L}$ is a subsystem of the Boolean algebra L and is closed under finite multiplication and product complementation, it is a Boolean algebra. Since $a \leftrightarrow (a-0)$, the image of K is in $\bar{L}$, $\bar{L}$ is a canonical extension of K to a Boolean algebra, and L is isomorphic to $\bar{L}$. It is easy to show directly that each element of L is equal to an element of $\bar{L}$. For each element of K' this is true by 13.10. Hence,

$$\sum_{i=0}^{n} (a_i, \bar{a}_i) = \sum_{i=0}^{n} (a_i, a_i\bar{a}_i) = \left[\prod_{i=0}^{n} [(1 - a_i) + (a_i\bar{a}_i - 0)] \right]',$$

where the latter expression can be reduced to the required form by 13.20 and 13.21. We next make a direct attack on the ordering relation in L and $\bar{L}$.

13.22 LEMMA. *$\sum_{i=0}^{n}(a_i, \bar{a}_i) = (0, 0)$ if, and only if, $a_i \subset \bar{a}_i$ for $i = 0, \cdots, n$.*

Proof: If $a_i \subset \bar{a}_i$, then $(a_i, \bar{a}_i) = (0, 0)$ and $\sum_{i=0}^{n}(a_i, \bar{a}_i) = (0, 0)$. If $\sum_{i=0}^{n}(a_i, \bar{a}_i) = (0, 0)$, then $(a_i, \bar{a}_i) \subset (0, 0)$ and $a_i \subset \bar{a}_i$.

13.23 THEOREM. *$\sum_{i=0}^{n}(a_i, \bar{a}_i) \subset \sum_{j=0}^{m}(b_j, \bar{b}_j)$ if, and only if, $a_i \prod \bar{b}_j \subset \bar{a}_i + \sum b_j$ for $i = 0, \cdots, n$ and for all combinations of the indices $j = 0, \cdots, m$ such that each index occurs either in $\bar{b}_j$ or in b_j but not in both, and $\prod \bar{b}_j = I$ if no index occurs on the left and $\sum b_j = 0$ if no index occurs on the right.*

Proof: By 7.27, $\sum_{i=0}^{n}(a_i, \bar{a}_i) \subset \sum_{j=0}^{m}(b_j, \bar{b}_j)$ if, and only if, $[\sum_{i=0}^{n}(a_i, \bar{a}_i)] \prod_{j=0}^{m}(I, b_j) + (\bar{b}_j, 0)] = (0, 0)$. Distributing this product,

$$\left[\sum_{i=0}^{n} (a_i, \bar{a}_i) \right] \prod_{j=0}^{m} [(I, b_j) + (\bar{b}_j, 0)] = \sum (a_i \prod \bar{b}_j, \bar{a}_i + \sum b_j),$$

where the sum is taken over i from 0 to n and over all combinations of j as specified in the statement of the theorem. Applying 13.22, $\sum(a_i \prod \bar{b}_j, \bar{a}_i + \sum b_j) = (0, 0)$ if, and only if, $a_i \prod \bar{b}_j \subset \bar{a}_i + \sum b_j$ for the combinations of indices specified.

13.24 COROLLARY. *$\sum_{i=0}^{n}(a_i - \bar{a}_i) \subset \sum_{j=0}^{m}(b_j - \bar{b}_j)$ if, and only if, $\sum_{i=0}^{k} a_i b_{k-i-1} \subset \sum_{i=0}^{k}(\bar{a}_i b_{k-i-1} + a_i b_{k-i})$ for $k = 0, 1, \cdots, m+n+1$.*

13.25 COROLLARY. *$\sum_{i=0}^{n}(a_i - \bar{a}_i) \subset \sum_{j=0}^{m}(b_j - \bar{b}_j)$ if, and only if, $a_i \bar{b}_{j-1} \subset \bar{a}_i + b_j$ for $i = 0, \cdots, n$ and $j = 0, \cdots, m+1$.*

13.26 COROLLARY. *$\sum_{i=0}^{n}(a_i, \bar{a}_i) \subset \sum_{j=0}^{m}(\bar{b}_j, b_j)$ if, and only if, for any two elements x and y of K, $x \subset y$ whenever $x \subset a_i$, $\bar{a}_i \subset y$ for some subscript i and either $x \subset \bar{b}_j$ or $b_j \subset y$ for every subscript j.*

Corollary 13.26 is in a form which extends immediately to infinite sums as it depends only on the ordering of K, whereas Theorem 13.23 depends on sums and products in K which might not exist in the infinite case.

13.27 THEOREM. *$\sum_{i=0}^{n}(a_i, \bar{a}_i) = (a, \bar{a})$, where $a = \sum_{i=0}^{n} a_i$ and $\bar{a} = \prod_{i=0}^{n} \bar{a}_i$, if, and only if, for any two elements x and y of K, $x \subset y$ whenever $x \subset a$, $\bar{a} \subset y$ and either $x \subset \bar{a}_i$ or $a_i \subset y$ for each subscript i.*

If $\sum a_i$ and $\prod \bar{a}_i$ exist and are distributive, Theorem 13.27 can be generalized at once to the infinite case yielding both necessary and sufficient conditions for the validity of the addition formula of Theorem 13.6.

If we define the *degree* of $\sum_{i=0}^{n}(a_i, \bar{a}_i)$ as n, then the degree of $[\sum_{i=0}^{n}(a_i, \bar{a}_i)]'$ is $2^{n+1}-1$, that of $\sum_{i=0}^{n}(a_i, \bar{a}_i) + \sum_{j=0}^{m}(b_j, \bar{b}_j)$ is $m+n+1$, that of $[\sum_{i=0}^{n}(a_i, \bar{a}_i)][\sum_{j=0}^{m}(b_j, \bar{b}_j)]$ is $(m+1)(n+1)-1$, and the number of conditions required to establish $\sum_{i=0}^{n}(a_i, \bar{a}_i) \subset \sum_{j=0}^{m}(b_j, \bar{b}_j)$ is $(n+1)2^{m+1}$, assuming no reductions are made. However, in $\bar{L}$, the degree of $[\sum_{i=0}^{n}(a_i - \bar{a}_i)]'$ is only $n+1$, that of $[\sum_{i=0}^{n}(a_i - \bar{a}_i)][\sum_{j=0}^{m}(b_j - \bar{b}_j)]$ is $m+n$, and the number of conditions required to establish $\sum_{i=0}^{n}(a_i - \bar{a}_i) \subset \sum_{j=0}^{m}(b_j - \bar{b}_j)$ is $m+n+2$, though it is frequently easier, in practice, to apply the $(n+1)(m+2)$ conditions of Corollary 13.25. We now agree to make the following *elementary reductions* in the degree of $\sum_{i=0}^{n}(a_i - \bar{a}_i)$ whenever possible. Whenever $\sum_{i=0}^{n}(a_i - \bar{a}_i) = (0, 0)$, by Lemma 13.22, it will be written $(0, 0)$. Whenever $\sum_{i=0}^{n}(a_i - \bar{a}_i) \neq (0, 0)$, $(a_i - \bar{a}_i)$ is omitted if $a_i = \bar{a}_i$ as, in this case, $(a_i - \bar{a}_i) = (0, 0)$ and $(a_i - \bar{a}_i) + (a_{i+1} - \bar{a}_{i+1})$ is written $(a_i - \bar{a}_{i+1})$ if $\bar{a}_i = a_{i+1}$, applying Corollary 13.11. The order in which these latter two reductions are applied does not effect the final result which is to strengthen the relations on the a's to $a_k < \bar{a}_j < a_j$ for $j < k$. When these reductions have been performed, it follows that the degree of $\sum_{i=0}^{n}(a_i - \bar{a}_i) + \sum_{j=0}^{m}(b_j - \bar{b}_j)$ is not greater than $m+n+1$ and that for every element of L an equal element can be found in $\bar{L}$ of equal or lower degree. Hence the apparent advantages of operating in $\bar{L}$, derived from considerations of degree, are real. We remark that the elementary reductions in $\bar{L}$ are so called because they are the only reductions which are obtainable from the application of the operations and ordering relation of K upon the elements a_i and $\bar{a}_i$. In the reduction so far we have preserved the absolute character of the extension, the final step is necessarily relative to the entire set, K.

To continue with the reduction to normal form, we must assume that K has completely distributive products. This involves no real loss in generality, since any distributive lattice may be so extended by the dual of the extension from a multiplicative system to a lattice with completely distributive sums and this extension can be put in normal form. As a preliminary to finding a normal form we discuss the theory of minimal covers.

13.28 DEFINITION. An element, c, of K is a *minimal cover* of $\sum_{i=0}^{n}(a_i-\bar{a}_i)$ in $\bar{L}$ if $\sum_{i=0}^{n}(a_i-\bar{a}_i) \subset (c-0)$ and if $\sum_{i=0}^{n}(a_i-\bar{a}_i) \subset (x-0)$ implies $c \subset x$.

13.29 THEOREM. *If c and d are minimal covers of $\sum_{i=0}^{n}(a_i-\bar{a}_i)$, then $c=d$.*

13.30 THEOREM. *If $\sum_{i=0}^{n}(a_i-\bar{a}_i)$ is in $\bar{L}$, then c, the minimal cover of $\sum_{i=0}^{n}(a_i-\bar{a}_i)$, exists in K.*

Proof: Let $c=\prod x_j$ where the product is taken over all elements x_j in K such that $\sum_{i=0}^{n}(a_i-\bar{a}_i) \subset (x_j-0)$. Since $\prod x_j$ is distributive, it is preserved, by 13.14, in K' and hence in L and $\bar{L}$. Therefore, $\sum_{i=0}^{n}(a_i-\bar{a}_i) \subset (c-0)$. Also, $\sum_{i=0}^{n}(a_i-\bar{a}_i) \subset (x-0)$ implies $c \subset x$, for every such x is an x_j. The set x_j is not void, for I is an x_j.

13.31 THEOREM. *If c is the minimal cover of $\sum_{i=0}^{n}(a_i-\bar{a}_i)$, then $c \subset a_0$.*

Proof: $\sum_{i=0}^{n}(a_i-\bar{a}_i) \subset (a_0-0)$.

13.32 THEOREM. *If c is the minimal cover of $\sum_{i=0}^{n}(a_i-\bar{a}_i)$, d the minimal cover of $\sum_{j=0}^{n}(b_j-\bar{b}_j)$, and $\sum_{i=0}^{n}(a_i-\bar{a}_i) \subset \sum_{j=0}^{m}(b_j-\bar{b}_j)$, then $c \subset d$.*

Proof: $\sum_{i=0}^{n}(a_i-\bar{a}_i) \subset \sum_{j=0}^{m}(b_j-\bar{b}_j) \subset (d-0)$.

13.33 COROLLARY. *If $\sum_{i=0}^{n}(a_i-\bar{a}_i)=\sum_{j=0}^{m}(b_j-\bar{b}_j)$, then $c=d$.*

13.34 COROLLARY. *$c+d$ is the minimal cover of $\sum_{i=0}^{n}(a_i-\bar{a}_i)+\sum_{j=0}^{m}(b_j-\bar{b}_j)$.*

13.35 COROLLARY. *$f \subset cd$, where f is the minimal cover of $[\sum_{i=0}^{n}(a_i-\bar{a}_i)]$ $[\sum_{j=0}^{m}(b_j-\bar{b}_j)]$.*

Thus the correspondence $\sum_{i=0}^{n}(a_i-\bar{a}_i) \to c$ from $\bar{L}$ to K is a homomorphism which preserves sums but not, in general, products.

13.36 DEFINITION. $\sum_{k=0}^{p}(c_k-\bar{c}_k)$ is the normal form of $\sum_{i=0}^{n}(a_i-\bar{a}_i)$ $\neq(0-0)$ if c_k is the minimal cover of $\sum_{i=k}^{n}(\bar{c}_{k-1}a_i-\bar{c}_{k-1}\bar{a}_i)$, $\bar{c}_k$ is the minimal cover of $\sum_{i=k}^{n}(c_k\bar{a}_i-c_ka_{i+1})$, and p is the greatest subscript for which $c_p \neq 0$. $(0-0)$ is the normal form of zero.

13.37 THEOREM. *$\sum_{k=0}^{p}(c_k-\bar{c}_k)$ is in $\bar{L}$ and $p \leqq n$.*

Proof: By 13.31, $c_k \subset \bar{c}_{k-1}a_k \subset \bar{c}_{k-1}$ and $\bar{c}_k \subset c_ka_k \subset c_k$. Also $p \leqq n$ for $c_{n+1}=0$.

13.38 LEMMA. *If $\sum_{i=k}^{n}(\bar{c}_{k-1}a_i-\bar{c}_{k-1}\bar{a}_i)=\sum_{j=k}^{m}(\bar{c}_{k-1}b_j-\bar{c}_{k-1}\bar{b}_j)$, then $\sum_{i=k}^{n}(c_k\bar{a}_i-c_ka_{i+1})=\sum_{j=k}^{m}(c_k\bar{b}_j-c_kb_{j+1})$.*

Proof: By 13.33, c_k is the minimal cover of both sides of the hypothesis. Also, $c_k \subset \bar{c}_{k-1}$ and $c_ka_k=c_kb_k=c_k$. Taking complements multiplying through by (c_k-0) on both sides, the first term on each side drops out, and the conclusion remains.

13.39 LEMMA. *If* $\sum_{i=k}^{n}(c_k\bar{a}_i - c_k a_{i+1}) = \sum_{j=k}^{m}(c_k\bar{b}_j - c_k b_{j+1})$, *then* $\sum_{i=k+1}^{n}(\bar{c}_k a_i - \bar{c}_k\bar{a}_i) = \sum_{j=k+1}^{m}(\bar{c}_k b_j - \bar{c}_k\bar{b}_j)$.

Proof: The proof is analogous to that of 13.38.

13.40 THEOREM. *If* $\sum_{k=0}^{p}(c_k - \bar{c}_k)$ *is the normal form of* $\sum_{i=0}^{n}(a_i - \bar{a}_i)$, $\sum_{k=0}^{q}(d_k - \bar{d}_k)$ *is the normal form of* $\sum_{j=0}^{m}(b_j - \bar{b}_j)$, *and* $\sum_{i=0}^{n}(a_i - \bar{a}_i) = \sum_{j=0}^{m}(b_j - \bar{b}_j)$, *then* $c_k = d_k$, $\bar{c}_k = \bar{d}_k$, *and* $p = q$.

Proof: The proof is by induction. Since $\sum_{i=0}^{n}(a_i - \bar{a}_i) = \sum_{j=0}^{m}(b_j - \bar{b}_j)$, $c_0 = d_0$ and the hypothesis of 13.38 is satisfied for $k=0$. Hence, the conclusion of 13.38 for $k=0$, which is in turn the hypothesis, for $k=0$, of 13.39, is fulfilled. In general, if for any value of k, $\bar{c}_{k-1} = \bar{d}_{k-1}$ and the hypothesis of 13.38 holds, then $c_k = d_k$ and the hypothesis of 13.39 holds for k. In an analogous manner, if for any value of k, $c_k = d_k$ and the hypothesis of 13.39 holds, then $\bar{c}_k = \bar{d}_k$ and the hypothesis of 13.38 holds for $k+1$. Since $c_p \neq 0$ and $c_{p+1} = 0$, $d_p \neq 0$ and $d_{p+1} = 0$. Hence, $p = q$.

13.41. COROLLARY. *If* $\sum_{k=0}^{p}(c_k - \bar{c}_k)$ *and* $\sum_{k=0}^{q}(d_k - \bar{d}_k)$ *are normal forms of* $\sum_{i=0}^{n}(a_i - \bar{a}_i)$, *then* $c_k = d_k$, $\bar{c}_k = \bar{d}_k$, *and* $p = q$.

13.42 LEMMA. *Let* $\sum_{j=0}^{p}(c_j - \bar{c}_j)$ *be the normal form of* $\sum_{i=0}^{n}(a_i - \bar{a}_i)$ *and define expressions* $A(k)$ *and* $B(k)$ *as follows*:

$A(k)$ $$\left[\sum_{i=0}^{n}(a_i - \bar{a}_i)\right]' = \sum_{j=0}^{k}(\bar{c}_{j-1} - c_j) + \sum_{i=k}^{n}(c_k\bar{a}_i - c_k a_{i+1}),$$

$B(k)$ $$\sum_{i=0}^{n}(a_i - \bar{a}_i) = \sum_{j=0}^{k}(c_j - \bar{c}_j) + \sum_{i=k+1}^{n}(\bar{c}_k a_i - \bar{c}_k\bar{a}_i).$$

Then $A(k)$ *implies* $B(k)$ *and* $B(k)$ *implies* $A(k+1)$.

Proof: By 13.36, $\sum_{i=k}^{n}(c_k\bar{a}_i - c_k a_{i+1}) = \sum_{i=k}^{n}(\bar{c}_k\bar{a}_i - \bar{c}_k a_{i+1})$. Substituting this in $A(k)$ and taking the complement, since $\bar{c}_k\bar{a}_k = \bar{c}_k$, $B(k)$ results. Similarly, since $\sum_{i=k+1}^{n}(\bar{c}_k a_i - \bar{c}_k\bar{a}_i) = \sum_{i=k+1}^{n}(c_{k+1}a_i - c_{k+1}\bar{a}_i)$ and $c_{k+1}a_{k+1} = c_{k+1}$, $A(k+1)$ results from substituting and taking the complement.

13.43 THEOREM. *If* $\sum_{j=0}^{p}(c_j - \bar{c}_j)$ *is the normal form of* $\sum_{i=0}^{n}(a_i - \bar{a}_i)$, *then* $\sum_{j=0}^{p}(c_j - \bar{c}_j) = \sum_{i=0}^{n}(a_i - \bar{a}_i)$.

Proof: Since $\sum_{i=0}^{n}(a_i - \bar{a}_i) = \sum_{i=0}^{n}(c_0 a_i - c_0\bar{a}_i)$ and $c_0 a_0 = c_0$, by substituting and taking the complement $A(0)$ is established. $B(p)$ follows from this by induction employing 13.42. But $\sum_{i=p+1}^{n}(\bar{c}_p a_i - \bar{c}_p\bar{a}_i) = \sum_{i=p+1}^{n}(c_{p+1}a_i - c_{p+1}\bar{a}_i)$ $= (0-0)$, since $c_{p+1} = 0$. Hence, $B(p)$ reduces to the required result.

In each class of equal elements in $\bar{L}$ there must be at least one element of degree not greater than that of any other element in the class. The normal form of this element falls within the class by Theorem 13.43, hence cannot be

of lower degree than the element of least degree. However, by Theorem 13.37, its degree cannot be greater. Furthermore, each class of equal elements has but one normal form, by Theorem 13.40. Hence, we conclude that the degree of an element in normal form is not greater than the degree of any equal element. This implies that no elementary reductions can be performed on a normal form.

In general, there is no upper bound for the degrees of elements in normal form in $\bar{L}$. Consider Example [4.11, 4.7]. After elementary reductions have been made, each element is in normal form and such elements of any degree can be constructed.

Since the reduction of $\bar{L}$ to normal form also reduces K' to normal form, either of the methods of extending K to a complete Boolean algebra can be put into normal form by using the normal forms of the component extensions employed.

If the given distributive lattice, K, does not contain a unit I, then the extension to $\bar{L}$, with trivial changes, yields a Boolean ring. However, the dependence of the extension upon the unit 0 is essential.

Bibliography

The bibliography which follows lists only those papers referred to in footnotes and is, only in this sense, complete.

1. Alexandroff, Paul, *Sur les espaces discrets*, Comptes Rendus, vol. 200 (1936), pp. 1649–1651.
2. Bennett, A. A., *Semi-serial order*, American Mathematical Monthly, vol. 37 (1930), pp. 418–423.
3. Birkhoff, Garrett, *On the combination of subalgebras*, Proceedings of the Cambridge Philosophical Society, vol. 29 (1933), pp. 441–464.
4. Birkhoff, Garrett, *On the lattice theory of ideals*, Bulletin of the American Mathematical Society, vol. 40 (1934), pp. 613–619.
5. Dedekind, R., *Stetigkeit und irrationale Zahlen*, authorized translation entitled *Essays on Number*, Chicago, Open Court, 1901.
6. Dedekind, R., *Ueber die von drei Moduln erzeugte Dualgruppe*, Mathematische Annalen, vol. 53 (1900), pp. 371–403.
7. Fraenkel, A., *Sur l'axiome du choix*, L'Enseignement Mathématique, vol. 34 (1935), pp. 32–51.
8. Freudenthal, Hans, *Zur Abstraktion des Integralbegriffs*, Proceedings, Akademie van Wetenschappen te Amsterdam, vol. 39 (1936), pp. 741–746.
9. Grell, H., *Beziehungen den Idealen verschiedener Ringe*, Mathematische Annalen, vol. 97 (1927), pp. 490–523.
10. Hausdorff, F., *Grundzüge der Mengenlehre*, 1st edition, Leipzig, Von Veit, 1914.
11. Hausdorff, F., *Mengenlehre*, 2d edition, Leipzig, de Gruyter, 1927.
12. Huntington, E. V., *Sets of independent postulates for the algebra of logic*, these Transactions, vol. 5 (1904), pp. 288–309.
13. Huntington, E. V., *New sets of independent postulates for the algebra of logic*, these Transactions, vol. 35 (1933), pp. 274–304, 557, 971.
14. Klein, Fritz, *Grundzüge der Theorie der Verbände*, Mathematische Annalen, vol. 111 (1935), pp. 596–621.

15. Klein, Fritz, *Über ausgeglichene Verbände*, Mathematische Annalen, vol. 112 (1936), pp. 411–418.

16. Menger, Karl, *New foundations of projective and affine geometry*, Annals of Mathematics, (2), vol. 37 (1936), pp. 456–482.

17. von Neumann, J., *Continuous geometry*, also, *Examples of continuous geometries*, Proceedings of the National Academy of Sciences, vol. 22 (1936), pp. 92–108.

18. Ore, O., *On the foundations of abstract algebra*, Annals of Mathematics, (2), (I) vol. 36 (1935), pp. 406–437, (II) vol. 37 (1936), pp. 265–292.

19. Peirce, C. S., *On the algebra of logic*, American Journal of Mathematics, vol. 3 (1880), pp. 15–57.

20. Quine, W. V., *A System of Logistic*, Cambridge, Harvard University Press, 1934.

21. Schröder, Ernst, *Vorlesungen über die Algebra der Logik*, Leipzig, Teubner, 1890.

22. Stone, M. H., *Linear Transformations in Hilbert Space*, American Mathematical Society Colloquium Publications, vol. 15, 1932.

23. Stone, M. H., *Postulates for Boolean algebras and generalized Boolean algebras*, American Journal of Mathematics, vol. 57 (1935), pp. 703–732.

24. Stone, M. H., *The theory of representations for Boolean algebras*, these Transactions, vol. 39 (1936), pp. 37–111.

25. Tarski, A., *Zur Grundlegung der Boolschen Algebra*, Fundamenta Mathematicae, vol. 24 (1935), pp. 177–198.

26. Tucker, A. W., *Cell spaces*, Annals of Mathematics, (2), vol. 37 (1936), pp. 92–100.

27. Zermelo, E., *Neuer Beweis für die Möglichkeit einer Wohlordnung*, Mathematische Annalen, vol. 65 (1908), pp. 107–128.

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ON MECHANICAL QUADRATURES, IN PARTICULAR, WITH POSITIVE COEFFICIENTS*

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I. Introduction

The present paper is an extension in various directions of a paper by Fejér.†

Consider a system of orthonormal polynomials (*OP*)

$$(1)\qquad \begin{aligned}\phi_n(x;\,a,\,b;\,d\psi) &\equiv \phi_n(x;\,d\psi) \equiv \phi_n(x) = a_n(x^n - S_{n-1}x^{n-1} + \cdots)\\ &\equiv a_n\Phi_n(x); \qquad n = 0,\,1,\,2,\,\cdots;\ a_n \equiv a_n(d\psi) > 0,\ddagger\end{aligned}$$

with the orthogonality property

$$(2)\qquad \int_a^b \phi_m(x)\phi_n(x)d\psi = \delta_{mn} = \begin{cases}0, & m \neq n;\\ 1, & m = n;\end{cases} \qquad m,\,n = 0,\,1,\,\cdots.$$

Here and hereafter $\psi(x)$ denotes a bounded non-decreasing function in (a, b), with infinitely many points of increase, including the end-points a, b. The limits a and b may be finite or infinite, but such that all moments

$$(3)\qquad \alpha_n = \int_a^b x^n d\psi, \qquad n = 0,\,1,\,\cdots, \text{ with } \alpha_0 > 0$$

exist. From (2) it follows that

$$(4)\qquad \int_a^b \Phi_n(x)G_{n-1}(x)d\psi = 0, \qquad \int_a^b \Phi_n(x)G_n(x)d\psi = \frac{g_n}{a_n^2}, \quad n = 0,\,1,\,\cdots.$$

Here $G_{-1}(x) \equiv 0$ and $G_s(x) \equiv \sum_{i=0}^{s} g_i x^i$ is a generic notation for an arbitrary polynomial of degree $\leqq s$, subject, in some instances, to certain explicitly stated conditions. The polynomials $\Phi_n(x) \equiv \Phi_n(x;\,d\psi) \equiv \Phi_n(x;\,a,\,b;\,d\psi)$ satisfy the recurrence relation

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† L. Fejér, *Mechanische Quadraturen mit positiven Cotesschen Zahlen*, Mathematische Zeitschrift, vol. 37 (1933), pp. 287–309.

‡ The notations employed are those of my monograph, *Théorie générale des polynomes orthogonaux de Tchebycheff* (hereafter referred to as M), Mémorial des Sciences Mathématiques, fasc. 66 (1934), to which the reader is referred for further details.

$$\Phi_n(x) = (x - c_n)\Phi_{n-1}(x) - \lambda_n \Phi_{n-2}(x), \qquad n \geqq 2,$$
$$(\Phi_0(x) = 1, \ \Phi_1(x) = x - c_1),$$
$$\lambda_n = \frac{a_{n-2}^2}{a_{n-1}^2} > 0, \qquad c_n = S_n - S_{n-1}. \tag{5}$$

The zeros of $\Phi_n(x)$ are known to be real and distinct; they lie in (a, b) and will be denoted by

$$x_{i,n}(d\psi) \equiv x_{i,n} \equiv x_i, \quad \text{with} \quad a < x_1 < x_2 < \cdots < x_n < b. \tag{6}$$

Using the points (6) as abscissas in the Lagrange interpolation formula (*LIF*) for a given function $f(x)$ which is finite at every point of $[a, b]$ and for which $\int_a^b f(x)d\psi$ exists,* we are led to a Gaussian formula of mechanical quadratures (*GMQ* formula)

$$\int_a^b f(x)d\psi \approx \sum_{i=1}^{n} H_{i,n} f(x_{i,n}), \quad H_{i,n} = \int_a^b \frac{\phi_n(x)d\psi}{(x - x_{i,n})\phi_n'(x_{i,n})}, \tag{7}$$

with the following properties:

All "coefficients" $H_{i,n}$ are positive, namely,

$$H_{i,n} \equiv H_{i,n}(d\psi) = \int_a^b \left[\frac{\phi_n(x)}{(x - x_{i,n})\phi_n'(x_{i,n})}\right]^2 d\psi, \quad i = 1, 2, \cdots, n; \tag{8.1}$$

Formula (7) is exact for any polynomial of degree $\leqq 2n-1$, i.e.,

$$\int_a^b f(x)d\psi = \sum_{i=1}^{n} H_{i,n} f(x_{i,n}) + R_n(f), \quad \text{with} \quad R_n(G_{2n-1}) = 0. \tag{8.2}$$

The property (8.1) is of importance in connection with the convergence properties of (7), as shown by Fejér (loc. cit.) and as will be developed below. The question then naturally arises: *does there exist, besides* (6), *some other choice of the points* $x_{i,n}$ *which yields a mechanical quadratures formula* (*MQF*) *with positive coefficients*? Fejér's answer is in the affirmative in the case where (a, b) is finite, $d\psi \equiv dx$, and the abscissas $x_{i,n}$ are the zeros of the polynomial

$$P_n(x) + AP_{n-1}(x) + BP_{n-2}(x),$$

where $P_n(x)$ is the Legendre polynomial of degree n, A and B are real constants, with $B \leqq 0$, provided the zeros in question are real and distinct and belong to the (closed) interval $[a, b]$.†

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* Throughout this paper integrals like $\int_a^b f(x)d\psi$ are understood to be taken in the sense of Stieltjes.

† The various constants dealt with in this paper are assumed to be real, unless explicitly stated to the contrary.

The object of the present paper is to give a more general answer to the foregoing question. We may mention the following direct generalization of Fejér's result. The polynomial $\Phi_n(x)+A\Phi_{n-1}(x)+B\Phi_{n-2}(x)$, where $\{\Phi_n(x)\}$ is any sequence of *OP* and the constants A, B are arbitrarily chosen, subject only to the limitation $B \leqq 0$, has all zeros real and distinct. Employing these zeros as abscissas, we get *MQF* (7), *with all coefficients positive.* Moreover, if (a, b) is finite the *MQF* in question converges, i.e., $\lim_{n\to\infty} R_n(f)=0$, for any bounded $f(x)$ for which $\int_a^b f(x)d\psi$ exists, regardless of the distribution of abscissas relative to (a, b). We obtain similar results concerning the more general polynomial

$$\omega_n(x) = \Phi_n(x) + A_1\Phi_{n-1}(x) + \cdots + A_{k-1}\Phi_{n-k+1}(x).$$

Various properties of *OP* are essential for our discussion. The most important point is to show that under sufficiently general conditions the terms in our *MQF* corresponding to abscissas outside (a, b) do not affect the convergence properties for polynomials.

In connection with our main objective we make a general study of the zeros of $\omega_n(x)$. We also link the *MQ* formulae under consideration with the theory of algebraic continued fractions, and we show that the *MQ* formulae related to *OP* are a powerful tool in the general study of such polynomials. At the outset we give some properties of the coefficients and the abscissas of any *MQF* based on *LIF*, extending and generalizing results due to Stekloff.†

II. Mechanical quadratures formulae based on Lagrange interpolation

1. **Construction.** Let $f(x)$ be single-valued and finite at every point of a certain interval (a, b) (closed, if it is finite). Choose n distinct points

$$c_1 < c_2 < \cdots < c_n, \tag{9}$$

and construct the associated *LIF*

$$f(x) \approx \sum_{i=1}^{n} \frac{\omega_n(x)}{(x-c_i)\omega_n'(c_i)} f(c_i), \qquad f(x) = \sum_{i=1}^{n} \frac{\omega_n(x)}{(x-c_i)\omega_n'(c_i)} + \rho_n(f), \tag{10}$$

$$\omega_n(x) = (x-c_1)(x-c_2)\cdots(x-c_n), \qquad \rho_n(G_{n-1}) = 0.$$

By integration, we obtain from (10) a Lagrangian *MQF* (*LMQ* formula)

† W. Stekloff, *Sur le problème de représentation des fonctions à l'aide de polynomes, du calcul approché des intégrales définies, du développment de fonctions en séries infinies suivant les polynomes et de l' interpolation, considérés au point de vue de Tchebycheff.* Proceedings of the International Mathematical Congress, Toronto, 1924, vol. I, pp. 631–640.

which may be spoken of as "generated" by the polynomial $\omega_n(x)$.* Assuming the existence of $\int_a^b f(x)dx$, we have

$$\int_a^b f(x)d\psi \equiv \sum_{i=1}^{n} C_i f(c_i), \qquad C_i = \int_a^b \frac{\omega_n(x)d\psi}{(x - c_i)\omega_n'(c_i)}, \tag{11}$$

$$\int_a^b f(x)d\psi = Q_n(f) + R_n(f), \qquad Q_n(f) \equiv \sum_{i=1}^{n} C_i f(c_i). \tag{11.1}$$

In particular

$$R_n(G_{n-1}) = 0, \quad \text{so that} \quad \alpha_0 = \int_a^b d\psi = \sum_{i=1}^{n} C_i. \tag{11.2}$$

We call the points c_i "abscissas," the coefficients C_i "coefficients" (Cotes' numbers) of the *MQF* under consideration.

In dealing with sequences of *MQ* formulae corresponding to $n=N$, $N+1, \cdots$, we shall use the notations

$$c_{i,n}, \qquad C_{i,n}, \qquad i = 1, 2, \cdots, n. \tag{9.1}$$

The following remark is important. If some of the abscissas fall outside the closed interval $[a, b]$,† we may assign arbitrary finite values to the corresponding $f(c_i)$ in (11), for this evidently does not affect the value of $\int_a^b f(x)d\psi$, but only that of $R_n(f)$ which, by definition, represents the difference $\int_a^b f(x)d\psi - Q_n(f)$. We agree, in case $f(x)$ is not a polynomial, to let $f(c_i)=0$, if c_i is outside $[a, b]$. In other words, *if $f(x)$ is not a polynomial, the summation $Q_n(f) \equiv \sum_{i=1}^{n} C_i f(c_i)$ is extended over only such c_i as belong to* $[a, b]$. When dealing with polynomials we keep all terms in the above summation in order not to destroy the important property $R_n(G_{n-1})=0$.

2. **Degree of precision of *LMQ* formulae.** If the abscissas in (11) are so chosen that

$$R_n(G_q) = 0, \tag{12}$$

but for one, at least, $G_{q+1}(x)$, $R_n(G_{q+1}) \neq 0$, then q is called "degree of precision" of our *LMQ* formula‡ (Stekloff, loc. cit.). Here q may have any value from $n-1$, which corresponds to a random choice of abscissas, to $2n-1$

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* Pólya (*Ueber die Konvergenz von Quadraturverfahren*, Mathematische Zeitschrift, vol. 37 (1933), pp. 264–286) discusses *MQF* $\int_a^b f(x)dx = \sum_{i=1}^{n} \lambda_i f(c_i)$, where the coefficients λ_i are chosen according to a certain fixed rule, not necessarily connected with interpolation.

† If $c_i \leqq a$ (or $c_i \geqq b$), it will be assumed that a (or b) is finite.

‡ The expression "*MQ* formula" means here and hereafter a formula with n abscissas, where n is fixed, unless specified otherwise.

in *GMQ* formula (7). $2n-1$ is the highest possible degree of precision, for $R_n(G_{2n})=0$ leads to a contradiction, namely:

$$\int_a^b \omega_n^2(x)d\psi = \sum_{i=1}^{n} C_i\omega_n^2(c_i) = 0.$$

THEOREM I. *A necessary and sufficient condition that $q=2n-k$ be the degree of precision of LMQ formula* (11) *is that*

$$\int_a^b \omega_n(x)G_{q-n}(x)d\psi = 0, \tag{13}$$

which is equivalent to the statement that $\omega_n(x)$ admits of the following representation

$$\omega_n(x) = \Phi_n(x) + A_1\Phi_{n-1}(x) + \cdots + A_{k-1}\Phi_{n-k+1}(x),^* \tag{14}$$

where the A_i are arbitrary constants.

The first part is readily proved by integration, upon combining (4) with the relations

$$G_q(x) = \omega_n(x)\,\overline{r}_{q-n}(x) + G_{n-1}(x), \qquad G_q(c_i) = \overline{r}_{n-1}(c_i), \qquad i = 1, 2, \cdots, n.$$

The second part is obvious. Observing that

$$G_{q+1}(x) = \omega_n(x)G_{q-n+1}(x) + G_{n-1}(x),$$

we immediately obtain the following formula:

$$\int_a^b G_{q+1}(x)d\psi = \frac{A_{k-1}g_{k+1}}{a_{n-k+1}^2} + \sum_{i=1}^{n} C_iG_{q+1}(c_i). \tag{15}$$

THEOREM II. *The polynomial*

$$\omega_n(x) = \Phi_n(x) + A_1\Phi_{n-1}(x) + \cdots + A_{k-1}\Phi_{n-k+1}(x), \tag{14}$$

satisfies the orthgonality relation (13), *with $q=2n-k$, and changes sign in (a, b) at least $n-k+1$ times.*

The first part is evident, while the second is easily proved by an argument well known in the theory of *OP*.

3. **Signs of the coefficients; location of the abscissas.** Denote by C_i' the positive among the coefficients C_i in (11) and by C_i'' the remaining ones (negative or vanishing). Let c_i', c_i'' be the corresponding abscissas, and P and N be the number of C_i' and C_i'' respectively.

THEOREM III. *If q is the degree of precision of LMQ formula* (11), *then $n \geqq P \geqq [(q+2)/2]$, so that $0 \leqq N \leqq [(2n-q-1)/2]$.*

* By virtue of the recurrence relation (5), a polynomial of the form $\Phi_n(x)+P_1(x)\Phi_{n-1}(x)+\cdots+P_\nu(x)P_{n-\nu}(x)$ ($P_i(x)$—polynomial of degree $\leqq i$) can be written in the form (14).

This result is due to Stekloff (loc. cit.). It is readily established by applying our *MQF* to the polynomial $G_{(q-1)/2}(x)$ if q is odd, or to $G_{q/2}(x)$ if q is even. These polynomials vanish at all c_i'.

Further indications as to the location of the c_i and more particularly, of the c_i' and c_i'', are given in the following theorems.

THEOREM IV. *Let $D_n(x)$ be a polynomial of degree n satisfying the orthogonality relation $\int_a^b D_n(x)G_\nu(x)d\psi=0$. If $\psi(x)$ is constant in $(\alpha, \beta) \subset (a, b)$, then $D_n(x)$ may change sign in (α, β) at most $n-\nu+\theta$ times, where $\theta=0$ or 1, according as n is odd or even. It follows that in an LMQ formula, with degree of precision q, at most $2n-q+\theta$ abscissas may lie inside an interval of constancy of $\psi(x)$, where $\theta=0$ or 1, according as $2n-q$ is odd or even.*

If (α, β) contains more than $n-\nu+\theta$ points where $D_n(x)$ changes sign, take any $\mu=n-\nu+\theta+1$ of these, say $\beta_{j+1}, \beta_{j+2}, \cdots, \beta_{j+\mu}$. We have, by hypothesis,

$$0 = \int_a^b D_n(x) \frac{D_n(x)d\psi}{(x-\beta_{j+1})\cdots(x-\beta_{j+\mu})}$$
$$= \int_a^\alpha \frac{D_n^2(x)d\psi}{(x-\beta_{j+1})\cdots(x-\beta_{j_\mu})} + \int_\beta^b \cdots,$$

which is impossible, both integrals on the right being >0. This is a generalization of a known property of *OP* $(n-\nu=1)$, which is usually proved by means of Tchebycheff inequalities for the coefficients $H_{i,n}$ in (7), (see below, §11). The above proof (suggested by A. N. Milgram) is a simple application of the orthogonality property of $D_n(x)$.

THEOREM V. (i) *If a certain c_j'' coincides with an end-point a or b, then there exists a $c_i'>a$ or $<b$ respectively.* (ii) *If 2ν denotes the greatest integer contained in $q-2N-1$, then neither of the intervals $(-\infty, a]$, $[b, \infty)$ may contain more than $P-\nu-1$ of the c_i'.**

To prove part (i), assume for definiteness $c_j''=a$, n odd. By Theorem III, $N-1 \leqq (n-3)/2$, and we may construct a $G_{(n-3)/2}(x)$ such that $G_{(n-3)/2}(c_i'')=0$ at all c_i'', except for $c_j''=a$. Moreover, $q \geqq n-1>n-2$,

$$0 > \int_a^b (a-x)G_{(n-3)/2}^2(x)d\psi = \sum C_i'(a-c_i')G_{(n-3)/2}^2(c_i'),$$

which shows that not all c_i' lie to the left of a.

To prove part (ii), note that $q \geqq 2N+2\nu+1$. Construct $G_{N+\nu}(x)$ such that $G_{N+\nu}(x)=0$ at all c_i'' and at any ν of the c_i', say, at $c_{i_1}', c_{i_2}', \cdots, c_{i_\nu}'$. Apply-

* We necessarily have $P-\nu-1 \geqq 0$, for $\nu \leqq (q-2N-1)/2 \leqq (2n-2-2N)/2=n-N-1=P-1$.

ing our *LMQ* formula (11) to $(x-a)G_{N+\nu}^2(x)$ and $(b-x)G_{N+\nu}^2(x)$ we obtain

$$0 < \int_a^b (x-a)G_{N+\nu}^2(x)d\psi = \sum C_i'(c_i'-a)G_{N+\nu}^2(c_i'),$$

$$0 < \int_a^b (b-x)G_{N+\nu}^2(x)d\psi = \sum C_i'(b-c_i')G_{N+\nu}^2(c_i'), \qquad c_i' \neq c_{i_1}', \cdots, c_{i_\nu}',$$

which shows that the remaining $P-\nu$ of the c_i' cannot all lie in $(-\infty, a]$ or in $[b, \infty)$.

The above theorems readily yield the following corollaries.

COROLLARY 1. *Every LMQ formula has at least* $[(n+1)/2]$ *positive coefficients.*

COROLLARY 2. *If the polynomial* (14) *has all zeros real and distinct, the LMQ formula with degree of precision* $q=2n-k$, *which uses these zeros as abscissas, has at least* $[(2n-k+2)/2]$ *positive coefficients, hence, at most,* $[(k-1)/2]$ *negative or vanishing coefficients.*

COROLLARY 3. *If all zeros of the polynomial* (14) *are real and distinct, and the associated LMQ formula has all coefficients positive, then neither of the intervals* $(-\infty, a]$, $[b, \infty)$ *may contain more than* $[k/2]$ *abscissas.*

Some special cases. (α) $k=1$. Here $q=2n-1$, $n-k+1=n$, $P=n$, $N=0$, and we have a *GMQ* formula.

(β) $k=2$, i.e.,

$$\omega_n(x) = \Phi_n(x) + A_1\Phi_{n-1}(x), \qquad A_1 \neq 0. \tag{16}$$

Here $q=2n-2$, $n-k+1=n-1$, $[k/2]=1$, $[(k-1)/2]=0$, $P=n$, $N=0$; all zeros are real and simple, with one, at most, $\leqq a$ or $\geqq b$, and all coefficients are positive.

(γ) $k=3$, i.e.,

$$\omega_n(x) = \Phi_n(x) + A_1\Phi_{n-1}(x) + A_2\Phi_{n-2}(x), \qquad A_2 \neq 0. \tag{17}$$

Here at least $n-2$ zeros are real and lie between a and b. Assuming further that all zeros are real and simple, we get for the corresponding *LMQ* formula

$$q = 2n-3, \qquad P \geqq n-1, \qquad N \leqq 1, \tag{17.1}$$

so that one, at most, of the coefficients C_i may become negative or vanish.

(δ) $k=4$, i.e.,

$$\omega_n(x) = \Phi_n(x) + A_1\Phi_{n-1}(x) + A_2\Phi_{n-2}(x) + A_3\Phi_{n-3}(x), \qquad A_3 \neq 0. \tag{18}$$

Here at least $n-3$ zeros are real and lie between a and b. If all zeros are real and simple, then for the corresponding *LMQ* formula

$$q = 2n - 4, \qquad P \geqq n - 1, \qquad N \leqq 1.$$

By virtue of Theorem V, we find that in both cases (γ) and (δ) neither interval $(-\infty, a]$, $[b, \infty)$ may contain more than two c_i', and not more than one c_i' if $N=0$.

4. **Second method for studying the coefficients and abscissas.** We now proceed to study the c_i, C_i by another method which tells us more about the coefficients if we know more about the abscissas. Thus, the results previously obtained will be supplemented and extended.

Consider again an *LMQ* formula with degree of precision $q=2n-k$. We change the previous notations and divide the abscissas into two groups as follows.

$$\begin{aligned} &s = s_1 + s_2 \text{ exterior abscissas :} \\ &\quad a^{(1)} < a^{(2)} < \cdots < a^{(s_1)} \leqq a, \qquad b^{(s_2)} > b^{(s_2-1)} > \cdots > b^{(1)} \geqq b, \\ &\nu = n - s \text{ interior abscissas :} \\ &\quad a < c^{(1)} < c^{(2)} < \cdots < c^{(\nu)} < b, \end{aligned} \tag{19}$$

with the corresponding coefficients

$$C_{a^{(i)}} \equiv C^{(i)}, \qquad C_{b^{(j)}} \equiv C^{(j)}, \qquad C_{c^{(l)}} \equiv C^{(l)},$$

so that

$$\begin{aligned} &\omega_n(x) = (-1)^{s_2} \prod(x)\Phi(x), \\ &\prod(x) = \prod_{i=1}^{s_1}(x - a^{(i)}) \prod_{j=1}^{s_2}(b^{(j)} - x) \equiv \prod_a(x) \prod_b(x), \\ &\Phi(x) = \prod_{l=1}^{\nu}(x - c^{(l)}). \end{aligned} \tag{20}$$

Note that $\prod(x) \geqq 0$ in $[a, b]$ and >0 in (a, b). We agree to replace $\prod_a(x)$ or $\prod_b(x)$ by unity in case $s_1=0$ or $s_2=0$. Introduce a new system of *OP*:

$$\Phi_n(x; d\psi_1) \equiv \Phi_n(x; a, b; d\psi_1), \qquad d\psi_1(x) = \prod(x)d\psi(x).^* \tag{21}$$

Equation (13) can be rewritten as

$$\int_a^b \Phi(x)G_{q-n}(x)d\psi_1 = 0. \tag{22}$$

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* We recall that $\Phi_n(x)$ stands for the *OP* with the highest coefficient unity, and $\phi_n(x)$ stands for the normalized *OP* (see Introduction).

We get at once the previous result:

$$q - n = n - k < \nu = n - s, \quad \text{i.e.,} \quad s < 2n - q = k$$

(for otherwise we obtain a contradiction by taking in (22) $G_{q-n}(x) \equiv \Phi(x)$). Moreover, (22) leads to a new representation of $\omega_n(x)$ in terms of $\Phi_n(x; d\psi_1)$ (supplementing (14)):

$$\Phi(x) = \Phi_\nu(x; d\psi_1) + B_1\Phi_{\nu-1}(x; d\psi_1) + \cdots + B_{k_1}\Phi_{\nu-k_1+1}(x; d\psi_1),$$

$$\omega_n(x) = (-1)^{s_3}\prod(x)\left[\Phi_\nu(x; d\psi_1) + B_1\Phi_{\nu-1}(x; d\psi_1) + \cdots + B_{k_1}\Phi_{\nu-k_1+1}(x; d\psi_1)\right], \tag{23}$$

$$(B_i = \text{const.}, \nu = n - s, k_1 = k - s).$$

Upon rewriting (22) as

$$\int_a^b \Phi(x)G_{q-s-(n-s)}(x)d\psi_1 = 0 \tag{22.1}$$

and recalling that $\Phi(x)$ is of degree $\nu = n - s$, we conclude that *the interior abscissas in* (19) *originate a new LMQ formula, with* $d\psi(x)$ *replaced by* $d\psi_1(x) = \prod(x)d\psi(x)$, namely:

$$\int_a^b f(x)d\psi_1 \approx \sum_{l=1}^{\nu} C_l' f(c^{(l)}), \tag{24}$$

with degree of precision $q_1 = q - s = 2\nu - k_1$, $k_1 = k - s$. Since $q_1 \leqq 2(n-s) - 1$, we see that

$$q \leqq 2n - s - 1. \tag{25}$$

Hence, $2n - s - 1$ is the highest degree of precision attainable by the *LMQ* formula (11) having s preassigned exterior abscissas.

We proceed to establish relations between the coefficients of the two *LMQ* formulae (11) and (24). Applying (11), where $\omega_n(x)$ is now replaced by $(-1)^{s_3}\prod(x)\Phi(x)$, we get:

$$C^{(l)} = \frac{1}{\prod(c^{(l)})}\int_a^b \frac{\Phi(x)d\psi_1}{(x - c^{(l)})\Phi'(c^{(l)})} = \frac{C_l'}{\prod(c^{(l)})}, \quad l = 1, 2, \cdots, \nu, \tag{26}$$

$$C^{(l)} \cdot C_l' > 0, \text{ for all interior coefficients } C^{(l)}. \tag{27}$$

Letting s be fixed, consider some special cases.

(i) q attains its highest value, i.e.,

$$q = 2n - s - 1, \qquad k = s + 1. \tag{28}$$

Here $\Phi(x) = \Phi_\nu(x; d\psi_1)$, $k_1 = 1$, $q_1 = 2\nu - 1$. Hence, (24) is a *GMQ* formula, and

$$C_l' = H_{l,\nu}(d\psi_1), \qquad l = 1, 2, \cdots, \nu.$$

Furthermore, applying the original *LMQ* formula (11) to

$$G_{2n-s-1}(x) = \prod_a(x)\prod_b(x)\Phi(x)\Phi_\nu(x; d\psi_{1,i})/(x - a^{(i)}), \qquad d\psi_{1,i} = \frac{d\psi_1}{x - a^{(i)}},$$

we get

$$C^{(i)} = 1/\{a_\nu^2(d\psi_{1,i})\Phi(a^{(i)})\Phi_\nu(a^{(i)}; d\psi_{1,i})\prod_a{}'(a^{(i)})\prod_b(a^{(i)}),\ i = 1, 2, \cdots, s_1,$$

and similarly for the $C^{(j)}$. But (M, p. 27)

$$\Phi_n(\xi; d\psi)\Phi_n(\xi; d\bar{\psi}) = K_n(\xi; d\psi)/a_n^2(d\psi)$$
$$(\xi \leqq a \text{ or } \geqq b,\ d\bar{\psi} = |x - \xi|\, d\psi),$$

where, by Darboux's formula,

$$(29)\ K_n(x; d\psi) \equiv K_n(x) \equiv \sum_{i=0}^{n}\phi_i^2(x) = \frac{a_n}{a_{n+1}}[\phi_{n+1}'(x)\phi_n(x) - \phi_n'(x)\phi_{n+1}(x)].$$

We thus finally get the following formulae for the coefficients C_i in (11):

Interior coefficients:

$$C^{(l)} = \frac{H_{l,\nu}(d\psi_1)}{\prod(c^{(l)})}, \qquad l = 1, 2, \cdots, \nu;$$

(30) Exterior coefficients:

$$C^{(i)} = 1/\{\prod_a{}'(a^{(i)})\prod_b(a^{(i)})K_\nu(a^{(i)}; d\psi_{1,i})\},^*$$

$$C^{(j)} = 1/\{\prod_a(b^{(j)})\prod_b{}'(b^{(j)})K_\nu(b^{(j)}; d\psi_{1,j})\}$$

$$(1 \leqq i \leqq s_1;\ 1 \leqq j \leqq s_2;\ d\psi_{1,i} = d\psi/(x - a^{(i)}),\ d\psi_{1,j} = d\psi/(b^{(j)} - x)).$$

This leads to

THEOREM VI. *Consider an LMQ formula with s_1 abscissas $a^{(1)} < a^{(2)} < \cdots < a^{(s_1)} \leqq a$ and s_2 abscissas $b^{(s_2)} > b^{(s_2-1)} > \cdots > b^{(1)} \geqq b$, having the highest possible degree of precision $q = 2n-s-1 = 2n-k$, with $s = s_1+s_2$, $k = s+1$. Then all interior coefficients are positive; the exterior coefficients alternate in sign, namely*: sgn $C^{(i)} = (-1)^{s_1-i}$, sgn $C^{(j)} = (-1)^{j-1}$ $(1 \leqq i \leqq s_1,\ 1 \leqq j \leqq s_2)$.

COROLLARY. *If $q = 2n-k$, with $k = s_1+s_2+1$, then all the coefficients are positive if and only if $s_1 \leqq 1$, $s_2 \leqq 1$.*

(ii) $q = 2n-s-2$, i.e., $q = 2n-k$, with $k = s+2$. Here again all interior coefficients are positive. In fact,

$$k = 2, \qquad q_1 = 2\nu - 2, \qquad \Phi(x) = \Phi_\nu(x; d\psi_1) + B_1\Phi_{\nu-2}(x; d\psi_1),$$

* Cf. C. Winston, *On mechanical quadratures formulae involving the classical orthogonal polynomials*, Annals of Mathematics, vol. 35 (1934), pp. 658–677.

so that, by the preceding discussion, all C_l', and hence, by (27), all $C^{(l)}$, are positive. We have

$$(31) \qquad C^{(l)}\prod(c^{(l)}) = \int_a^b \prod(x)\left\{\frac{\Phi(x)}{(x-c^{(l)})\Phi'(c^{(l)})}\right\}^2 d\psi, \quad l = 1, 2, \cdots, \nu.$$

In particular, for $s=0$,

$$(31.1) \qquad C^{(l)} \equiv C_i = \int_a^b \left\{\frac{\omega_n(x)}{(x-c^{(l)})\omega_n'(c^{(l)})}\right\}^2 d\psi, \quad i = l = 1, 2, \cdots, \nu = n,$$

so that for $q=2n-1$, $2n-2$, we have the same expression for C_i (see (8.1)). Apply now (15) to the polynomials

$$G_{2n-s-1}(x) = \frac{\prod(x)}{x-a^{(i)}}\Phi^2(x), \quad \frac{\prod(x)}{b^{(j)}-x}\Phi^2(x), \quad i = 1, 2, \cdots, s_1; j = 1, 2, \cdots, s_2.$$

Then

$$(-1)^{s_1}A_{k-1}/a_{n-k+1}^2 + C^{(i)}(-1)^{s_1-i}\left|\prod_a'(a^{(i)})\right|\Phi^2(a^{(i)}) > 0, \quad i = 1, 2, \cdots, s_1,$$

$$(-1)^{s_2}A_{k-1}/a_{n-k+1}^2 + C^{(j)}(-1)^{j}\left|\prod'(b^{(j)})\right|\Phi^2(b^{(j)}) < 0, \qquad j = 1, 2, \cdots, s_2.$$

Hence, $(-1)^{s_1}A_{k-1}<0$ implies that the $C^{(i)}$ alternate in sign; $(-1)^{s_2}A_{k-1}>0$ implies that the $C^{(j)}$ alternate in sign.*

(iii) $q=2n-k$, with $k=s+3$, $s+4$. Here we may have no more than one negative or vanishing coefficient, as is seen from

$$0 < \int_a^b \prod(x)\left\{\frac{\Phi(x)}{(x-c^{(l_1)})(x-c^{(l_2)})}\right\}^2 d\psi = h_1C^{(l_1)}\prod(c^{(l_1)}) + h_2C^{(l_2)}\prod(c^{(l_2)}),$$

$$1 \leqq l_1, \quad l_2 \leqq \nu, \quad l_1 \neq l_2,$$

where h_1, h_2 are certain positive constants. Similarly we prove the more general result: *if $q=2n-k$, with $k=s+r$ $(r\geqq 1)$, then we may have no more than $[(r-1)/2]$ negative or vanishing interior coefficients.*

LMQ formulae, where *all* coefficients are positive, enjoy special important properties as was indicated by Fejér (loc. cit.). It seems proper to call such formulae "*MQ* formulae of Fejér's type" (*FMQ* formulae). The following section is devoted to their discussion.

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* In particular, in case $q=2n-4$, $s_1=s_2=1$, $A_3>0$ implies $C_{a^{(1)}}>0$, and $A_3<0$ implies $C_{b^{(1)}}>0$. If $C_{b^{(1)}}<0$ or $C_{a^{(1)}}<0$, then respectively:

$$|C_{b^{(1)}}| < |A_3|/\{a_{n-3}^2(d\psi_1)(b^{(1)}-a^{(1)})\Phi^2(a^{(1)})\},$$
$$|C_{a^{(1)}}| < |A_3|/\{a_{n-3}^2(d\psi_1)(b^{(1)}-a^{(1)})\Phi^2(b^{(1)})\}.$$

III. *FMQ* FORMULAE

5. Existence; properties of the abscissas. Using the previously introduced s_1, s_2, $s=s_1+s_2$, we state the following results.

(α) $\omega_n(x)=\Phi_n(x)+A\Phi_{n-1}(x)$, A arbitrary, always generates an *FMQ* formula ($A=0$ yields a *GMQ* formula).

(β) $\omega_n(x)=\Phi_n(x)+A_1\Phi_{n-1}(x)+A_2\Phi_{n-2}(x)$ ($A_2\neq 0$) generates an *FMQ* formula, if $s_1=s_2=1$ or $s_1+s_2=1$, $A_2<0$. It cannot generate such a formula if $s_1=2$ or $s_2=2$ (here $s_1+s_2\leqq 2$).

(α) has been shown above; (β) follows from the Corollary to Theorem VI and from (31), (32). It will be improved below.

The following result is important, since it deals with the A_i only.

THEOREM VII. *The polynomial*

$$\omega_n(x) = \Phi_n(x) + A_1\Phi_{n-1}(x) + A_2\Phi_{n-2}(x), \qquad A_2 < 0, \tag{33}$$

*has, with any $\Phi_n(x; a, b; d\psi)$, all zeros real and distinct. It generates an FMQ formula.**

We already know that $n-2$ zeros are real and distinct and lie in (a, b). It thus remains to prove that no double or imaginary zero may occur. This is readily achieved by means of Darboux's formula

$$K_n(x, y; d\psi) \equiv K_n(x, y) \equiv \sum_{i=0}^{n} \phi_i(x)\phi_i(y) = \frac{a_n}{a_{n+1}} \frac{\phi_{n+1}(x)\phi_n(y) - \phi_n(x)\phi_{n+1}(y)}{x-y}. \tag{34}$$

Rewrite (33):

$$a_n\omega_n(x) = \phi_n(x) + A_1'\phi_{n-1}(x) + A_2'\phi_{n-2}(x), \qquad A_2' < 0.$$

Let $\omega_n(x)$ have a double zero ξ or two conjugate imaginary zeros $\xi, \bar{\xi}$. In the first case

$$\phi_n(\xi) + A_1'\phi_{n-1}(\xi) + A_2'\phi_{n-2}(\xi) = 0, \quad \phi_n'(\xi) + A_1'\phi_{n-1}'(\xi) + A_2'\phi_{n-2}'(\xi) = 0,$$

whence, making use of (29),

$$A_2' = \frac{a_n a_{n-2}}{a_{n-1}^2} \cdot \frac{K_{n-1}(\xi)}{K_{n-2}(\xi)} > 0.$$

In the second case we have

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* Cf. Fejér (loc. cit., pp. 302–303), where the last part of this theorem is established for the case $\Phi_n(x)\equiv P_n(x)$—Legendre polynomial.

$$\phi_n(\xi) + A_1'\phi_{n-1}(\xi) + A_2'\phi_{n-2}(\xi) = 0, \qquad \phi_n(\bar{\xi}) + A_1'\phi_{n-1}(\bar{\xi}) + A_2'\phi_{n-2}(\bar{\xi}) = 0,$$

and we can use (34) with the result

$$A_2' = \frac{a_n a_{n-2}}{a_{n-1}^2} \cdot \frac{K_{n-1}(\xi, \bar{\xi})}{K_{n-2}(\xi, \bar{\xi})} > 0.$$

The second part of Theorem VII is proved in precisely the same manner as in the special case considered by Fejér.

COROLLARY. *For the polynomial* (33) *we can have neither* $s_1 = 2$, *nor* $s_2 = 2$.

This follows from what was said in (β) above.

The following simple example shows that if the condition $A_2 < 0$ is violated, the polynomial (33) may have a double zero. Take $a = -1$, $b = 1$ and a "symmetric" sequence of *OP*, i.e., a sequence for which all c_n in (5) vanish. The polynomial

$$\omega_{2n}(x) = \Phi_{2n}(x) + \lambda_{2n}\Phi_{2n-2}(x) \equiv x\Phi_{2n-1}(x)$$

has a double zero at the origin.

Still more can be said about the abscissas in case $A_2 = 0$. We make use of the known property of *OP* by which (see (6))

$$(35) \qquad a < x_{1,n} < x_{1,n-1} < x_{2,n} < x_{2,n-1} < \cdots < x_{n-1,n-1} < x_{n,n} < b,$$

and state

THEOREM VIII. *The zeros* $c_{i,n}$ *of the polynomial*

$$\omega_n(x) = \Phi_n(x) + A_1\Phi_{n-1}(x),$$

where $A_1 (\neq 0)$ *is an arbitrary constant which may depend on* n, *are distributed as follows*:

In case $A_1 > 0$,

$$c_{1,n} < x_{1,n}; \; x_{1,n-1} < c_{1,n-1} < x_{2,n}; \cdots; \; x_{n-1,n-1} < c_{n,n} < x_{n,n}.$$

In case $A_1 < 0$,

$$x_{1,n} < c_{1,n} < x_{1,n-1}; \; x_{2,n} < c_{2,n} < x_{2,n-1}; \cdots;$$
$$x_{n-1,n-1} < c_{n-1,n-1} < x_{n-1,n}; \; x_{n,n} < c_{n,n}.$$

The proof follows from the relations

$$\omega_n(x_{i,n})\omega_n(x_{i+1,n}) = A_1^2\,\Phi_{n-1}(x_{i+1,n})\Phi_{n-1}(x_{i,n}) < 0, \qquad 1 \leqq i \leqq n-1,$$
$$\omega_n(x_{i,n-1})\omega_n(x_{i+1,n-1}) = \Phi_n(x_{i,n-1})\Phi_n(x_{i+1,n-1}) < 0, \qquad 1 \leqq i \leqq n-2,$$
$$\operatorname{sgn}\{\omega_n(x_{1,n})\omega_n(x_{1,n-1})\} = -\operatorname{sgn}\{\omega_n(x_{n,n})\omega_n(x_{n-1,n-1})\} = \operatorname{sgn} A_1,$$
$$\operatorname{sgn}\{\omega_n(x_{1,n})\omega_n(-\infty)\} = -\operatorname{sgn}\{\omega_n(x_{n,n})\omega_n(+\infty)\} = -\operatorname{sgn} A_1.$$

6. **Upper bounds for the coefficients.** The following theorem plays an important part in the study of the convergence properties of *FMQ* formulae.

THEOREM IX. *Any coefficient* C_i $(i=1, 2, \cdots, n)$ *of an FMQ formula, with degree of precision* q, *satisfies the inequality*

$$C_i \leqq \min \int_a^b G_{[q/2]}^2(x)d\psi$$

for all $G_{[q/2]}(x)$ *such that* $G_{[q/2]}(x_i)=1$. *It follows (by known properties of OP) that*

$$C_i \leqq 1/K_{[q/2]}(c_i; d\psi), \qquad i = 1, 2, \cdots, n.$$

In fact, for $G_{[q/2]}(x)$ with the above property we have

$$\int_a^b G_{[q/2]}^2(x)d\psi = \sum_{l=1}^{n} C_l G_{[q/2]}^2(c_l) \geqq C_i.$$

7. **The convergence properties.** Hereafter, *the interval* (a, b) *is assumed to be finite.* The case of an infinite interval will be treated elsewhere, but many of the results here obtained hold for an infinite interval as well.

We further assume to have given an infinite sequence of polynomials

$$\omega_n \equiv \omega_n(x) = \Phi_n(x) + A_1\Phi_{n-1}(x) + \cdots + A_{k-1}\Phi_{n-k+1}(x) \tag{36}$$

$$(n = 1, 2, \cdots; \Phi_{-i}(x) \equiv 0; A_i \text{ are constants})$$

with the following properties: (i) the zeros $c_{1,n}<c_{2,n}<\cdots<c_{n,n}$ of each ω_n are real and distinct; (ii) each ω_n generates an *FMQ* formula (11). In discussing the convergence of (11), i.e., the relation $\lim_{n\to\infty}R_n(f)=0$, we consider two cases.

Case I. All abscissas belong to $[a, b]$. Here we have

THEOREM X. *The FMQ formula* (11), *with all abscissas in* $[a, b]$, *converges for any bounded* $f(x)$ *for which* $\int_a^b f(x)d\psi$ *exists.*

This is a direct extension of an identical theorem of Fejér stated for the special case of the ordinary Riemann integral $\int_a^b f(x)dx$.* The present case requires more care, due to possible discontinuities of $\psi(x)$.† Let F denote the class of functions to which our *FMQ* formula can be applied. Thus, the statement $f(x) \subset F$ implies that $\int_a^b f(x)d\psi$ exists and

* Fejér, loc. cit., pp. 303–307. A different proof for this special case was given earlier by Stekloff (Bulletin of the Russian Academy of Sciences, 1916). The same result was derived recently by R. Bailey, *Convergence of sequences of positive linear functional operators*, Duke Mathematical Journal, vol. 2 (1936), pp. 287–303.

† The following version of the proof is due to J. D. Tamarkin. It is identical in principle with, but is an improvement in form over, the original proof of the author.

$$Q_n(f) \to \int_a^b f(x)d\psi, \text{ as } n \to \infty.$$

We now have the following

LEMMA. *If $\int_a^b f(x)d\psi$ exists and if, corresponding to any $\epsilon>0$, there exist two functions $f_{1,\epsilon}(x)$ and $f_{2,\epsilon}(x)$, both belonging to F and such that*

$$f_{1,\epsilon}(x) \leqq f(x) \leqq f_{2,\epsilon}(x) \text{ in } [a, b], \quad 0 \leqq \int_a^b [f_{2,\epsilon}(x) - f_{1,\epsilon}(x)]d\psi < \epsilon,$$

then also $f(x) \subset F$.

In fact, due to the positiveness of the coefficients C_i in

$$Q_n(f) \equiv \sum_{i=1}^{n} C_i f(c_i),$$

we have for each n,

$$Q_n(f_{1,\epsilon}) \leqq Q_n(f_{2,\epsilon}).$$

Letting here $n \to \infty$ and observing that, by hypothesis,

$$Q_n(f_{j,\epsilon}) \to \int_a^b f_{j,\epsilon}(x)d\psi, \qquad j = 1, 2,$$

we conclude that both $\limsup_{n\to\infty} Q_n(f)$ and $\liminf_{n\to\infty} Q_n(f)$ lie between $\int_a^b f_{1,\epsilon}(x)d\psi$ and $\int_a^b f_{2,\epsilon}(x)d\psi$, each of which, tends to $\int_a^b f(x)d\psi$, as $\epsilon \to 0$. It follows that

$$Q_n(f) \to \int_a^b f(x)d\psi, \text{ as } n \to \infty,$$

and this proves our Lemma.

In order to prove Theorem X, we observe that all polynomials belong to F. Since any continuous function $f(x)$ can be approximated uniformly on $[a, b]$ by polynomials, the above lemma shows that all continuous functions also belong to F. Now let $f(x)$ be a step-function taking the constant value 1 in (α, β) $(a<\alpha<\beta<b)$ and the value zero elsewhere in $[a, b]$, where α and β are points of continuity of $\psi(x)$. Let $\delta<b-a$ be such that $a<\alpha-\delta$, $\beta+\delta<b$. Define two continuous functions $f_{1,\delta}(x)$, $f_{2,\delta}(x)$ as follows:

$$f_{1,\delta}(x) = \begin{cases} 1 \text{ in } [\alpha, \beta], \ 0 \text{ in } [a, \alpha - \delta], \ [\beta + \delta, b], \\ \text{linear in } (\alpha - \delta, \alpha), \ (\beta, \beta + \delta), \end{cases}$$

$$f_{2,\delta}(x) = \begin{cases} 1 \text{ in } [\alpha + \delta, \beta - \delta], \ 0 \text{ in } [a, \alpha], \ [\beta, b], \\ \text{linear in } (\alpha, \alpha + \delta), \ (\beta - \delta, \beta). \end{cases}$$

It is clear that

$$f_{1,\delta}(x) \leqq f(x) \leqq f_{2,\delta}(x) \text{ in } [a, b].$$

On the other hand,

$$0 \leqq \int_a^b [f_{2,\delta}(x) - f_{1,\delta}(x)]d\psi \leqq \psi(\alpha + \delta) - \psi(\alpha - \delta) + \psi(\beta + \delta) - \psi(\beta - \delta).$$

Since α, β are points of continuity of $\psi(x)$, the right-hand member tends to 0 as $\delta \to 0$; hence, by the above lemma, $f(x) \subset F$. It follows that any linear combination of a finite number of functions of the type of the above $f(x)$ belongs to F, and so does, therefore, any step-function with a finite number of steps, whose points of discontinuity in (a, b) are points of continuity of $\psi(x)$. (The values of such a function at its points of discontinuity may be chosen arbitrarily, since they do not affect the value of $\int_a^b f(x)d\psi$.)

Assume now that $f(x)$ is any function for which $\int_a^b f(x)d\psi$ exists. This means that

$$\int_a^b f(x)d\psi = \lim \sum_{i=0}^{n-1} f(\xi_i)[\psi(x_i) - \psi(x_{i-1})], \text{ as } \max (x_i - x_{i-1}) \to 0.$$

Here $x_0 = a$, $x_n = b$, $x_{i-1} < x_i$, and ξ_i is taken arbitrarily in $[x_{i-1}, x_i]$. Since the set of points of continuity of $\psi(x)$ is dense in $[a, b]$, we may assume all interior points of subdivision $x_1, x_2, \cdots, x_{n-1}$ to be points of continuity of $\psi(x)$. Let m_i, M_i denote respectively $\inf_x f(x)$, $\sup_x f(x)$ for x in $[x_{i-1}, x_i]$. Let $f_{1,n}(x)$, $f_{2,n}(x)$ be two step-functions assuming constant values m_i, M_i respectively in $[x_{i-1}, x_i]$, their values at $x = x_i$ being arbitrary, subject to the only condition

$$f_{1,n}(x_i) \leqq f(x_i), \qquad f_{2,n}(x_i) \geqq f(x_i), \qquad i = 1, 2, \cdots, n - 1.$$

Then it is clear that

$$f_{1,n}(x) \leqq f(x) \leqq f_2(x), \qquad a \leqq x \leqq b,$$

$$\int_a^b f_{j,n}(x)d\psi \to \int_a^b f(x)d\psi, \text{ as } n \to \infty, \qquad j = 1, 2.$$

A direct application of the lemma shows that $f(x) \subset F$, and this completes the proof of our theorem.

Remark. For $A_1 = A_2 = \cdots = A_{k-1} = 0$, the preceding proof shows the convergence of any *GMQ* formula (in a finite interval), without imposing any restriction on $\psi(x)$.*

* Stieltjes in his classical paper, *Quelques recherches sur les quadratures dites mécaniques*, Oeuvres, vol. I, pp. 377–394, has proved the convergence of the *GMQ* formula for the case $d\psi(x) = p(x)dx$, $p(x) \geqq 0$, with the restriction $\int_\alpha^\beta p(x)dx > 0$, $a \leqq \alpha < \beta \leqq b$. It was the desire to lift this restriction that prompted Stieltjes to introduce a new concept of integral which we now call the Stieltjes Integral.

Case II. Some abscissas fall outside $[a, b]$. Without loss of generality, we may assume $a=-1$, $b=1$.

THEOREM XI. *The convergence Theorem* X *for the FMQ formula generated by the polynomial*

$$\omega_n(x) = \Phi_n(x) + A_1\Phi_{n-1}(x) + \cdots + A_{k-1}\Phi_{n-k+1}(x)$$

holds, regardless of the location of the abscissas relative to $(-1, 1)$, *even if the coefficients* $A_1, \cdots, A_{k-1}$ *and the integer* k *vary with* n, *provided*: (i) $\psi(x)$ *is continuous at* $x=\pm 1$, *and* (ii) *the number* l *of abscissas* $c_{i,n}$ *such that* $1 \leqq |c_{i,n}| < 1+h$, *where* $h>0$ *is arbitrarily small but fixed, is* $o[K_\sigma(\pm 1; d\psi)]$, $\sigma=[(2n-k)/2]$. (The latter condition is obviously satisfied if k is bounded, for $K_\sigma(\pm 1; d\psi) \to \infty$, as $n\to\infty$; see Lemma I below.)

An analysis of the proof of Theorem X, in connection with our agreement concerning $f(c_{i,n})$ for $|c_{i,n}|>1$, shows at once that Theorem XI will be proved if we prove the following lemmas.

LEMMA I. *Let* c *denote an end-point of* $[a, b]$, *where* $\psi(x)$ *is continuous. Then*

$$\lim_{n\to\infty} K_n(c; d\psi) = \infty.$$

Reduce, without loss of generality, $[a, b]$ to $[-1, 1]$. Take, to be definite, $c=1$. Then (M, p. 52)

$$\begin{aligned}\frac{1}{K_n(1; d\psi)} \leqq \int_0^1 x^{2n}d\psi &= \left(\int_0^{1-n^{-2/3}} + \int_{1-n^{-2/3}}^1\right) x^{2n}d\psi \\ &= o(1) + \int_{1-n^{-2/3}}^1 d\psi(x) \\ &= o(1) + \psi(1) - \psi(1-n^{-2/3}) \to 0, \text{ as } n\to\infty.\end{aligned}$$

Combining the above lemma with Theorem IX, we derive

LEMMA II. *If* c, *with the same significance and property as in Lemma* I, *is one of the abscissas, then the corresponding coefficient, say,* $C_{i,n}\to 0$, *as* $n\to\infty$.

LEMMA III. *Under the conditions of Theorem* XI, *the part of the summation* $Q_n(f)$ *which extends over abscissas not in* $[-1, 1]$ *tends to* 0, *as* $n\to\infty$, *if* $f(x)$ *is a polynomial of arbitrarily fixed degree* r. *Hence, making use of Lemma* II,

$$\sum_{|c_{i,n}|<1} C_{i,n}G_r(c_{i,n}) \to \int_a^b G_r(x)d\psi, \text{ as } n\to\infty \ (r \text{ fixed}).$$

For brevity denote by ξ any abscissa not in $[-1, 1]$, with the correspond-

ing coefficient C_ξ. We have, by Theorem IX,

$$(37)\quad C_\xi \leqq \frac{1}{K_\sigma(\xi; d\psi)} < \left\{\frac{2\alpha_0^{1/2}}{[\xi + (\xi^2-1)^{1/2}]^\sigma + [\xi - (\xi^2-1)^{1/2}]^\sigma}\right\}^2, \quad |\xi| > 1.*$$

Moreover, according to Tchebyheff,

$$(38)\quad \begin{aligned} &|G_\tau(x)| \leqq g \text{ on } [-1, 1] \text{ implies} \\ &|G_\tau(\xi)| \leqq \frac{g}{2}\left|[\xi + (\xi^2-1)^{1/2}]^\tau + [\xi - (\xi^2-1)^{1/2}]^\tau\right|, \qquad |\xi| > 1. \end{aligned}$$

Hence,

$$(39)\quad \begin{aligned} &|C_\xi G_\tau(\xi)| \leqq \frac{4g\alpha_0}{[\xi + (\xi^2-1)^{1/2}]^{2\sigma-\tau}}, \quad |\xi + (\xi^2-1)^{1/2}| > 1, \\ &\left|\sum_{|\xi|\geqq 1+h} C_\xi G_\tau(\xi)\right| \leqq 4gk\alpha_0\{1 + h + [(1+h)^2 - 1]^{1/2}\}^{\tau-2\sigma}. \end{aligned}$$

Furthermore,

$$(40)\quad C_\xi \leqq \frac{1}{K_\sigma(\xi; d\psi)} \leqq \frac{1}{K_\sigma(d; d\psi)}, \quad \text{if} \quad 1 \leqq |\xi| < 1 + h,\dagger$$

where d is that of the points ± 1 which is nearest to ξ, and

$$(41)\quad \left|\sum_{1\leqq|\xi|<1+h} C_\xi G_\tau(\xi)\right| \leqq \frac{l\bar{g}}{K_\sigma(1; d\psi)} + \frac{l\bar{g}}{K_\sigma(-1; d\psi)},$$
$$\bar{g} = \max |G_\tau(x)| \text{ in } [-1-h, 1+h].$$

Upon combining (39) and (41), we obtain a proof of our Lemma.

Remark. If we assume that the inequality $|c_{i,n}| \geqq 1+h$ holds for all abscissas outside $(-1, 1)$ (i.e., the number l in Theorem XI is zero), then the condition that $\psi(x)$ be continuous at $x = \pm 1$ is unnecessary, as seen from (39). It is interesting to note that the same reasoning which yields the proof of Theorem X proves also the following

THEOREM XII. *Let* $\{\psi_n(x)\}$, $n = 1, 2, \cdots$, *be a sequence of functions monotonic and uniformly bounded in the finite interval* $[a, b]$, *such that, as* $n \to \infty$,

* J. Shohat, *On a general formula in the theory of Tchebycheff polynomials and its applications*, these Transactions, vol. 29 (1927), pp. 569–583; p. 575. The considerations therein employed for $d\psi(x) = q(x)dx$, apply, without modification, to the general $d\psi(x)$.

† $K_n'(x) = 2\sum_{i=1}^{n}\phi_i(x)\phi_i'(x) > 0$, $x > x_{n,n}$, and < 0, $x < x_{1,n}$, so that, as x increases, $K_n(x)$ increases for $x > x_{n,n}$ and decreases for $x < x_{1,n}$.

$\psi_n(x)$ converges to a monotonic function $\psi(x)$ over a set of points dense in $[a, b]$ and containing a, b. Then

$$\lim_{n\to\infty} \int_a^b f(x)d\psi_n(x) = \int_a^b f(x)d\psi(x)$$

for any $f(x)$ for which $\int_a^b f(x)d\psi_n(x)$, $\int_a^b f(x)d\psi(x)$ exist.

This constitutes an extension of the Helly-Bray Theorem,* where $f(x)$ is assumed to be continuous, but, to compensate, $\psi(x)$, $\psi_n(x)$ are functions of bounded variation.

8. **New representation of C_i and $Q_n(f)$; relation to algebraic continued fractions.** Given the *LMQ* formula (11), introduce the polynomial of degree $n-1$,

$$\sigma_n(x) = \int_{-1}^{1} \frac{\omega_n(x) - \omega_n(y)}{x - y}\, d\psi(y), \tag{42}$$

and we get the following representation for C_i, $Q_n(f)$:

$$C_i = \frac{\sigma_n(c_i)}{\omega_n'(c_i)}, \quad i = 1, 2, \cdots, n; \qquad Q_n(f) = \sum_{i=1}^{n} \frac{\sigma_n(c_i)}{\omega_n'(c_i)} f(c_i). \tag{43}$$

In particular, for any fixed z, real or complex, but not on $[-1, 1]$, we have, assuming that all c_i are on $[-1, 1]$:

$$Q_n\left(\frac{1}{z - x}\right) = \frac{\sigma_n(z)}{\omega_n(z)}. \tag{44}$$

Returning to *FMQ* formulae, we observe that our convergence theorem evidently holds for complex functions of the real variable x, hence, also for the function $1/(z-x)$, z being fixed as in (44). Thus, for the *FMQ* formula as described in Theorem X, with all abscissas on $[-1, 1]$,

$$\lim_{n\to\infty} \frac{\sigma_n(z)}{\omega_n(z)} = \int_{-1}^{1} \frac{d\psi(x)}{z - x}, \tag{45}$$

z given, real or complex, not on $[-1, 1]$.

Now introduce a sequence of horizontal step-functions $\psi_n(x)$, $n = 1, 2, \cdots$, as follows: $\psi_n(-1) = 0$, $\psi_n(x)$ is constant in each interval $[-1, c_{1,n})$, $[c_{1,n}, c_{2,n})$, $\cdots$, $[c_{n,n}, 1]$, and has a saltus at $x = c_{1,n}, c_{2,n}, \cdots, c_{n,n}$, the saltus at $x = c_{i,n}$ being $C_{i,n}$, $i = 1, 2, \cdots, n$. It follows that $\psi_n(x)$ is bounded and

* Theorem XII is known to hold for continuous $f(x)$ (cf. G. C. Evans, *The Logarithmic Potential*, American Mathematical Society Colloquium Publications, vol. 6, pp. 14–15), and the above reasoning is applicable, for $\int_a^b f(x)d\psi_n(x)$, which here replaces $Q_n(f)$, has the same positive linear character as the latter.

non-decreasing in $[-1, 1]$, with $\psi_n(1)=C_{1,n}+\cdots+C_{n,n}=\alpha_0$. With the aid of the sequence $\{\psi_n(x)\}$, we may rewrite our *FMQ* formula (11) as

$$\int_{-1}^{1} f(x)d\psi(x) = \int_{-1}^{1} f(x)d\psi_n(x) + R_n(f), \tag{46}$$

and our convergence Theorem X states that

$$\lim_{n\to\infty} \int_{-1}^{1} f(x)d\psi_n(x) = \int_{-1}^{1} f(x)d\psi(x). \tag{47}$$

It is interesting to note that (47) has been obtained without further investigating the nature of $\psi_n(x)$.

The application of the above considerations to *GMQ* formulae is especially interesting. Here $\sigma_n(z)/\omega_n(z)\equiv\Omega_n(z)/\Phi_n(z)$ is the $(n+1)$st convergent to the continued fraction

$$\frac{\lambda_1|}{|z-c_1} - \frac{\lambda_2|}{|z-c_2} - \cdots - \frac{\lambda_n|}{|z-c_n} - \cdots \tag{48}$$

"associated" with the integral $\int_{-1}^{1} d\psi(y)/(z-y)$, and thus (47) *yields at once the convergence of the continued fraction* (48) *to the above integral, for any z real or complex, not on* $[-1, 1]$.*

9. **The remainder.** Write

$$R_n(f) = \int_{-1}^{1} [f(x) - G_q(x)]d\psi - \sum_{i=1}^{n} C_{i,n}[f(c_{i,n}) - G_q(c_{i,n})], \tag{49}$$

and consider several cases.

(i) $f(x)$ *is continuous in* $[-1, 1]$. (49) gives at once:

$$|R_n(f)| < 2\alpha_0 E_q(f), \tag{50}$$

where $E_q(f)$ is the "best approximation" of $f(x)$ on $[-1, 1]$ by polynomials of degree $\leqq q$, i.e.,

$$E_q(f) = \min_{G_q} \max_{|x|\leqq 1} |f(x) - G_q(x)|.$$

In particular, with $f(x)=1/(z-x)$, z real and $|z|>1$, and all $c_{i,n}$ on $[-1, 1]$:†

* The same convergence theorem shows immediately that the "moment-problem"

$$\int_a^b x^n d\psi(x) = \text{given } \alpha_n,\ n=0, 1, \cdots,$$

for the finite interval (a, b) is "determined," i.e., has at most one solution, for any one solution gives rise to the same sequence of *OP*, hence, to the same *GMQ* formula.

† The expression for $E_n(1/(z-x))$ is due to S. Bernstein (*Leçons sur les Propriétes Extrémales et la Meilleure Approximation des Fonctions Analytiques d'une Variable Réelle*, Paris, 1926, p. 121).

$$(51)\qquad \left| R_n\left(\frac{1}{z-x}\right)\right| \equiv \left|\int_{-1}^{1}\frac{d\psi(x)}{z-x} - \frac{\sigma_n(z)}{\omega_n(z)}\right| < 2\alpha_0/(z^2-1)[z+(z^2-1)^{1/2}]^q \quad (|z+(z^2-1)^{1/2}|>1).$$

(More generally, (49) shows the convergence, for any continuous $f(x)$, of every *LMQ* formula, for which the sum $\sum_{i=1}^{n}|C_{i,n}|$ is bounded, as $n\to\infty$.) For $q=2n-1$, (51) gives the degree of convergence of the continued fraction (48) to $\int_{-1}^{1}d\psi(x)/(z-x)$ for real z, outside $[-1, 1]$.

If, in addition, $f(x)$ has in $[-1, 1]$ derivatives of various orders, we may take in (49) for $G_q(x)$ a properly chosen interpolation polynomial for $f(x)$. Thus, with $q=2n-4$, $s_1=s_2=1$, $c_{1,n}=-1$, $c_{n,n}=1$, choose $G_q(x)$ so that

$$G_q(\pm 1)=f(\pm 1),\quad G_q^{(\alpha)}(c_{i,n})=f^{(\alpha)}(c_{i,n}),\quad \alpha=0, 1;\ i=2, 3, \cdots, n-1.$$

Then, by virtue of (23),

$$(52)\qquad R_n(f) = -\frac{f^{(2n-2)}(\xi)}{(2n-2)!}\left\{\frac{1}{a_{n-1}^2(d\psi_1)}+\frac{B_1}{a_{n-3}^2(d\psi_1)}\right\},$$

where $|\xi|\leqq 1$ and $d\psi_1=(1-x^2)d\psi$.

(ii) *$f(x)$ is analytic.* Let $f(t)$ $(t=x+iy)$ be analytic in a certain region, bounded by a simple closed rectifiable curve C which contains in its interior the line-segment $[-1, 1]$. Then

$$f(x)=\frac{1}{2\pi i}\int_C \frac{f(t)dt}{t-x},$$

and, by (44),

$$(53)\qquad \int_{-1}^{1}f(x)d\psi(x)=\frac{1}{2\pi i}\int_{-1}^{1}d\psi(x)\int_C\frac{f(t)dt}{t-x}=\frac{1}{2\pi i}\int_C\frac{f(t)\sigma_n(t)}{\omega_n(t)}dt+R_n(f).$$

It is readily proved that

$$(54)\qquad \int_{-1}^{1}d\psi(x)\left\{\int_\alpha^\beta F(x, y)dy\right\}=\int_\alpha^\beta dy\left\{\int_{-1}^{1}F(x, y)d\psi(x)\right\},$$

if $F(x, y)$ is continuous in x, y for $-1\leqq x\leqq 1$; $\alpha\leqq y\leqq\beta$; (α, β) finite. Hence,

$$(55)\qquad \frac{1}{2\pi i}\int_{-1}^{1}d\psi(x)\int_C\frac{f(t)dt}{t-x}=\frac{1}{2\pi i}\int_C f(t)dt\int_{-1}^{1}\frac{d\psi(x)}{t-x},$$

$$(56)\qquad R_n(f)=\frac{1}{2\pi i}\int_C f(t)\left[\int_{-1}^{1}\frac{d\psi(x)}{t-x}-\frac{\sigma_n(t)}{\omega_n(t)}\right]dt.$$

We proceed to transform the expression (56). Denoting generally

$$\frac{\alpha'}{x^s}+\frac{\beta'}{x^{s+1}}+\cdots=\left(\frac{1}{x^s}\right),\qquad s>0,\qquad \alpha'\neq 0,$$

we have, by the definition of degree of precision,

$$\omega_n(t)F(t)-\sigma_n(t)=\left(\frac{1}{x^{q-n+2}}\right),\qquad F(t)=\int_{-1}^{1}\frac{d\psi(x)}{t-x},$$

$$F(x)-\frac{\sigma_n(t)}{\omega_n(t)}=\left(\frac{1}{x^{q+2}}\right). \tag{57}$$

Upon writing

$$\frac{\sigma_n(t)}{\omega_n(t)}=\frac{1}{\omega_n(t)}\int_{-1}^{1}\frac{\omega_n(t)-\omega_n(x)}{t-x}\,d\psi(x)=F(t)-\frac{1}{\omega_n(t)}\int_{-1}^{1}\frac{\omega_n(x)d\psi(x)}{t-x}$$

and comparing with (57), we conclude that

$$F(t)-\frac{\sigma_n(t)}{\omega_n(t)}=\frac{1}{\omega_n(t)}\int_{-1}^{1}\frac{\omega_n(x)d\psi(x)}{t-x}\equiv\frac{S_n(t)}{\omega_n(t)},$$

$$S_n(t)=\int_{-1}^{1}\frac{\omega_n(x)d\psi(x)}{t-x}, \tag{58}$$

and (56) gives

$$R_n(f)=\frac{1}{2\pi i}\int_C\frac{f(t)S_n(t)dt}{\omega_n(t)}=\frac{1}{2\pi i}\int_C f(t)\left[F(t)-\frac{\sigma_n(t)}{\omega_n(t)}\right]dt. \tag{59}$$

In the special case of the *GMQ* formula, $S_n(t)$ is the so-called Tchebycheff function of second kind,

$$S_n(t)=\int_{-1}^{1}\frac{\Phi_n(x)d\psi(x)}{t-x}, \tag{60}$$

and (59) gives an expression for the remainder of the *GMQ* formula in terms of the remainder of the associated continued fraction (48). Namely,

$$R_n(f)=\frac{1}{2\pi i}\int_C\frac{f(t)S_n(t)dt}{\Phi_n(t)}=\frac{1}{2\pi i}\int_C f(t)\left[F(t)-\frac{\Omega_n(t)}{\Phi_n(t)}\right]dt, \tag{59.1}$$

where, we recall, $\Omega_n(z)/\Phi_n(z)$ is the $(n+1)$st convergent to (48). In the general case of an *FMQ* formula with degree of precision $q=2n-k$, we may substitute into (42) the expression (14) for $\omega_n(x)$, which gives, by (60),

$$R_n(f) = \frac{1}{2\pi i}\int_C f(t)\,\frac{S_n(t) + A_1S_{n-1}(t) + \cdots + A_{k-1}S_{n-k+1}(t)}{\Phi_n(t) + A_1\Phi_{n-1}(t) + \cdots + A_{k-1}\Phi_{n-k+1}(t)}\,dt$$

(61)

$$= \frac{1}{2\pi i}\int_C f(t)\left\{F(t) - \frac{\Omega_n(t) + A_1\Omega_{n-1}(t) + \cdots + A_{k-1}\Omega_{n-k+1}(t)}{\Phi_n(t) + A_1\Phi_{n-1}(t) + \cdots + A_{k-1}\Phi_{n-k+1}(t)}\right\}dt.$$

Formulae (56), (61) hold for any LMQ formula with all abscissas on $[-1, 1]$, $f(t)$ being analytic inside and on C.

The above formulae show once more the close connection between the theory of mechanical quadratures and that of algebraic continued fractions. The latter enables us to estimate $R_n(f)$ if an estimate is known for $S_n(t)$.

For real t we may proceed as follows. Formula (60) shows that the $\{S_n(t)\}$ are the coefficients in the expansion

$$\frac{1}{t - x} \sim \sum_{n=0}^{\infty} a_nS_n(t)\phi_n(x),$$

whence, from known results,

$$|S_n(t)| \leqq \alpha_0^{1/2}/a_nE_n\left(\frac{1}{t-x}\right) \leqq \alpha_0/\left\{2^{n-1}(t^2-1)\,|\,t \pm (t^2-1)^{1/2}\,|^{n-1}\right\}.$$

Here we assume that t is real, $|t| > 1$, and $\pm$ is so chosen that $|t \pm (t^2-1)^{1/2}| > 1$. Furthermore, if δ denotes the minimum distance from t to the line-segment $[-1, 1]$, then

$$|\Phi_n(t)| \equiv |(t - x_{1,n}) \cdots (t - x_{n,n})| > \delta^n,$$

whence,

$$\left|F(t) - \frac{\Omega_n(t)}{\Phi_n(t)}\right| \equiv \left|\frac{S_n(t)}{\Phi_n(t)}\right| \leqq \alpha_0/2^{n-1}(t^2-1)\delta^n\,|\,t \pm (t^2-1)^{1/2}\,|^{n-1}.$$

10. **The case of $d\psi(x) = p(x)\,dx$. $p(x)$ an S-function.** The Lebesgue integral

$$\int_{-1}^{1} \frac{p(x)dx}{(1-x^2)^{1/2}} \tag{62}$$

then exists. In this important special case we can go much further in the discussion of the abscissas and the remainder in our FMQ formula (11), for here $\Phi_n(x)$ possesses many asymptotic properties. Thus, as $n \to \infty$,

$$a_n = 2^nA[1 + o(1)],\ \lambda_n \to \tfrac{1}{4},\ c_n \to 0;\ A > 0,\ \text{independent of } n. \tag{63}$$

$$\Phi_n(\pm 1) = o[(\tfrac{1}{2} + \epsilon)^n],\ \epsilon > 0,\ \text{arbitrarily small}; \tag{64}$$

$$\Phi_n(\pm 1)/\Phi_{n-1}(\pm 1) \to \pm\tfrac{1}{2}.$$

(65) $$\Phi_n(x) \sim w^n\Phi(w),\ K_n(x) \sim (2w)^{2n}K(w);\ \Phi(w), K(w) \text{ independent of } n, \qquad w = \{x + (x^2 - 1)^{1/2}\}/2,$$

x is not on $[-1, 1]$ (M, pp. 50, 52, 54). The determination of the radical $(x^2-1)^{1/2}$ here and hereafter is so chosen that $|w|\to\infty$, as $|x|\to\infty$. These asymptotic relations for $\Phi_n(x)$ hold uniformly in any finite closed region in the x-plane whose minimum distance from the line-segment $[-1, 1]$ is positive. We denote such a region by D.

Now, the relations

$$\frac{\Omega_n(x)}{\Phi_n(x)} \to F(x), \quad \frac{S_n(x)}{\Phi_n(x)} \to 0, \quad \text{as} \quad n \to \infty \qquad \left(F(x) = \int_{-1}^{1} \frac{d\psi(y)}{x - y}\right)$$

imply

$$\frac{\Omega_{n+1}(x)}{\Omega_n(x)} \to w, \quad \text{as} \quad n \to \infty,$$

uniformly in D. Note that the transformation

(66) $$w = \{x + (x^2 - 1)^{1/2}\}/2, \qquad x = \tfrac{1}{2}\{2w + 1/(2w)\},$$

maps conformally the complex x-plane onto the w-plane outside the circle $|w| \leqq \frac{1}{2}$, so that the line-segment $[-1, 1]$ corresponds to the circumference $|w| = \frac{1}{2}$, and the line-segments $(-\infty, -1]$ and $[1, \infty)$ correspond to the line-segments $(-\infty, -\frac{1}{2}]$ and $[\frac{1}{2}, \infty)$ respectively. We now proceed to investigate the zeros $c_{i,n}$ of the polynomial

(14) $$\omega_n(x) = \Phi_n(x) + A_1\Phi_{n-1}(x) + \cdots + A_{k-1}\Phi_{n-k+1}(x),$$

under the following assumptions, which we call "assumptions P":

(α) k is fixed, independent of n;

(β) $A_1, A_2, \cdots, A_{k-1}$, if dependent on n, have finite limits $h_1, h_2, \cdots, h_{k-1}$, as $n\to\infty$.

Note in passing that, by virtue of (α), (β), we have here

(67) $$\lim_{n\to\infty} \frac{\sigma_n(x)}{\omega_n(x)} = \lim_{n\to\infty} \frac{\Omega_n(x) + A_1\Omega_{n-1}(x) + \cdots + A_{k-1}\Omega_{n-k+1}(x)}{\Phi_n(x) + A_1\Phi_{n-1}(x) + \cdots + A_{k-1}\Phi_{n-k+1}(x)} = F(x)$$

at any point x not on $[-1, 1]$, which does not coincide with one of the zeros of the polynomial

(68) $$H(w) \equiv w^{k-1} + h_1w^{k-2} + \cdots + h_{k-1} \qquad (w = \{x + (x^2 - 1)^{1/2}\}/2).$$

We have

(69) $$\omega_n(x) = \Phi_{n-k+1}F_n(x),\ F_n(x) \equiv \frac{\Phi_n(x)}{\Phi_{n-k+1}(x)} + A_1\frac{\Phi_{n-1}(x)}{\Phi_{n-k+1}(x)} + \cdots + A_{k-1},$$

and the rational function $F_n(x)$ is analytic outside $(-1, 1)$; its zeros outside $[-1, 1]$ are zeros for $\omega_n(x)$, and vice versa, and its only singularities are poles in $(-1, 1)$. The following asymptotic relations form the basis of the subsequent discussion.

As $n \to \infty$, we have uniformly in D, by virtue of (63), (64), (65),

$$\frac{\omega_n(x)}{\Phi_{n-k+1}(x)} \equiv F_n(x) \to H(w), \qquad \omega_n(x) \sim \Phi_{n-k+1}(x)H(w), \tag{70}$$

$$\frac{\omega_n(\pm 1)}{\Phi_{n-k+1}(\pm 1)} = F_n(\pm 1) \to H(\pm \tfrac{1}{2}). \tag{71}$$

It follows that in all cases

$$\omega_n(\pm 1) \to 0, \quad \text{as} \quad n \to \infty. \tag{72}$$

We set

$$w_{i,n} = \{c_{i,n} + (c_{i,n}^2 - 1)^{1/2}\}/2, \qquad (i = 1, 2, \cdots, n; n = 1, 2, \cdots) \tag{73}$$

corresponding to the zeros of $\omega_n(x)$ (at least $n-k+1$ of which lie in $(-1, 1)$). We denote by w_j', $j=1, 2, \cdots, k-1$, the zeros of $H(w)$, and set

$$x_j' = \frac{1}{2}\left(2w_j' + \frac{1}{2w_j'}\right), \qquad j = 1, 2, \cdots, k-1. \tag{74}$$

The relations (69), (70) lead to

THEOREM XIII. *Let D be a finite closed region in the x-plane at a minimum positive distance from the line-segment $[-1, 1]$, and let D' be the corresponding region in the w-plane. If D' contains in its interior zeros w_{i_1}', w_{i_2}', $\cdots$, w_{i_μ}' of $H(w)$, but no zeros w_j' on the boundary $(k-1 \geqq \mu \geqq 0)$, then for $n \geqq N = N(D')$ sufficiently large, $\omega_n(x)$ has in D' precisely μ zeros $c_{j_1,n}', \cdots, c_{j_\mu,n}'$ (a zero of multiplicity ρ being counted ρ times). Moreover,*

$$\lim_{n\to\infty} c_{j_s,n}' = x_{j_s}' = \frac{1}{2}\left\{2w_{j_s}' + \frac{1}{2w_{j_s}'}\right\}, \quad s = 1, 2, \cdots, \mu.$$

We readily draw the following conclusions:

(i) The zeros of $\omega_n(x)$, $n=1, 2, \cdots$, are all bounded.

(ii) Denote by ν the number of zeros w_j'' outside the circle $|w| \leqq 1/2$. For n sufficiently large, $\omega_n(x)$ has precisely ν zeros $c_{i,n}''$ outside the line-segment $[-1, 1]$ converging to $\frac{1}{2}\{2w_i'' + 1/2w_i''\}$, as $n \to \infty$. If $\nu = k-1$, these $c_{i,n}''$ account for all zeros of $\omega_n(x)$ outside $[-1, 1]$. If $\nu < k-1$, the remaining zeros of $\omega_n(x)$, if they do not actually belong to $[-1, 1]$, belong to it asymptotically, i.e., all their limit-points lie on $[-1, 1]$.

(iii) A necessary condition for the reality of all zeros of $\omega_n(x)$, for n sufficiently large (hereafter expressed by $n \geqq N$, N being properly chosen in the case under discussion) is that $H(w)$ shall not have imaginary zeros w_j', with $|w_j'| > \frac{1}{2}$. In other words, *if $\omega_n(x)$ generates an LMQ formula, then the zeros w_j' of $H(w)$ outside the circle $|w| \leqq \frac{1}{2}$ are necessarily all real.* The corresponding points $x_j' = \frac{1}{2}\{2w_j' + 1/(2w_j')\}$ are the limiting positions, as $n \to \infty$, for the corresponding abscissas of our *LMQ* formula. The most interesting case is that of $\omega_n(x)$ generating an *FMQ* formula.

THEOREM XIV. *A necessary condition that*

$$\omega_n(x) = \Phi_n(x) + A_1\Phi_{n-1}(x) + \cdots + A_{k-1}\Phi_{n-k+1}(x)$$

satisfying the "assumptions P" generates an FMQ formula for $n \geqq N$ is that all zeros of $H(w)$ lying outside the circle $|w| \leqq 1/2$ shall be real and that either interval $(-\infty, -\frac{1}{2}]$, $[\frac{1}{2}, \infty)$ shall contain no more than $[k/2]$ of these zeros.

As an illustration, we take up once more the special cases $k=2, 3$, A_i independent of n.

(i) $k=2$, i.e.,

$$\omega_n(x) = \Phi_n(x) + A_1\Phi_{n-1}(x), \qquad A_1 \neq 0.$$

The discussion centers around $c_{1,n}$, $c_{n,n}$. Here

$$H(w) \equiv w + A_1, \qquad w_1' = -A_1. \tag{75}$$

$A_1 > \frac{1}{2}$ implies $c_{1,n} < -1 (n \geqq N)$, $c_{1,n} \to \frac{1}{2}(2A_1 + 1/(2A_1))$, as $n \to \infty$.

$A_1 < -\frac{1}{2}$ implies $c_{n,n} > 1 (n \geqq N)$, $c_{n,n} \to \frac{1}{2}(2A_1 + 1/(2A_1))$, as $n \to \infty$.

$|A_1| \leqq \frac{1}{2}$ implies $-1 < c_{i,n} < 1$, $i = 1, 2, \cdots, n$.

$$\lim_{n\to\infty} \omega_n(\pm 1)/\Phi_n(\pm 1) = A_1 \pm \tfrac{1}{2}$$

in all cases.* We may add that since $x_{1,n} \to -1$, $x_{n,n} \to +1$, as $n \to \infty$, we get from Theorem VIII

$A_1 > \frac{1}{2}$ implies $c_{n,n} \to 1$; $A_1 < -\frac{1}{2}$ implies $c_{1,n} \to -1$; (76)

$|A_1| \leqq \frac{1}{2}$ implies $c_{1,n} \to -1$, $c_{n,n} \to 1$.

(ii) $k=3$, i.e.,

$$\omega_n(x) = \Phi_n(x) + A_1\Phi_{n-1}(x) + A_2\Phi_{n-2}(x), \qquad A_2 \neq 0.$$

* Write

$$\omega_n(x) = \Phi_{n-1}(x)\left[\frac{\Phi_n(x)}{\Phi_{n-1}(x)} + A_1\right],$$

and note that $\Phi_n(x)/\Phi_{n-1}(x)$ always increases with x and $\to \pm\frac{1}{2}$ at $x = \pm 1$, as $n \to \infty$.

The discussion centers around two of the zeros of $\omega_n(x)$. Here

$$H(w) \equiv w^2 + A_1 w + A_2. \tag{77}$$

We discuss the following possibilities.

$$A_1^2 - 4A_2 < 0, \quad |w_j'| > \tfrac{1}{2}, \quad j = 1, 2. \tag{78.1}$$

$\omega_n(x)$ has two imaginary zeros, which converge to $\frac{1}{2}\{2w_j' + 1/(2w_j')\}$, $j=1, 2$.

$$A_1^2 - 4A_2 > 0, \quad |w_j'| > \tfrac{1}{2}, \quad j = 1, 2. \tag{78.2}$$

The zeros of $\omega_n(x)$ are all real, two lying outside $(-1, 1)$ and converging to $\frac{1}{2}\{2w_j' + 1/(2w_j')\}$, $j=1, 2$. If $A_2>0$, then $s_1=2$, $s_2=0$, or $s_1=0$, $s_2=2$. If $A_2<0$, then $s_1=s_2=1$. Thus, under the assumption (78.2), $\omega_n(x)$ generates an *LMQ* formula which becomes an *FMQ* formula, if and only if $A_2<0$ (see case (β), §5).

$$A_1^2 - 4A_2 > 0, \quad |w_1'| \leqq \tfrac{1}{2}, \quad |w_2'| > \tfrac{1}{2}. \tag{78.3}$$

Here again all zeros of $\omega_n(x)$ are real, one necessarily outside $(-1, 1)$ and converging to $\frac{1}{2}\{2w_2' + 1/(2w_2')\}$.

$$A_1^2 - 4A_2 > 0, \quad |w_j'| \leqq \tfrac{1}{2}, \quad j = 1, 2. \tag{78.4}$$

The zeros of $\omega_n(x)$, for $n \geqq N$, either all belong to $[-1, 1]$, or else, the limit-points of the sequence of the two exterior zeros are on $[-1, 1]$.

$$A_1^2 - 4A_2 = 0, \quad |A_1| > 1. \tag{78.5}$$

Here again all zeros of $\omega_n(x)$ are real, with two outside $[-1, 1]$ and both converging to $-(A_1^2+1)/(2A_1)$. There may be a double root, or we may have $s_1=2$, $s_2=0$, or $s_1=0$, $s_2=2$; $\omega_n(x)$ cannot generate an *FMQ* formula.

In a similar manner we may discuss the case $k=4$.

In the above discussion we have encountered cases where $\omega_n(x)$ generates an *LMQ* formula which cannot have all coefficients positive. However, we are dealing here with polynomials $\Phi_n(x)$ which possess special properties, and the question arises: do these special properties compensate for the presence of negative coefficients in our *MQ* formula? The answer is in the affirmative, as is shown in

THEOREM XV. *Consider the polynomial* (14) $\omega_n(x)$, *where, in addition to the "assumptions P," it is assumed*: (i) *the zeros* w_j' *of the polynomial* (68) $H(w)$ *are all real and distinct,* $|w_j'| > 1/2$. *If* $\omega_n(x)$ *generates an LMQ formula, the latter converges, as* $n \to \infty$, *for any* $f(x)$ *for which* $\int_{-1}^{1} f(x)\,d\psi$ *exists.*

The proof follows the same lines as that of Theorems X, XI, making use of the following properties of the *MQ* formula under discussion:

(i) the number of negative coefficients is bounded;

(ii) $\omega_n(x) \sim w^{n-k+1}\omega(w)$, x real or complex, not on $[-1, 1]$, $\omega(w)$ independent of n;

(iii) A, B denoting certain fixed finite quantities independent of n, we have:

$$|a^{(i)} - a^{(j)}| \geqq h_1 > 0, \quad |b^{(l)} - b^{(m)}| \geqq h_1 > 0, \quad -1 - h_2 \geqq a^{(i)} \geqq A,$$

$$1 + h_2 \leqq b^{i} \leqq B \qquad (n \geqq N;\ h_1, h_2 \text{ independent of } n;$$

$$i, j = 1, 2, \cdots, s_1;\ l, m = 1, 2, \cdots, s_2;\ i \neq j;\ l \neq m).$$

(i) is known; (ii) follows from (71), (72), (73); (iii) follows from the fact that all $a^{(i)}$, $b^{(l)}$ converge, as $n \to \infty$, to certain *fixed* points $\frac{1}{2}\{2w_i' + 1/(2w_i')\}$.

(iv) $s = k-1$, so that $q = 2n-s-1$, hence (by Theorem VI), all interior coefficients are positive.

We have thus obtained a wide class of convergent *LMQ* formulae, the existence of which is assured by the considerations developed above.

Remark. We may liberalize considerably the conditions imposed upon $H(w)$, without impairing the validity of Theorem XV. Thus, under (62), $\psi(x)$ is continuous at $x = \pm 1$, so that $K_n(\pm 1; d\psi) \to \infty$, as $n \to \infty$, and we could modify accordingly the condition requiring that all $|w_i'| > \frac{1}{2}$, permitting a certain number (necessarily finite) of $a^{(i)}$, $b^{(j)}$ to converge to ∓ 1 respectively (see (40), (41)). We shall not dwell here upon this and other possible modifications, except the following one.

THEOREM XV (bis). *Given any $d\psi(x)$ and an LMQ formula employing $s = s_1 + s_2$ (s_1, s_2 fixed) arbitrarily preassigned abscissas, with degree of precision $q = 2n - s - 1$. The formula converges for any $f(x)$ for which $\int_{-1}^{1} f(x)d\psi$ exists; $\psi(x)$ is supposed to be continuous at $x = \pm 1$, if these points (one or both) are among the preassigned abscissas.*

In fact, referring to §4, we see that the conditions (i), (ii), (iii), (iv), given above and the expressions (30) for the exterior coefficients hold for the *LMQ* formula under discussion.

11. **Tchebycheff inequalities for the coefficients of some classes of *FMQ* formulae.** These important inequalities, given without proof by Tchebycheff in 1874 for *GMQ* formulae, were proved in 1884 by Stieltjes and Markoff independently.* Markoff went somewhat further than Stieltjes, extending Tchebycheff inequalities to certain classes of *FMQ* formulae. The following

* Stieltjes, loc. cit., pp. 384–392; A. Markoff, *On Certain Applications of Algebraic Continued Fractions* (in Russian), Thesis, St. Petersburg, 1884. It is curious to note that both use the same proof, namely, applying properly constructed *MQ* formulae to suitably chosen polynomials. In what follows the proof is omitted.

is taken, with slight modifications and extensions (in order to cover the cases $q=2n-3,\ 2n-4$) from Markoff's Thesis.

THEOREM XVI. *The coefficients C_i in an FMQ formula, with degree of precision $q=2n-1,\ 2n-2$, or $q=2n-3$ with $s_1+s_2=1,\ 2$, or $q=2n-3,\ 2n-4$, $s_1=s_2=1$, satisfy the following Tchebycheff inequalities*:

$$(1)\quad \int_a^{c_i} d\psi(x) > C_1+C_2+\cdots+C_{i-1}, \quad i=2,3,\cdots,n,$$

$$(2)\quad \int_a^{c_i} d\psi(x) < C_1+C_2+\cdots+C_i, \quad i=1,2,\cdots,n,$$

$$(3)\quad \int_{c_i}^{b} d\psi(x) > C_{i+1}+\cdots+C_n, \quad i=2,3,\cdots,n,$$

$$(4)\quad \int_{c_i}^{b} d\psi(x) < C_i+\cdots+C_n, \quad i=1,2,\cdots,n,$$

$$(5)\quad \int_{c_i}^{c_k} d\psi(x) > C_{i+1}+\cdots+C_{k-1}, \quad i=1,2,\cdots,n-1;\ k=2,3,\cdots,n,$$

$$(6)\quad \int_{c_i}^{c_k} d\psi(x) < C_i+\cdots+C_k, \quad i,k=1,2,\cdots,n.$$

In particular,

$$(7)\quad \int_a^{c_1} d\psi(x) < C_1, \qquad \int_{c_n}^{b} d\psi(x) < C_n,$$

$$(8)\quad \int_{c_i}^{c_{i+1}} d\psi(x) > 0, \qquad i=1,2,\cdots,n-1,$$

$$(9)\quad \int_{c_i}^{c_{i+2}} d\psi(x) > C_{i+1}, \qquad i=1,2,\cdots,n-2.$$

Remarks. (i) These inequalities hold for any (a, b), finite or infinite.

(ii) If $c_i \neq a, b$, we may assign in the above inequalities to $\psi(x)$ at $x=c_i$ any value in $[\psi(c_i-0), \psi(c_i+0)]$. Thus, (8) may be rewritten as

$$(8.1)\quad \int_{c_i+0}^{c_{i+1}-0} d\psi(x) > 0, \qquad i=1,2,\cdots,n-1,$$

which shows that a subinterval of constancy of $\psi(x)$ cannot contain more than one abscissa—a property proved above in a different manner.

(iii) We agree to let $\psi(x)=\psi(a)$ or $\psi(x)=\psi(b)$ for $x \leqq a$ or $x \geqq b$ respectively. Then it is not necessary to take (with Markoff) $c_1 \geqq a$, $c_n \leqq b$.

(iv) The above inequalities hold for an LMQ formula, with $q=2n-3$ and $s_1=2$ or $s_2=2$ (then necessarily $s_2=0$ or $s_1=0$ respectively), with the following exceptions: $c_1<c_2\leqq a$ invalidates the inequalities

$$\int_a^{c_1} d\psi(x) < C_1, \qquad \int_{c_1}^{c_2} d\psi(x) > 0; \tag{79.1}$$

$c_n>c_{n-1}\geqq b$ invalidates the inequalities

$$\int_{c_n}^{b} d\psi(x) < C_n, \qquad \int_{c_{n-1}}^{c_n} d\psi(x) > 0. \tag{79.2}$$

In fact, by Theorem VI, $C_1<0$ or $C_n<0$ respectively, while the integrals on the left vanish, according to our agreement.

(v) For an LMQ formula, with $q=2n-4$, $s_1=2$, $s_2=0$, or $s_1=0$, $s_2=2$, the above inequalities hold, with the same possible exceptions as stated for $q=2n-3$. By virtue of (32), $c_1<c_2\leqq a$ actually invalidates (79.1) if $A_3<0$ in (14), and $c_n>c_{n-1}\geqq b$ actually invalidates inequalities (79.2) if $A_3>0$.

(vi) Assuming all $C_i>0$ and all c_i in (a, b), choose

$$a < \xi_1 < \xi_2 < \cdots < \xi_n \leqq b,$$

so that

$$C_1 + C_2 + \cdots + C_i = \int_a^{\xi_i} d\psi(x), \qquad i = 1, 2, \cdots, n.$$

It follows from the inequalities (1, 2) that

$$a < c_1 < \xi_1 < c_2 < \xi_2 < \cdots < c_n < \xi_n = b.$$

12. **Discussion of the abscissas and the coefficients based on Theorem XI.*** First, assume $q\leqq 2n-4$. We make use of the inequality (8), from which follows an important property of the abscissas which we call "C-property," namely: *a sub-interval (c, d) of constancy of $\psi(x)$, contains, including its end-points, at most one abscissa.*

We now turn to Theorem XI. Suppose that for a certain fixed i and with certain l, m, we have, as $n\to\infty$:

$$\begin{aligned} &(80.1) \quad c_{i,n}, c_{i+1,n}, \cdots, c_{i+l,n} \to \xi', \\ (80)\quad &(80.2) \quad c_{i-1,n}\, c_{i-2,n}, \cdots, c_{i-m,n} \to \xi'', \qquad a \leqq \xi'' < \xi' \leqq b, \\ &(80.3) \quad |\, c_{i+l+1,n} - \xi'|, \quad |\, c_{i-m-1,n} - \xi''| > h > 0, \end{aligned}$$

* The reasoning of the second part of this section generally follows that of Fejér, but the results are considerably modified in many respects, due to our dealing with $\psi(x)$ not necessarily continuous, and, a fortiori, not necessarily absolutely continuous.

where h is fixed, independent of n.* The interval $[\xi''+\delta, \xi'-\delta]$ now contains no zeros of $\omega_n(x)$, for $n \geqq N$. If ξ', ξ'' do not coincide with b or a respectively, choose δ so that $\xi' \pm \delta$, $\xi'' \pm \delta$ are points of continuity of $\psi(x)$. Apply Theorem XI to the following functions:

$$f_1(x) = 1 \text{ in } [\xi'' + \delta, \xi' - \delta], \quad f_2(x) = 1 \text{ in } [\xi' - \delta, \xi' + \delta],$$

$$f_3(x) = 1 \text{ in } [\xi'' - \delta, \xi'' + \delta]; \quad f_j(x) = 0 \text{ elsewhere, } j = 1, 2, 3.$$

We get:

$$\lim_{n\to\infty} Q_n(f_1) = \int_{\xi''+\delta}^{\xi'-\delta} d\psi(x) = \psi(\xi' - \delta) - \psi(\xi'' + \delta) = 0,$$

$$\lim_{n\to\infty} Q_n(f_2) = \int_{\xi'-\delta}^{\xi'+\delta} d\psi = \lim_{n\to\infty} (C_{i,n} + C_{i+1,n} + \cdots + C_{i+l,n}),$$

$$\lim_{n\to\infty} Q_n(f_3) = \int_{\xi''-\delta}^{\xi''+\delta} d\psi = \lim_{n\to\infty} (C_{i-1,n} + C_{i-2,n} + \cdots + C_{i-m,n}).$$

It follows that the values of the integrals $\int_{\xi''+\delta}^{\xi'-\delta} d\psi (=0)$, $\int_{\xi'-\delta}^{\xi'+\delta} d\psi$, $\int_{\xi''-\delta}^{\xi''+\delta} d\psi$ do not depend on δ. This means that $\psi(x)$ is constant in the intervals (ξ'', ξ'), $(\xi', \xi'+\delta)$, $(\xi''-\delta, \xi'')$, so that

$$\int_{\xi'-\delta}^{\xi'+\delta} d\psi = \psi(\xi' + 0) - \psi(\xi' - 0), \qquad \int_{\xi''-\delta}^{\xi''+\delta} d\psi = \psi(\xi'' + 0) - \psi(\xi'' - 0).$$

Hence, by virtue of the "C-property," $m=1$, $l=0$. Furthermore, the assumption $\psi(\xi'+\delta)-\psi(\xi'-\delta)=0$ implies $\psi(x)=$const. in (ξ'', ξ'), which requires that $c_{i-1,n}$ be to the left of ξ'', with a similar conclusion from the assumption $\psi(\xi''+\delta)-\psi(\xi''-\delta)=0$. Finally, we notice that we cannot have simultaneously

$$\psi(\xi'' + \delta) - \psi(\xi'' - \delta) = 0, \qquad \psi(\xi' + \delta) - \psi(\xi' - \delta) = 0,$$

for then $\psi(x)=$const. in $(\xi''-\delta, \xi'+\delta)$, and this, by the "$C$-property," contradicts (80). Thus, if (80) is satisfied, then: (α) $l=0$, $m=1$; (β) $\psi(x)$ is constant in (ξ'', ξ'), $(\xi', \xi'+\delta)$, $(\xi''-\delta, \xi'')$; (γ) $\lim_{n\to\infty} C_{i,n} = \psi(\xi'+0)-\psi(\xi'-0) = \sigma_1$, $\lim_{n\to\infty} C_{i-1,n} = \psi(\xi''+0)-\psi(\xi''-0) = \sigma_2$; ($\delta$) $\sigma_1^2+\sigma_2^2 \neq 0$, i.e., one at least of the points ξ', ξ'' is a point of discontinuity of $\psi(x)$ and the saltus is the limiting value of the corresponding coefficient. Moreover, if, for instance, $\psi(x)$ is continuous at $x=\xi'$, then $c_{i-1,n} \leqq \xi''$ $(n \geqq N)$; (ϵ) $c_{i,n}$ and $c_{i-1,n}$ cannot

* All limits ξ', ξ'', $\cdots$ here considered are assumed to belong to $[a, b]$. If $\xi'=b$ or $\xi''=a$, then $\xi'+\delta$, $\xi''-\delta$ should be replaced by b or a respectively. Here and hereafter δ or N denote properly chosen sufficiently small or sufficiently large numbers respectively, which may be different in different formulae.

both belong to the interval (ξ'', ξ') $(n \geqq N)$. It follows that if one at least of the conditions $l>0$ or $m>1$ is satisfied, then the third condition (80.3) cannot hold.

By the foregoing, any subinterval $(c, d) \subset (a, b)$ which has no abscissas $c_{i,n}$, for $n \geqq N$, is an interval of constancy for $\psi(x)$. More precisely, the zeros of $\omega_n(x)$ are everywhere dense in any subinterval $[a_1, b_1]$ which is not an interval of constancy for $\psi(x)$, i.e., $[a_1, b_1]$ contains at least one zero of $\omega_n(x)$, for $n \geqq N$, in its interior (or at its end-points, if $\psi(x)$ is continuous at $x=a_1, b_1$). (We omit the proof, since it is quite similar to that of Fejér, loc. cit., pp. 308–309.)

This property can be generalized as follows: Let a subinterval $[c, d]$ contain no more than one abscissa c_{i_ν,n_ν} for $n=n_1, n_2, \cdots, n_\nu, \cdots$.* Then, if $\psi(x)$ is continuous at $x=c_1$, $d_1 (c \leqq c_1 < d_1 \leqq d)$

$$\lim_{\nu\to\infty} C_{i_\nu,n_\nu} = \int_{c_1}^{d_1} d\psi(x).$$

Hence, $\psi(x)=\text{const.}$ in (c, d); $\lim_{\nu\to\infty} C_{i_\nu,n_\nu}=0$.

The above results hold for any convergent *FMQ* formula. Suppose now that *all abscissas lie in* $[a, b]$. We can state some more properties as follows (again derived by reasoning similar to that of Fejér).

In a convergent FMQ formula corresponding to a finite interval either all coefficients $\to 0$, *as* $n\to\infty$, *or else,* $\psi(x)$ *has discontinuities in* (a, b). *Hence, if* $\psi(x)$ *is continuous throughout* $[a, b]$ (*and a fortiori, if* $d\psi(x)=p(x)dx$, $p(x)\geqq 0$ *and integrable in* (a, b)), *then* $\lim_{n\to\infty} C_{i,n}=0$, $i=1, 2, \cdots, n$.

We may go still further if we assume $q=2n-1$, i.e., when dealing with a *GMQ* formula. Here we have the following result.

If in a GMQ formula

$$\lim_{n\to\infty} c_{i,n} = \xi_i, \qquad i = 1, 2, \cdots, \tag{81}$$

then $\psi(x)$ *is a step-function with saltus* σ_i *at* $x=\xi_i$, *and* $\lim_{n\to\infty} C_{i,n}=\sigma_i$, $i=1, 2, \cdots$.

The statement concerning $C_{i,n}$ follows as above in (γ). As to the behavior of $\psi(x)$, (81) implies, by the preceding, that $\psi(x)$ is constant in the intervals (ξ_{i-1}, ξ_i). Moreover, were $\psi(x)$ continuous at a certain point $x=\xi_i$, ξ_{i-1} and

* This is possible. Consider, for example, $\omega_n(x)=\Phi_n(x; d\psi)$, where $(a, b)\equiv(-h, h)$, $d\psi=p(x)dx$, $p(-x)\equiv p(x)$, $p(x)=0$ in $(-h_1, h_1) \subset (-h, h)$. Then $c_{n+1,2n+1}=0$, $n=0, 1, \cdots$, and this is the only zero of $\omega_{2n+1}(x)$ in $[-h_1, h_1]$, while $\omega_{2n}(x)$ has no zeros in this interval.

ξ_{i+1} would have been points of discontinuity, so that $\psi(x)=$const. in the interval (ξ_{i-1}, ξ_{i+1}), which could contain one abscissa only, namely, $c_{i,n}$, and

$$c_{i-1,n} \leqq \xi_{i-1}, c_{i-2,n} \leqq \xi_{i-2}, \cdots ; c_{i+1,n} \geqq \xi_{i+1}, c_{i+2,n} \geqq \xi_{i+2}, \cdots .$$

This contradicts (81), for all $c_{i,n}$ lie in (a, b), while $c_{1,n}$ decreases and $c_{n,n}$ increases, as n increases. *It follows that if $\psi(x)$ is continuous throughout $[a, b]$, with no subintervals of constancy, then neither* (80) *nor* (81) *is possible.*

The above results generalize those known for *GMQ* formulae. It is interesting to note that in their proof there was no need to invoke the theory of algebraic continued fractions nor that of the Moment-Problem (on the contrary, the preceding results are applicable to the latter problem). We have made use of but a small part of our convergence theorems, namely, of the fact that the *FMQ* formula under discussion converges for any $f(x)$ which is constant in any finite subinterval, arbitrarily chosen, and vanishes elsewhere. Thus, our results hold for an infinite interval as well, once the above convergence property is established for such functions.

13. **The sequence $\{\omega_n(x)\}$ forms an *OP* sequence.** We close with a brief discussion of the following interesting question.

Under what conditions is the sequence

$$\omega_n(x) = \Phi_n(x) + A_1\Phi_{n-1}(x) + \cdots + A_{k-1}\Phi_{n-k+1}(x) \qquad (n = 0,1, \cdots ; \Phi_{-i}(x) \equiv 0, A_1, A_2, \cdots , A_{k-1} = \text{const.}) \tag{14}$$

itself a sequence of OP?

The solution offered below, although incomplete, is of interest, for it introduces a rather unusual kind of *OP*, where the corresponding $\psi(x)$ has a subinterval of constancy. It also offers a good illustration of the preceding results. In what follows we discuss—and incompletely—the following case only:

$$\omega_n(x) = \Phi_n(x) + A\Phi_{n-1}(x) \qquad (n = 0, 1, \cdots ; \Phi_{-1}(x) \equiv 0; A \neq 0, \text{independent of } n). \tag{82}$$

The answer to the proposed question will be in the affirmative if and only if a recurrence relation of the form (5) exists for the $\{\omega_n(x)\}$:

$$\omega_{n+2}(x) \equiv \omega_{n+2} = (x - c'_{n+2})\omega_{n+1}(x) - \lambda'_{n+2}\omega_n(x) \qquad (n \geqq 0, \omega_0 = 1, \omega_1 = x - c_1'), \tag{83}$$

where $\lambda_1' (>0)$ and c_1' are arbitrary, and *all* λ_n' *are positive.* Substituting here for $\omega_n, \omega_{n+1}, \omega_{n+2}$ their expressions (82) and making use of (5), we get:

$$c'_{n+2} - c_{n+2} = 0,\ \lambda'_{n+2} - \lambda_{n+2} + A(c'_{n+2} - c_{n+1}) = 0,\ n = 0, 1, \cdots,$$
$$\lambda'_{n+2} - \lambda_{n+1} = 0, \qquad n = 1, 2, \cdots,$$

whence,

$$(84)\qquad \begin{aligned} &c'_1 = c_1 - A; \qquad c'_n = c_n, \qquad n = 2, 3, \cdots, \\ &\lambda'_n = \lambda_{n-1}, \qquad n = 3, 4, \cdots; \qquad \lambda'_2 = \lambda_2 - A(c_2 - c_1), \end{aligned}$$

so that the given sequence of *OP*, $\{\Phi_n(x)\}$, must satisfy the following conditions:

$$(85)\qquad \lambda_{n+1} - \lambda_{n+2} + A(c_{n+2} - c_{n+1}) = 0, \qquad n = 1, 2, \cdots.$$

Here we try to satisfy (85) in the following special manner:

$$(86)\qquad c_{n+2} - c_{n+1} = h - \text{const., independent of } n \qquad (n = 1, 2, \cdots).$$

The above relations now yield

$$(87)\qquad \lambda_{n+1} - \lambda_{n+2} = -Ah, \qquad n = 1, 2, \cdots,$$

$$(88)\qquad c_n = (n-2)h + c_2, \qquad \lambda_n = (n-2)Ah + \lambda_2, \qquad n = 3, 4, \cdots,$$

$\lambda_1, \lambda_2 > 0$ and c_1, c_2 arbitrary; $Ah \geqq 0$. Taking (without loss of generality) $\lambda'_1 = \lambda_1 = 1$ and changing notations, we state the following result.

Consider the continued fractions

$$(89)\qquad F(x) \equiv \frac{1|}{|x-c} - \frac{\lambda|}{|x} - \frac{\lambda + Ah|}{|x-h} - \cdots - \frac{\lambda + nAh|}{|x-nh} - \cdots,$$

$$(90)\qquad F_1(x) \equiv \frac{1|}{|x-c+A} - \frac{\lambda + Ac|}{|x} - \frac{\lambda|}{|x-h} - \frac{\lambda + Ah|}{|x-2h} - \cdots - \frac{\lambda + (n-1)Ah|}{|x-nh} - \cdots,$$

where the constants λ, A, c, h *are such that*

$$(91)\qquad \lambda > 0, \qquad \lambda + Ac > 0, \qquad Ah \geqq 0.$$

The denominators of the successive convergents to (89) *and* (90), *which we denote respectively by* $\{\Phi_n(x)\}$, $\{\omega_n(x)\}$ *give a solution of our problem, i.e., each sequence* $\{\Phi_n(x)\}$, $\{\omega_n(x)\}$ *is an OP sequence (which is not new), and their mutual relation is expressed in* (82).

What can be said of the orthogonality intervals and of the corresponding $\psi(x), \psi_1(x)$ for each sequence?

Leaving aside the case $Ah > 0$ which leads to infinite intervals of orthogonality, assume $h = 0$. We find from (89), (90):

$$(92) \qquad F(x) = 2/\{x - 2c + (x^2 - 4\lambda)^{1/2}\}$$

$$(93) \qquad F_1(x) = 2\lambda^2/\{2\lambda^2(x - A - c) - (\lambda + Ac)(x + (x^2 - 4\lambda)^{1/2})\}.$$

For the sake of brevity, we make the additional assumption $c=0$. Making use of some results of Stieltjes,* we state our final results as follows. For the sequence $\{\Phi_n(x)\}$:

$$F(x) = \frac{1|}{|x} - \frac{\lambda|}{|x} - \frac{\lambda|}{|x} - \cdots = \{x - (x^2 - 4\lambda)^{1/2}\}/2\lambda$$

$$= \frac{1}{2\pi\lambda}\int_{-2\lambda^{1/2}}^{2\lambda^{1/2}} (4\lambda - y^2)^{1/2}\frac{dy}{x - y},$$

($\lambda > 0$; x real or complex, not on $[-2\lambda^{1/2}, 2\lambda^{1/2}]$).

The interval of orthogonality is $(-2\lambda^{1/2}, 2\lambda^{1/2})$ and

$$\psi(x) = \frac{1}{2\pi\lambda}\int_{-2\lambda^{1/2}}^{x} (4\lambda - y^2)^{1/2}dy.$$

For the sequence $\{\omega_n(x)\}$:

$$F_1(x) = \frac{1 - \lambda/A^2}{x - \alpha} + \frac{1}{A}\int_{-2\lambda^{1/2}}^{2\lambda^{1/2}} \frac{(4\lambda - y^2)^{1/2}}{y - \alpha}\frac{dy}{x - y}$$

$$= \{x + 2A + (x^2 - 4\lambda)^{1/2}\}/2A(x - \alpha), \qquad \alpha = -A - \lambda/A.$$

In case $A > 0$ the interval of orthogonality is $(\alpha, 2\lambda^{1/2})$, and

$$\psi_1(\alpha) = 0, \quad \psi_1(x) = \text{const.} = 1 - \lambda/A^2 \text{ on } (\alpha, -2\lambda^{1/2}],$$

$$\psi_1(x) = 1 - \frac{\lambda}{A^2} + \frac{1}{A}\int_{-2\lambda^{1/2}}^{x} (4\lambda - y^2)^{1/2}\frac{dy}{y - \alpha} \text{ on } [-2\lambda^{1/2}, 2\lambda^{1/2}].$$

In case $A < 0$ the interval of orthogonality is $(-2\lambda^{1/2}, \alpha)$, and

$$\psi_1(x) = -\frac{1}{A}\int_{-2\lambda^{1/2}}^{x} \frac{d\psi(y)}{\alpha - y} \text{ on } [-2\lambda^{1/2}, 2\lambda^{1/2}],$$

$$\psi_1(x) = \text{const.} = -\frac{1}{A}\int_{-2\lambda^{1/2}}^{2\lambda^{1/2}} \frac{d\psi(y)}{\alpha - y}$$

on $[2\lambda^{1/2}, \alpha)$,

$$\psi_1(\alpha) = -\frac{1}{A}\int_{-2\lambda^{1/2}}^{2\lambda^{1/2}} \frac{d\psi(y)}{\alpha - y} + 1 - \frac{\lambda}{A^2}.$$

Thus, in both cases

* Stieltjes, *Recherches sur les fractions continues*, Oeuvres, vol. II, pp. 402–566; pp. 509–510.

$$d\psi_1(x) = \frac{1}{A}\,\frac{d\psi(x)}{x-\alpha} \text{ in } (-2\lambda^{1/2}, 2\lambda^{1/2}).$$

Note that all zeros $c_{i,n}$ of $\omega_n(x)$ lie in $(-2\lambda^{1/2}, 2\lambda^{1/2})$, except one. Moreover (in accordance with the general theory and with the preceding discussion), as $n\to\infty$,

$$c_{1,n}\to\alpha,\ c_{n,n}\to 2\lambda^{1/2},\ \text{if } A>0,$$

$$c_{1,n}\to -2\lambda^{1/2},\ c_{n,n}\to\alpha,\ \text{if } A<0,$$

where α, introduced above, is the zero of the polynomial

$$H(w)\equiv w+A\equiv\frac{x+(x^2-4\lambda)^{1/2}}{2}+A \qquad (|w|\to\infty,\ \text{as } |x|\to\infty).$$

$\psi_1(x)$ has no saltus if and only if $A^2=\lambda$; then the two intervals of orthogonality obtained above coincide with $(-2\lambda^{1/2}, 2\lambda^{1/2})$, and

$$d\psi_1(x)=\frac{1}{\lambda^{1/2}}\,\frac{d\psi(x)}{2\lambda^{1/2}-x}.$$

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EXISTENCE THEOREM FOR THE FLOW OF AN IDEAL INCOMPRESSIBLE FLUID IN TWO DIMENSIONS*

BY

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We shall prove the existence of the flow of an ideal incompressible fluid in two dimensions under certain initial conditions.

Suppose that R is an open connected plane region with a sufficiently smooth boundary. $u(x, y, t)$ and $v(x, y, t)$ are velocity components parallel to the x and y axes respectively and in the positive direction, $p(x, y, t)$ is the pressure, and σ is the density.

The flow of an ideal homogeneous incompressible fluid is said to exist if $u(x, y, t)$, $v(x, y, t)$, $p(x, y, t)$ and $\sigma = \text{constant}$ are defined in R throughout some time interval, possessing sufficient differentiability and satisfying the conditions:

(1) The divergence is zero and the density is constant.

(2) The normal component of the velocity at the boundary is zero.

(3) The Euler dynamical equations are satisfied

$$-\frac{1}{\sigma}\frac{\partial p}{\partial x} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial u}{\partial t},$$

$$-\frac{1}{\sigma}\frac{\partial p}{\partial y} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial v}{\partial t}.$$

It will be proved that if sufficiently smooth velocity components $u(x, y)$ and $v(x, y)$ are defined in R at time $t=0$ such that the divergence is zero and the normal component of the velocity at the boundary is zero, then a flow exists for all time which at time $t=0$ has the given velocity components $u(x, y)$ and $v(x, y)$. That is, $u(x, y, t)$, $v(x, y, t)$, and $p(x, y, t)$ are defined for all time $-\infty < t < \infty$ and for all x, y in R such that at time $t=0$

$$u(x, y, 0) = u(x, y), \qquad v(x, y, 0) = v(x, y).$$

In the case of the plane flow of an ideal incompressible fluid the vortex density, $\zeta(x, y, t) = \partial u/\partial y - \partial v/\partial x$, is, according to Helmholtz, constant along the moving point. With initial velocity components $u(x, y)$ and $v(x, y)$ given and satisfying the same conditions as above it is shown that velocity components $u(x, y, t)$ and $v(x, y, t)$, and paths of the moving point $dx/dt = u$,

* Presented to the Society, September 10, 1937; received by the editors September 4, 1936.

$dy/dt=v$, $x=a$, $y=b$ at $t=0$ exist for all time such that conditions (1) and (2) above are satisfied, the vortex density is constant along the path of the moving point, and at time $t=0$, $u(x, y, 0)=u(x, y)$ and $v(x, y, 0)=v(x, y)$. Then if the region R and the given velocity components have the properties demanded of them in Theorem I, it will follow that $p(x, y, t)$ exists so that the Euler dynamical equations are satisfied. If R is multiply-connected, we have, in order to insure single-valuedness of the pressure, to impose additional conditions (constancy of certain circulations).

Similar results have been obtained by Wolibner and Hölder. For given initial conditions Wolibner* sets up approximating functions which are equicontinuous and, using the selection theorem of Ascoli, shows that a suitable subsequence of the approximating functions converges to a limit defining the flow of an ideal incompressible fluid. Lichtenstein has shown that for given initial conditions the flow of an ideal incompressible fluid exists for a sufficiently small time, and Hölder,† presupposing Lichtenstein's result, shows that the flow in truth exists for all time. Leray,‡ by a vastly different method, has obtained an existence theorem for the flow of an incompressible viscous fluid. The novelty of the present paper consists in setting up successive approximations which converge for all time.§

Let R be an open simply-connected|| plane region whose boundary, B, consists of a single Jordan curve given in the form $x=x(s)$, $y=y(s)$ where d^3y/ds^3 and d^3x/ds^3 exist and satisfy a uniform Hölder condition of some positive order less than one.

Green's function, $G(x, y, x', y')$, is defined for all pairs of points x, y and x', y' in R by:

$$G(x, y, x', y') = ln\frac{1}{r} + g(x, y, x', y')$$

where $r^2=(x-x')^2+(y-y')^2$ and, for every point x, y in R, $g(x, y, x', y')$ considered as a function of x', y' is continuous in $R+B$, harmonic in R, and has the value $-ln\ 1/r$ on B.

THEOREM I. *Let velocity components $u(x, y)$ and $v(x, y)$ be defined in R at time $t=0$ such that*:

* W. Wolibner, Mathematische Zeitschrift, vol. 37 (1933), pp. 698–726.

† E. Hölder, Mathematische Zeitschrift, vol. 37 (1933), pp. 727–738.

‡ J. Leray, Journal de Mathématiques, vol. 12 (1933), pp. 1–82.

§ This paper follows a doctor's thesis submitted at the Massachusetts Institute of Technology and written under the direction of Professor Eberhard Hopf to whom the author wishes to express his gratitude for many valuable suggestions. The thesis contained a proof of uniqueness which is not reproduced here as it has been published elsewhere. See Wolibner, loc. cit.

|| In the appendix multiply-connected regions are considered.

(a) *The partial derivatives of first order of u and v with respect to x and y exist and satisfy a uniform Lipschitz condition,*

$$\left| \frac{\partial u(x_1, y_1)}{\partial x} - \frac{\partial u(x_2, y_2)}{\partial x} \right| \leqq C\{(x_1 - x_2)^2 + (y_1 - y_2)^2\}^{1/2}, \cdots;$$

(b) *The divergence is zero,*

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0;$$

(c) *The normal component of the velocity at the boundary is zero,*

$$u\frac{dy}{ds} - v\frac{dx}{ds} = 0.$$

Then there exist velocity components $u(x, y, t)$ and $v(x, y, t)$ defined for all x, y in R and for all time $-\infty < t < \infty$ such that:
(d) $u(x, y, 0) = u(x, y)$, $v(x, y, 0) = v(x, y)$;
(e) *The partial derivatives of first order of $u(x, y, t)$ and $v(x, y, t)$ with respect to x and y exist and are continuous;*
(f) *The divergence is zero;*
(g) *The normal component of the velocity at the boundary is zero;*
(h) *The paths of the moving point, $dx/dt = u$, $dy/dt = v$, $x = a$, $y = b$ at $t = 0$ exist and at each instant there is a one to one, continuous, area-preserving relation between (a, b) and (x, y), (a, b) in $R+B$, (x, y) in $R+B$;*
(i) *The vortex density is constant along the moving point, and in each finite time interval satisfies a uniform Lipschitz condition with respect to x, y, t.*

Proof. Theorem I is proved by a method of successive approximations. For simplicity we consider a positive time interval $t \geqq 0$, although the same method applies to a negative time interval. Let

$$\zeta(a, b) = \frac{\partial u(a, b)}{\partial b} - \frac{\partial v(a, b)}{\partial a}.$$

Then by the hypotheses of Theorem I,

$$|\zeta(a, b)| \leqq C_1, \tag{1}$$

$$|\zeta(a_1, b_1) - \zeta(a_2, b_2)| \leqq C_2\{(a_1 - a_2)^2 + (b_1 - b_2)^2\}^{1/2}. \tag{2}$$

We define lines $x = x_n(a, b, t)$, $y = y_n(a, b, t)$, $n = 0, 1, 2, \cdots$, issuing from each point (a, b) of the region R at time $t = 0$ which will, as n becomes large, approximate the paths of the moving point in the moving fluid. Let

$$x_0(a, b, t) \equiv a, \qquad y_0(a, b, t) \equiv b.$$

Suppose that $x_{n-1}(a, b, t)$ and $y_{n-1}(a, b, t)$ have been defined for all time such that at each instant there is a one to one continuous correspondence between (a, b) and (x_{n-1}, y_{n-1}), (a, b) in $R+B$, (x_{n-1}, y_{n-1}) in $R+B$ and such that the first order partial derivatives of x_{n-1} and y_{n-1} with respect to a and b and the first order partial derivatives of a and b with respect to x_{n-1} and y_{n-1} are bounded in every finite time interval. As suggested by the vortex theorem of Helmholtz we define a vortex density which is constant along the lines $x=x_{n-1}$, $y=y_{n-1}$. Let

$$\zeta_{n-1}(x, y, t) = \zeta(a, b), \quad \text{where} \quad x = x_{n-1}(a, b, t), \quad y = y_{n-1}(a, b, t).$$

Then $\zeta_{n-1}(x, y, t)$ satisfies a Lipschitz condition with respect to x and y in every finite time interval, and for all time has the same bound as $\zeta(a, b)$. Let

$$\phi_n(x, y, t) = -\frac{1}{2\pi}\int_R \zeta_{n-1}(x', y', t)G(x, y, x', y')dx'dy'.$$

Then $\phi_n(x, y, t)$ vanishes on the boundary, and since ζ_{n-1} satisfies a Lipschitz condition, the second order derivatives of ϕ_n with respect to x and y exist, are bounded, and satisfy a uniform Hölder condition with respect to x and y in every finite time interval.* Let

$$u_n(x, y, t) = \frac{\partial}{\partial y}\phi_n(x, y, t), \qquad v_n(x, y, t) = -\frac{\partial}{\partial x}\phi_n(x, y, t).$$

Then the divergence of the velocity components u_n and v_n is zero and the vortex density is ζ_{n-1},

$$\frac{\partial u_n}{\partial x} + \frac{\partial v_n}{\partial y} = \frac{\partial^2 \phi_n}{\partial x \partial y} - \frac{\partial^2 \phi_n}{\partial y \partial x} = 0,$$

$$\frac{\partial u_n}{\partial y} - \frac{\partial v_n}{\partial x} = \frac{\partial^2 \phi_n}{\partial y^2} + \frac{\partial^2 \phi_n}{\partial x^2} = \zeta_{n-1}.$$

Let

$$x_n(a, b, t) = a + \int_0^t u_n\{x_n(a, b, \mu), y_n(a, b, \mu), \mu\}d\mu,$$

$$y_n(a, b, t) = b + \int_0^t v_n\{x_n(a, b, \mu), y_n(a, b, \mu), \mu\}d\mu.$$

Since ϕ_n is zero on the boundary and the derivatives of u_n and v_n with respect to x and y are bounded, it follows that $x_n(a, b, t)$ and $y_n(a, b, t)$ exist for all time, the point x_n, y_n remaining in $R+B$, and at each instant there is a one-to-one correspondence between (a, b) and (x_n, y_n), (a, b) in $R+B$, (x_n, y_n)

* Lichtenstein, *Potentialtheorie*, Encycklopädie der mathematischen Wissenschaften, vol. 2, part 3, p. 286.

in $R+B$. Since the derivatives of $u_n(x, y, t)$ and $v_n(x, y, t)$ with respect to x and y are continuous it follows that $x_n(a, b, t)$ and $y_n(a, b, t)$ have continuous first order derivatives with respect to a and b, and a and b have continuous first order derivatives with respect to x_n and y_n.

$$\frac{\partial}{\partial a} x_n(a, b, t) = 1 + \int_0^t \left\{ \frac{\partial u_n}{\partial x} \frac{\partial x_n}{\partial a} + \frac{\partial u_n}{\partial y} \frac{\partial y_n}{\partial a} \right\} d\mu, \cdots .$$

Then $x_n(a, b, t)$ and $y_n(a, b, t)$ have the same properties which $x_{n-1}(a, b, t)$ and $y_{n-1}(a, b, t)$ were supposed to possess, and in addition it is seen that since the divergence is zero the transformation defined by x_n and y_n is area-preserving.

We thus make the definitions:

$$x_0(a, b, t) \equiv a, \qquad y_0(a, b, t) \equiv b.$$

If x_{n-1} and y_{n-1} have been defined, $n \geqq 1$,

$$\zeta_{n-1}(x, y, t) = \zeta(a, b), \quad \text{where} \quad x = x_{n-1}(a, b, t), \quad y = y_{n-1}(a, b, t),$$

$$\phi_n(x, y, t) = -\frac{1}{2\pi} \int_R \zeta_{n-1}(x', y', t) G(x, y, x', y') dx' dy',$$

$$u_n(x, y, t) = \frac{\partial}{\partial y} \phi_n(x, y, t), \qquad v_n(x, y, t) = -\frac{\partial}{\partial x} \phi_n(x, y, t),$$

$$x_n(a, b, t) = a + \int_0^t u_n\{x_n(a, b, \mu), y_n(a, b, \mu), \mu\} d\mu,$$

$$y_n(a, b, t) = b + \int_0^t v_n\{x_n(a, b, \mu), y_n(a, b, \mu), \mu\} d\mu.$$

The aim now is to show that the approximating functions u_n and v_n converge as n becomes large to limits which are the velocity components of the moving fluid, and that the approximating lines $x=x_n$, $y=y_n$ converge to the paths of the moving points. But before proving convergence it will be shown that the approximating functions have a modulus of continuity which is independent of n, and this will follow readily from Lemma I.

LEMMA I. *Let r_0 be the distance between any two points a_1, b_1 and a_2, b_2 at time $t=0$ and r the distance between the corresponding points by the nth mapping function at a later time t, $0 \leqq t \leqq K$,*

$$r_0^2 = (a_1 - a_2)^2 + (b_1 - b_2)^2,$$

$$r^2 = (x_n(a_1, b_1, t) - x_n(a_2, b_2, t))^2 + (y_n(a_1, b_1, t) - y_n(a_2, b_2, t))^2.$$

Then

$$r_0 \leqq C_3 r, \qquad r \leqq C_3 r_0, \qquad 0 \leqq t \leqq K,$$

where C_3 is independent of n, but depends on K.

Proof. It is evident from the definition that $\zeta_n(x, y, t)$ is bounded independently of n and t. Solely from the bound of ζ_n, independently of any of its continuity properties, it is found that $r \leqq Cr_0^{\alpha}$, $r_0 \leqq Cr^{\alpha}$, where $0<\alpha<1$ and C and α are independent of n, and α is a decreasing function of t. From this first approximation it follows that $\zeta_n(x, y, t)$ does satisfy a uniform Hölder condition in every finite time interval independently of n, and from this fact a better approximation can be found.

From definition

$$(3) \qquad \begin{aligned} &|\zeta_n(x, y, t)| = |\zeta(a, b)| \leqq C_1, \qquad n = 0, 1, 2, \cdots, \\ &\phi_n(x, y, t) = -\frac{1}{2\pi}\int_R \zeta_{n-1}(x', y', t)G(x, y, x', y')dx'dy'. \end{aligned}$$

Then if $D_1\phi_n$ represents the first derivative of ϕ_n with respect to x or y, that is u_n or v_n,

$$(4) \qquad |D_1\phi_n(x_1, y_1, t) - D_1\phi_n(x_2, y_2, t)| \leqq A_1C_1 r \ln\frac{A_2}{r},$$

where $r^2=(x_1-x_2)^2+(y_1-y_2)^2$ and the constants A_1 and A_2 depend* only on the region R.

Let r_0 and r have the same meaning as in the statement of Lemma I, and let

$$x_1 = x_n(a_1, b_1, t), \qquad x_2 = x_n(a_2, b_2, t),$$
$$y_1 = y_n(a_1, b_1, t), \qquad y_2 = y_n(a_2, b_2, t).$$

Then from the definition of x_n and y_n,

$$x_1 - x_2 = a_1 - a_2 + \int_0^t \{u_n(x_1, y_1, \mu) - u_n(x_2, y_2, \mu)\}d\mu,$$

$$y_1 - y_2 = b_1 - b_2 + \int_0^t \{v_n(x_1, y_1, \mu) - v_n(x_2, y_2, \mu)\}d\mu,$$

$$\frac{dr}{dt} = \frac{x_1 - x_2}{r}\{u_n(x_1, y_1, t) - u_n(x_2, y_2, t)\} + \frac{y_1 - y_2}{r}\{v_n(x_1, y_1, t) - v_n(x_2, y_2, t)\},$$

* This follows from the fact that $|D_1G(x, y, x', y')| \leqq C/r$ and $|D_2G(x, y, x', y')| \leqq C/r^2$ where $r^2=(x-x')^2+(y-y')^2$ and C depends only on the region R. Then $(1/2\pi)\int_R |D_1G(x_1, y_1, x', y') - D_1G(x_2, y_2, x', y')|\, dx'dy' \leqq A_1 r \ln A_2/r$ where A_1 and A_2 depend only on R, and $r^2=(x_1-x_2)^2+(y_1-y_2)^2$.

$$\left|\frac{dr}{dt}\right| \leqq 2A_1C_1 r \ln\frac{A_2}{r},$$

$$\left|\frac{d}{dt}\ln\ln\frac{A_2}{r}\right| \leqq 2A_1C_1,$$

$$\left|\ln\frac{A_2}{r_0}\Big/\ln\frac{A_2}{r}\right| \leqq 2A_1C_1t,$$

$$r \leqq A_2 r_0^{\exp[-2C_1A_1t]} \qquad r_0 \leqq A_2 r^{\exp[-2C_1A_1t]} \tag{5}$$

where it is supposed that $A_2 \geqq 1$. Then it follows from (5) that $\zeta_n(x, y, t)$ satisfies a uniform Hölder condition with respect to x and y in every finite time interval. Suppose that $0 \leqq t \leqq K$ where K is an arbitrary but fixed positive number.

$$\begin{aligned}|\zeta_n(x_1, y_1, t) - \zeta_n(x_2, y_2, t)| &= |\zeta(a_1, b_1) - \zeta(a_2, b_2)| \\ &\leqq C_2 r_0 \leqq C_2 A_2 r^{\exp[-2A_1C_1K]}.\end{aligned} \tag{6}$$

To find a better approximation we consider the smoothness properties of $\phi_n(x, y, t)$ in the light of (3) and (6).

$$\phi_n(x, y, t) = -\frac{1}{2\pi}\int_R \zeta_{n-1}(x', y', t)G(x, y, x', y')dx'dy'.$$

$$\begin{cases} |D_1\phi_n(x, y, t)| \leqq A_3C_1, & 0 \leqq t < \infty, \\ |D_2\phi_n(x, y, t)| \leqq C_4(C_1 + C_2A_2), & 0 \leqq t \leqq K, \\ |D_2\phi_n(x_1, y_1, t) - D_2\phi_n(x_2, y_2, t)| \leqq C_5(C_1 + C_2A_2)r^{\exp[-2A_1C_1K]}, & \\ & 0 \leqq t \leqq K, \end{cases} \tag{7}$$

where $r^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$ and the constant A_3 depends* only on the region R, and C_4 and C_5 depend on the region R and on K, but are independent of n.

From the inequalities (7) it can be inferred that the first derivatives of x_n and y_n are bounded independently of n. First, since the derivatives of $u_n(x, y, t)$ and $v_n(x, y, t)$ with respect to x and y are continuous in x and y and bounded in each finite time interval, differentiation under the integral sign by a or b is justified.

$$\frac{\partial}{\partial a}x_n(a, b, t) = 1 + \int_0^t \left\{\frac{\partial u_n}{\partial x}\frac{\partial x_n}{\partial a} + \frac{\partial u_n}{\partial y}\frac{\partial y_n}{\partial a}\right\}d\mu, \cdots.$$

* Lichtenstein, loc. cit.

Then from the bound on the derivatives of u_n and v_n one finds:

$$\left|\frac{\partial x_n}{\partial a}\right|, \left|\frac{\partial x_n}{\partial b}\right|, \left|\frac{\partial y_n}{\partial a}\right|, \left|\frac{\partial y_n}{\partial b}\right| \leqq e^{2C_4(C_1+C_2A_2)t}, \qquad 0 \leqq t \leqq K$$

and in the same way

$$\left|\frac{\partial a}{\partial x_n}\right|, \left|\frac{\partial a}{\partial y_n}\right|, \left|\frac{\partial b}{\partial x_n}\right|, \left|\frac{\partial b}{\partial y_n}\right| \leqq e^{2C_4(C_1+C_2A_2)t}, \qquad 0 \leqq t \leqq K.$$

Lemma I follows from this since the bound $e^{2C_4(C_1+C_2A_2)K}$ is independent of n.

Note that another approximation shows that $\zeta_n(x, y, t)$ satisfies a Lipschitz condition which is independent of n but depends on the time. Letting (x_1, y_1) and (x_2, y_2) be two points at time t a distance r apart and (a_1, b_1) and (a_2, b_2) the corresponding points by the nth mapping function at time $t=0$ a distance r_0 apart we have from Lemma I

$$(8) \qquad |\zeta_n(x_1, y_1, t) - \zeta_n(x_2, y_2, t)| = |\zeta(a_1, b_1) - \zeta(a_2, b_2)| \leqq C_2 r_0 \leqq C_2 C_3 r,$$

where $r^2=(x_1-x_2)^2+(y_1-y_2)^2$. From (7)

$$(9) \quad \begin{cases} |u_n(x, y, t)|, \quad |v_n(x, y, t)| \leqq A_3C_1, & 0 \leqq t < \infty, \\ |u_n(x_1, y_1, t) - u_n(x_2, y_2, t)| \leqq C_6(|x_1 - x_2| + |y_1 - y_2|), & 0 \leqq t \leqq K, \\ |v_n(x_1, y_1, t) - v_n(x_2, y_2, t)| \leqq C_6(|x_1 - x_2| + |y_1 - y_2|), & 0 \leqq t \leqq K, \end{cases}$$

where A_3C_1 is independent of n and t, and C_6 depends on K but not on n.

It can now be shown that the approximating functions ζ_n, u_n, v_n, x_n, and y_n converge as n becomes large to limits which have the properties stated in Theorem I. Choosing any fixed positive number K consider the time interval $0 \leqq t \leqq K$. From the boundedness of $\zeta_n(x, y, t)$,

$$|\zeta_1(x, y, t) - \zeta_0(x, y, t)| \leqq 2C_1, \qquad 0 \leqq t < \infty.$$

Suppose it has been proved that

$$|\zeta_n(x, y, t) - \zeta_{n-1}(x, y, t)| \leqq 2C_1 \frac{(mt)^{n-1}}{(n-1)!}, \qquad 0 \leqq t \leqq K.$$

Then since $\int_R |D_1G(x, y, x', y')|\,dx'dy'$ is uniformly bounded in R one obtains from the definition of u_n and v_n,

$$(10) \quad \begin{aligned} |u_{n+1}(x, y, t) - u_n(x, y, t)| &\leqq 2C_1C_7 \frac{(mt)^{n-1}}{(n-1)!}, \qquad 0 \leqq t \leqq K, \\ |v_{n+1}(x, y, t) - v_n(x, y, t)| &\leqq 2C_1C_7 \frac{(mt)^{n-1}}{(n-1)!}, \qquad 0 \leqq t \leqq K, \end{aligned}$$

where C_7 depends only on the region R. From the inequality (10) we can find a bound on the distance between the lines $x=x_n$, $y=y_n$ and $x=x_{n+1}$, $y=y_{n+1}$ issuing from the same point (a, b) at time $t=0$ (the inequality (11)), and since the relation is defined by an integral the exponent $n-1$ is raised to n.

$$x_{n+1}(a, b, t) - x_n(a, b, t) = \int_0^t \{u_{n+1}(x_{n+1}, y_{n+1}, \mu) - u_n(x_{n+1}, y_{n+1}, \mu)\} d\mu + \int_0^t \{u_n(x_{n+1}, y_{n+1}, \mu) - u_n(x_n, y_n, \mu)\} d\mu,$$

and a similar expression for $y_{n+1}-y_n$. Applying (10) to the first term on the right and (9) to the second term on the right and adding the resulting inequalities one obtains

$$|x_{n+1} - x_n| + |y_{n+1} - y_n| \leqq 4C_1C_7m^{n-1}\frac{t^n}{n!} + 2C_6\int_0^t \{|x_{n+1} - x_n| + |y_{n+1} - y_n|\} d\mu,$$

where on the left the four variables are taken at time t, and on the right they are taken at time μ. To show that this implies (11) suppose it has been shown that the left-hand side of this inequality is less than $4C_1C_7m^{n-1}(t^n/n!)e^{2C_6t} + A(2C_6t)^p/p!$ in the time interval $0 \leqq t \leqq K$, where A is a positive constant and p is an integer or zero. On putting this in the right and performing the indicated integration it is found that the left-hand side is less than $4C_1C_7m^{n-1}(t^n/n!)e^{2C_6t} + A(2C_6t)^{p+1}/(p+1)!$. The constant A can be made so large that this is true for the case $p=0$, and then it follows for all p. Thus

$$|x_{n+1}(a, b, t) - x_n(a, b, t)| + |y_{n+1}(a, b, t) - y_n(a, b, t)| \leqq 4C_1C_7m^{n-1}\frac{t^n}{n!}e^{2C_6t}. \tag{11}$$

If a point (a, b) at time $t=0$ is considered, (11) gives a bound on the distance between the corresponding points (x_n, y_n) and (x_{n+1}, y_{n+1}) by the n and $n+1$ mapping functions. If a point (x, y) at time $t>0$ is considered, there correspond two points at time $t=0$, (a_1, b_1) and (a_2, b_2) by the n and $n+1$ mapping functions respectively. By the use of Lemma I, the inequality (11) can be transformed into a bound on the distance between (a_1, b_1) and (a_2, b_2). Let

$$x = x_n(a_1, b_1, t) = x_{n+1}(a_2, b_2, t),$$
$$y = y_n(a_1, b_1, t) = y_{n+1}(a_2, b_2, t).$$

We introduce the auxiliary point $x_n(a_2, b_2, t)$, $y_n(a_2, b_2, t)$ and use Lemma I.

$$\begin{aligned}|a_1 - a_2| + |b_1 - b_2| &\leqq 2C_3\{|x_n(a_1, b_1, t) - x_n(a_2, b_2, t)| \\ &\quad + |y_n(a_1, b_1, t) - y_n(a_2, b_2, t)|\} \\ &= 2C_3\{|x_{n+1}(a_2, b_2, t) - x_n(a_2, b_2, t)| \\ &\quad + |y_{n+1}(a_2, b_2, t) - y_n(a_2, b_2, t)|\},\end{aligned}$$

which by (11) is less than $8C_1C_3C_7m^{n-1}(t^n/n!)e^{2C_6t}$, $0 \leqq t \leqq K$. By construction, $\zeta_n(x, y, t) = \zeta(a_1, b_1)$ and $\zeta_{n+1}(x, y, t) = \zeta(a_2, b_2)$, so using (2) and the last found inequality we have

$$\begin{aligned}|\zeta_{n+1}(x, y, t) - \zeta_n(x, y, t)| &= |\zeta(a_2, b_2) - \zeta(a_1, b_1)| \\ &\leqq C_2\{|a_1 - a_2| + |b_1 - b_2|\} \leqq 8C_1C_2C_3C_7m^{n-1}\frac{t^n}{n!}e^{2C_6t}.\end{aligned}$$

The coefficient $4C_2C_3C_7e^{2C_6K}$ is independent of n, but depends on K; so, letting m be as large as this coefficient, we have

$$|\zeta_{n+1}(x, y, t) - \zeta_n(x, y, t)| \leqq 2C_1\frac{(mt)^n}{n!}, \qquad 0 \leqq t \leqq K.$$

This inequality then holds for all n, where C_1 and m are independent of n, but m depends on K. This also verifies the inequalities (10) and (11) so the functions $\zeta_n(x, y, t)$, $u_n(x, y, t)$, $v_n(x, y, t)$, $x_n(a, b, t)$, and $y_n(a, b, t)$ converge to continuous limits and uniformly in every finite time interval. Dropping the subscripts in the limits it follows from (8) that $\zeta(x, y, t)$ satisfies a Lipschitz condition with respect to x and y in every finite time interval.

$$|\zeta(x_1, y_1, t) - \zeta(x_2, y_2, t)| \leqq C_2C_3\{(x_1 - x_2)^2 + (y_1 - y_2)^2\}^{1/2}, \qquad 0 \leqq t \leqq K,$$

where C_2 is a given constant and C_3 depends on K.

The relations between the variables are preserved in the limits.

$$\phi(x, y, t) = -\frac{1}{2\pi}\int_R \zeta(x', y', t)G(x, y, x', y')dx'dy',$$

$$u(x, y, t) = \frac{\partial}{\partial y}\phi(x, y, t), \qquad u = \text{limit } u_n,$$

$$v(x, y, t) = -\frac{\partial}{\partial x}\phi(x, y, t), \qquad v = \text{limit } v_n,$$

$$\begin{aligned}D_2\phi &= -\frac{\zeta(x, y, t)}{2\pi}D_2\left\{\int_R G(x, y, x', y')dx'dy'\right\} \\ &\quad -\frac{1}{2\pi}\int_R \{\zeta(x', y', t) - \zeta(x, y, t)\}D_2G(x, y, x', y')dx'dy'.\end{aligned}$$

From this and similar expressions for $D_2\phi_n$ it follows that the first derivatives of $u_n(x, y, t)$ and $v_n(x, y, t)$ with respect to x and y converge to the corresponding derivatives of $u(x, y, t)$ and $v(x, y, t)$. The divergence is zero and $\zeta(x, y, t)$ is the vortex density of the moving fluid.

$$\frac{\partial}{\partial x}u(x, y, t) + \frac{\partial}{\partial y}v(x, y, t) = \frac{\partial^2}{\partial x \partial y}\phi(x, y, t) - \frac{\partial^2}{\partial y \partial x}\phi(x, y, t) = 0,$$

$$\frac{\partial}{\partial y}u(x, y, t) - \frac{\partial}{\partial x}v(x, y, t) = \frac{\partial^2}{\partial y^2}\phi(x, y, t) + \frac{\partial^2}{\partial x^2}\phi(x, y, t) = \zeta(x, y, t).$$

From the uniform convergence of x_n, y_n, u_n, and v_n to their respective limits and the equicontinuity of u_n and v_n expressed in (9) it follows that

$$x(a, b, t) = a + \int_0^t u(x(a, b, \mu), y(a, b, \mu), \mu)d\mu, \quad x = \text{limit } x_n,$$

$$y(a, b, t) = b + \int_0^t v(x(a, b, \mu), y(a, b, \mu), \mu)d\mu, \quad y = \text{limit } y_n.$$

At each instant there is a one-to-one continuous correspondence between (a, b) and $(x(a, b, t), y(a, b, t))$, (a, b) in $R+B$, (x, y) in $R+B$, and since the divergence is zero the transformation is area-preserving.

The vortex density is constant along the moving point, $\zeta(x, y, t) = \zeta(a, b)$, where $x = x(a, b, t)$, $y = y(a, b, t)$.

Since $\zeta(x, y, 0) = \zeta(x, y) = \partial u(x, y)/\partial y - \partial v(x, y)/\partial x$, where $u(x, y)$ and $v(x, y)$ are the given velocity components and since the divergence is zero,

$$u(x, y, 0) = u(x, y), \qquad v(x, y, 0) = v(x, y).$$

These functions have all the properties stated in the conclusions of Theorem I, except that it remains to show that $\zeta(x, y, t)$ satisfies a Lipschitz condition with respect to the time. Let t_1 be greater than t_2. To study the difference $\zeta(x, y, t_1) - \zeta(x, y, t_2)$ consider the moving point which passes through the point (x, y) at time t_1. Its coordinates, (x', y'), at time t_2 satisfy the relations

$$x = x' + \int_{t_2}^{t_1} u\, d\mu,$$

$$y = y' + \int_{t_2}^{t_1} v\, d\mu,$$

and since, by (9), u and v are bounded we have

$$|x - x'| + |y - y'| \leqq 2A_3C_1|t_1 - t_2|.$$

Then since ζ is constant along the moving point, we have from (8)

$$|\zeta(x, y, t_1) - \zeta(x, y, t_2)| = |\zeta(x', y', t_2) - \zeta(x, y, t_2)|$$
$$\leqq C_2C_3\{|x - x'| + |y - y'|\} \leqq 2A_3C_1C_2C_3|t_1 - t_2|,$$

where the constant $2A_3C_1C_2C_3$ depends on K, $0 \leqq t \leqq K$. It follows from this that the first order derivatives of $u(x, y, t)$ and $v(x, y, t)$ are continuous in x, y, t; and that $u(x, y, t)$ and $v(x, y, t)$ satisfy a Lipschitz condition with respect to x, y, t. This completes the proof of Theorem I.

From Theorem I it can be inferred that the velocity components $u(x, y, t)$ and $v(x, y, t)$ are differentiable with respect to the time. To show this we first outline the proof that $\partial\phi(x, y, t)/\partial t$ exists, ϕ being the flow function.

$$\phi(x, y, t) = -\frac{1}{2\pi}\int_R \zeta(x', y', t)G(x, y, x', y')dx'dy' = -\frac{1}{2\pi}\int_{R_0} - \frac{1}{2\pi}\int_{R-R_0},$$

where R_0 is a circle of radius $\epsilon > 0$ with center at x, y. The paths of the moving point issuing from points (a, b) of the region R at time $t=0$ and passing through points (x', y') at time t define a one-to-one, continuous, area-preserving relation between (a, b) and (x', y'), $x' = x'(a, b, t)$, $y' = y'(a, b, t)$. The points (x', y') lying in $R - R_0$ correspond to points (a, b) lying in a region which we call $R - R_1$. In the integral over the region $R - R_0$ at time t we transform the variables of integration from x', y' to a, b, the region of integration becoming $R - R_1$, and since the vortex density is constant along the moving point we have

$$-\frac{1}{2\pi}\int_{R-R_0} \zeta(x', y', t)G(x, y, x', y')dx'dy'$$
$$= -\frac{1}{2\pi}\int_{R-R_1} \zeta(a, b, 0)G(x, y, x', y')da\,db,$$

where in $G(x, y, x', y')$ on the right $x' = x'(a, b, t)$, $y' = y'(a, b, t)$. In the integral over the region R_0 we substitute the letters a, b for x', y', this being a change in nomenclature and not a transformation of variables. By definition

$$\phi(x, y, 0) = -\frac{1}{2\pi}\int_R \zeta(a, b, 0)G(x, y, a, b)da\,db.$$

Thus one obtains

$$\frac{\phi(x, y, t) - \phi(x, y, 0)}{t} = -\frac{1}{2\pi}\int_{R_0} \left\{\frac{\zeta(a, b, t) - \zeta(a, b, 0)}{t}\right\} G(x, y, a, b)da\,db$$
$$-\frac{1}{2\pi}\int_{R-R_0} \zeta(a, b, 0)\left\{\frac{G(x, y, x', y') - G(x, y, a, b)}{t}\right\} da\,db$$

$$+\left\{-\frac{1}{2\pi t}\int_{R_0-R_1}+\frac{1}{2\pi t}\int_{R_1-R_0}\right\}\{\zeta(a,b,0)G(x,y,x',y')da\,db\}$$
$$=\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}.$$

In the expression II,

$$\frac{x'-a}{t}\to u(a,b,0),\qquad \frac{y'-b}{t}\to v(a,b,0)\quad\text{as}\quad t\to 0.$$

By choosing first ϵ small and then t small it can be shown that I, III, and IV approach zero and II approaches a finite limit.

$$\lim_{t\to 0}\frac{\phi(x,y,t)-\phi(x,y,0)}{t}$$
$$=-\frac{1}{2\pi}\int_R\zeta(a,b,0)\left\{u(a,b,0)\frac{\partial G(x,y,a,b)}{\partial a}+v(a,b,0)\frac{\partial G(x,y,a,b)}{\partial b}\right\}da\,db.$$

This shows that $\partial\phi/\partial t$ exists at time $t=0$, and in the same way it exists for all time. Since ζ, u, and v satisfy a Lipschitz condition with respect to x, y, t, it is well known in potential theory that the first order derivatives of $\partial\phi/\partial t$ with respect to x and y exist and are continuous in x, y, t. Then the order of differentiation is immaterial,

$$\frac{\partial}{\partial y}\frac{\partial\phi}{\partial t}=\frac{\partial}{\partial t}\frac{\partial\phi}{\partial y}\equiv\frac{\partial u}{\partial t},$$

so that $\partial u/\partial t$ and $\partial v/\partial t$ exist and are continuous.

To construct the pressure, let C be any closed Jordan curve with continuously turning tangent moving with the fluid. If R_C is the interior of C we have

$$-\int_{R_C}\zeta(x,y,t)dx\,dy=\int_C u\,dx+v\,dy,$$

and since ζ is constant along the moving point and the fluid is incompressible $\int_{R_C}\zeta dxdy$ is a constant independent of the time,

$$\int_C u\,dx+v\,dy=\text{constant}.$$

Differentiating along the moving curve we have

$$\int_C\left\{\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{\partial u}{\partial t}\right)dx+\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{\partial v}{\partial t}\right)dy\right\}$$
$$+\frac{1}{2}\int_C\frac{\partial}{\partial s}(u^2+v^2)ds=0.$$

The second integral is zero around any closed curve so the first is also. Thus a pressure defined by

$$p(x, y, t) = -\sigma \int_{x_0, y_0}^{x, y} \left\{ \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial u}{\partial t} \right) dx + \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial v}{\partial t} \right) dy \right\}$$

exists and satisfies the Euler dynamical equations.

Appendix

It was assumed in Theorem I that the plane region R is simply connected, but this is an unnecessary restriction. Theorem I may be extended so as to apply to multiply-connected regions, but we shall consider only the extension to doubly-connected regions as this will illustrate the principle.

Let R be an open doubly-connected plane region whose boundary, B, consists of two distinct Jordan curves such that if $x=x(s)$, $y=y(s)$ defines either of the curves, d^3x/ds^3 and d^3y/ds^3 exist and satisfy a uniform Hölder condition of some positive order less than one.

In Theorem I the same hypotheses (a), (b), and (c) are made and all the conclusions (d), (e), (f), (g), (h), and (i) are drawn with the additional statement (j).

(j) *The circulation around the inner boundary is a given constant independent of the time.*

The constancy of this circulation will insure the existence of a single-valued pressure.

In the proof the following definitions are made:

$$x_0(a, b, t) \equiv a, \qquad y_0(a, b, t) \equiv b.$$

If $x_{n-1}(a, b, t)$ and $y_{n-1}(a, b, t)$ have been defined, $n \geqq 1$,

$$\zeta_{n-1}(x, y, t) = \zeta(a, b), \quad \text{where} \quad x = x_{n-1}(a, b, t), \quad y = y_{n-1}(a, b, t),$$

$$\phi_n(x, y, t) = -\frac{1}{2\pi}\int_R \zeta_{n-1}(x', y', t)G(x, y, x', y')dx'dy' + f_n(t)h(x, y),$$

$$u_n(x, y, t) = \frac{\partial}{\partial y}\phi_n(x, y, t), \qquad v_n(x, y, t) = -\frac{\partial}{\partial x}\phi_n(x, y, t),$$

$$x_n(a, b, t) = a + \int_0^t u_n(x_n(a, b, \mu), y_n(a, b, \mu), \mu)d\mu,$$

$$y_n(a, b, t) = b + \int_0^t v_n(x_n(a, b, \mu), y_n(a, b, \mu), \mu)d\mu.$$

Let $h(x, y)$ be continuous in $R+B$, harmonic in R, have the value one on

the inner boundary, and vanish on the outer boundary of R. $f_n(t)$ is a function to be defined presently.

It is seen that the only variation from the case of a simply connected region is that here the function $f_n(t)h(x, y)$ enters. At time $t=0$ the circulation around q_1 (q_1 is the inner boundary, and q_2 is the outer boundary of R) due to the velocity components $u(x, y)$ and $v(x, y)$ is a given constant determined by these given velocity components. Suppose the value of this circulation is W. We shall define $f_n(t)$ so that the circulation around q_1 due to the velocity components $u_n(x, y, t)$ and $v_n(x, y, t)$ is for all time the given constant W.

From definition of u_n and v_n the circulation around q_1 is

$$\int_{q_1} u_n dx + v_n dy = -\int_{q_1} \frac{\partial \phi_n}{\partial \nu} ds,$$

where ν is the direction of the inner normal. The right-hand side of this equation may be transformed by means of Green's second identity

$$\int_R h(x, y)\zeta_{n-1}(x, y, t)dx\,dy = -\int_{q_1} \frac{\partial \phi_n}{\partial \nu} ds + f_n(t)\int_{q_1} \frac{\partial h}{\partial \nu} ds,$$

where h is the harmonic function defined above. Defining $f_n(t)$ by

$$f_n(t) = \frac{1}{\lambda}\left\{-W + \int_R h(x, y)\zeta_{n-1}(x, y, t)dx\,dy\right\}, \tag{12}$$

where $\lambda = \int_{q_1}(\partial h/\partial \nu)ds \neq 0$, it follows that the circulation around q_1 due to the velocity components u_n and v_n is for all time the given constant W.

The proof of Theorem I is not essentially altered by the introduction of the function $f_n(t)h(x, y)$ in the definition of $\phi_n(x, y, t)$. First, from constructon it is seen that $\zeta_n(x, y, t)$ is uniformly bounded, and by (12) this implies that $f_n(t)$ is uniformly bounded,

$$|f_n(t)| \leqq \left|\frac{1}{\lambda}\right|\left\{|W| + C_1\int_R dx\,dy\right\} \leqq C_8,$$

where C_8 is independent of n, x, y, t. Due to the smoothness properties of the boundary of R the harmonic function $h(x, y)$ has continuous* (and bounded) third order derivatives in $R+B$, so the third order partial derivatives of $f_n(t)h(x, y)$ with respect to x and y are bounded independently of n, x, y, t.

Lemma I is proved as before. The inequality (4) is true with a larger constant in place of A_1C_1,

$$|D_1\phi_n(x_1, y_1, t) - D_1\phi_n(x_2, y_2, t)| \leqq C_9 r \ln\frac{A_2}{r},$$

* Kellogg, these Transactions, vol. 9 (1908), pp. 39–50 and vol. 13 (1912), pp. 108–132.

where C_9 is independent of n, x, y, t. From this one derives a first approximation, the inequality (5) with A_1C_1 replaced by C_9. Then using this result another approximation proves Lemma I, and the inequalities (8) and (9) with constants independent of n also follow.

To show that the sequence $\zeta_n(x, y, t)$ converges we have first from the boundedness of ζ_n,

$$|\zeta_1(x, y, t) - \zeta_0(x, y, t)| \leqq 2C_1, \qquad 0 \leqq t < \infty.$$

Suppose it has been shown that

$$|\zeta_n(x, y, t) - \zeta_{n-1}(x, y, t)| \leqq 2C_1 \frac{(mt)^{n-1}}{(n-1)!}, \qquad 0 \leqq t \leqq K.$$

Then from (12) it follows that

$$|f_{n+1}(t) - f_n(t)| \leqq \left|\frac{1}{\lambda}\right| \int_R |\zeta_n(x, y, t) - \zeta_{n-1}(x, y, t)|\, dx\, dy, \tag{13}$$

and finally

$$|u_{n+1}(x, y, t) - u_n(x, y, t)| \leqq 2C_1C_{10} \frac{(mt)^{n-1}}{(n-1)!},$$

with a similar inequality for $v_{n+1} - v_n$, where C_{10} is independent of n, x, y, t. The same procedure that was used in the proof of Theorem I shows that

$$|\zeta_{n+1}(x, y, t) - \zeta_n(x, y, t)| \leqq 2C_1 \frac{(mt)^n}{n!}, \qquad 0 \leqq t \leqq K,$$

if m is a sufficiently large constant which depends on K but not on n.

This implies that ζ_n, u_n, v_n, x_n, y_n and now, by (13), $f_n(t)$ converge uniformly in every finite time interval. By uniform convergence the relation (12) holds in the limits

$$f(t) = \frac{1}{\lambda}\left\{-W + \int_R \zeta(x, y, t)h(x, y)dx\, dy\right\},$$

where $f(t) = \text{limit } f_n(t)$, $\zeta = \text{limit } \zeta_n$. Then the circulation around q_1 due to the velocity components $u(x, y, t)$ and $v(x, y, t)$ is for all time the given constant W

$$\int_{q_1} u(x, y, t)dx + v(x, y, t)dy = \int_R h(x, y)\zeta(x, y, t)dx\, dy - f(t)\int_{q_1} \frac{\partial h}{\partial \nu} ds = W.$$

This verifies the statement (j), and the other conclusions of Theorem I are

verified as before. To infer the conclusion (d) we know that at time $t=0$ the velocity components $u(x, y, 0)$, $v(x, y, 0)$ and the given velocity components $u(x, y)$, $v(x, y)$ have the same vortex density $\zeta(x, y, 0)=\zeta(x, y)$, the same circulation, W, around the inner boundary, and divergence zero.

Using the transformation of variables that was employed in proving the existence of $\partial\phi/\partial t$ it can be shown that df/dt exists,

$$\frac{df(t)}{dt}=\frac{1}{\lambda}\int_R\left\{u(x, y, t)\frac{\partial h(x, y)}{\partial x}+v(x, y, t)\frac{\partial h(x, y)}{\partial y}\right\}\zeta(x, y, t)dx\,dy.$$

Then $\partial\phi/\partial t$ exists and is continuous,

$$\frac{\partial\phi(x, y, t)}{\partial t}=-\frac{1}{2\pi}\int_R\zeta(x', y', t)\left\{u\frac{\partial G}{\partial x'}+v\frac{\partial G}{\partial y'}\right\}dx'dy'+h(x, y)\frac{df}{dt}.$$

Then from potential theory we know that the first order partial derivatives of $\partial\phi/\partial t$ with respect to x and y exist and are continuous. By continuity the order of differentiation is immaterial so $\partial u(x, y, t)/\partial t$ and $\partial v(x, y, t)/\partial t$ exist and are continuous in x, y, t.

To construct the pressure, let C be a closed Jordan curve in R with continuously turning tangent and moving with the fluid. If C encloses the inner boundary, q_1, of R we have

$$\int_C u\,dx+v\,dy=\int_{q_1}u\,dx+v\,dy-\int_{R_C}\zeta(x, y, t)dx\,dy,$$

and it follows from the constancy of the circulation around q_1 and the constancy of $\int_{R_C}\zeta(x, y, t)\,dx\,dy$ that the circulation around C is a constant independent of the time. Thus if C is any closed curve moving with the fluid, whether or not C encloses the inner boundary of R, the circulation around C is a constant independent of the time. Then it follows that

$$\int_C\left\{\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{\partial u}{\partial t}\right)dx+\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{\partial v}{\partial t}\right)dy\right\}=0,$$

and a single valued pressure can be constructed satisfying the Euler dynamical equations.

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